Large Random Stochastic Matrices

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There exists a large literature, starting from the pioneering work of Eugene Wigner, about the statistics of spectra for large random matrices. Several kinds of ensembles have been thoroughly studied and characterised in high detail. Basic references are for instance

Mehta, M.L., Random Matrices, Academic Press, 2004

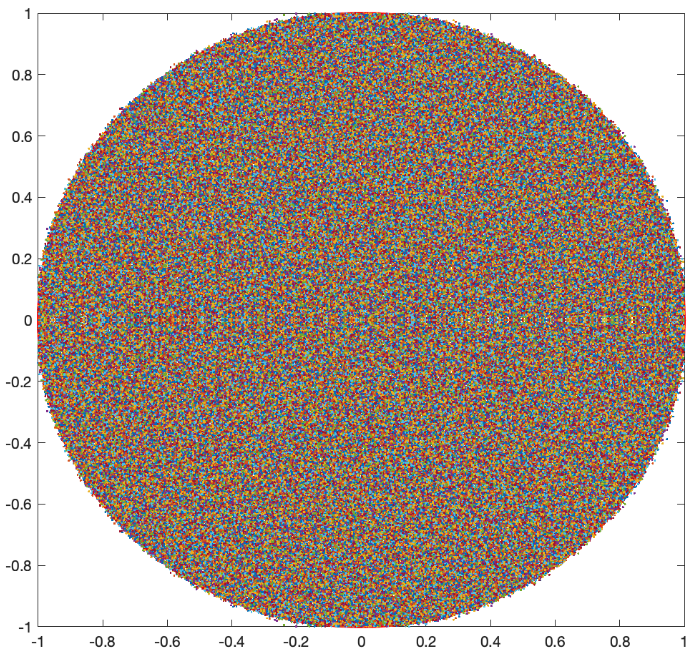
Akemann, G, Baik, J; Di Francesco P. (2011). The Oxford Handbook of Random Matrix Theory, Oxford U.P. 2011.

You may also consult the Wikipedia chapter on "Random Matrix".

Here we suggest to study the case of random stochastic matrices of large dimensions. To our knowledge is only limited information from the analytic standpoint, but it's easy to get some hint about their spectral properties through computer simulations (but see the recent reference at the end of this note).

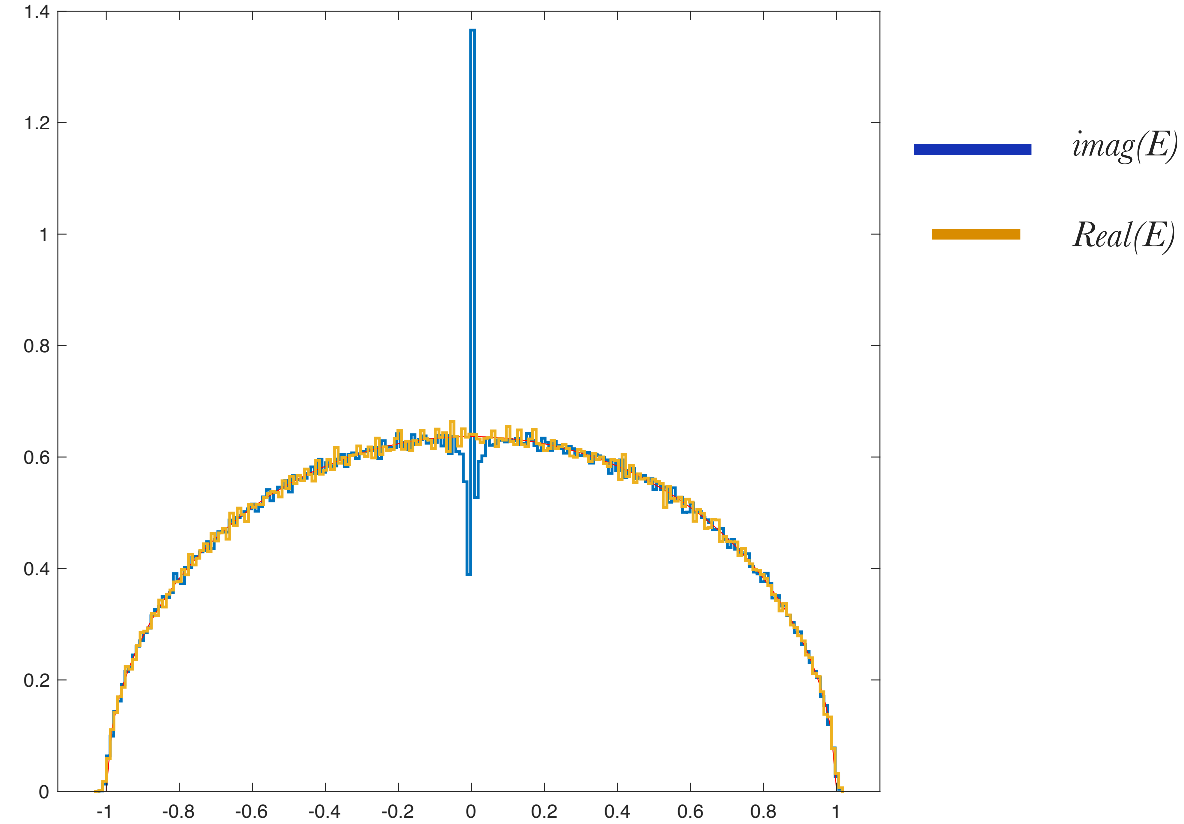
Evidence has it that the spectrum for such large positive matrices with column sum equal to one is confined to a disk in the complex plane of radius $1/\sqrt{N}$, N being the matrix dimension, except for the eigenvalue 1 which is always in the spectrum of a stochastic matrix. We performed several simulations by taking the following definition of the measure in the space of N-dimensional positive matrices: each column is uniformly distributed in the simplex

$Real(E)$

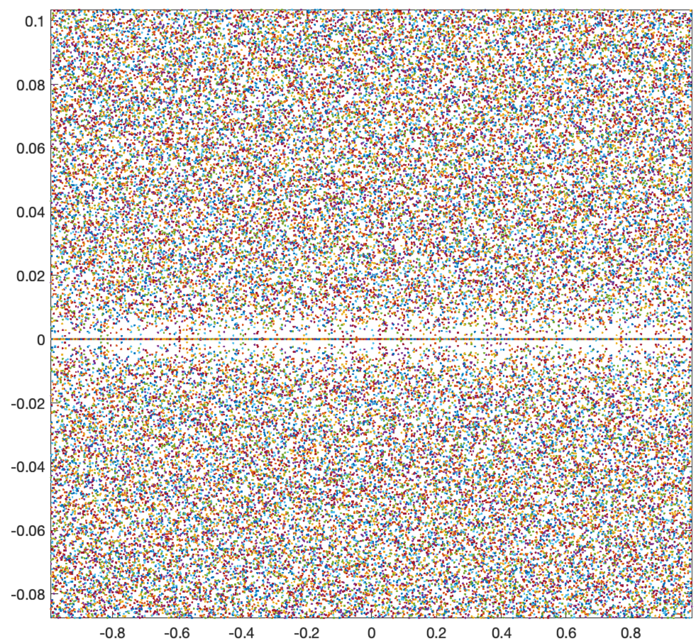


$S_j = \left\{ x_{ij} \geq 0, \sum_i x_{ij} = 1 \right\}$. A typical experiment gives a scatter plot as in the first picture (eigenvalue 1 is there, but outside the domain represented), and all eigenvalues have been scaled by $\sqrt{N}$.

A close up view of the spectrum near the real axis.



One is led to infer that the distribution of eigenvalues is uniform in the disk, in the limit of large dimension, except for a rather marked concentration on the real axis, as can be verified by the histogram of $Im(E)$.

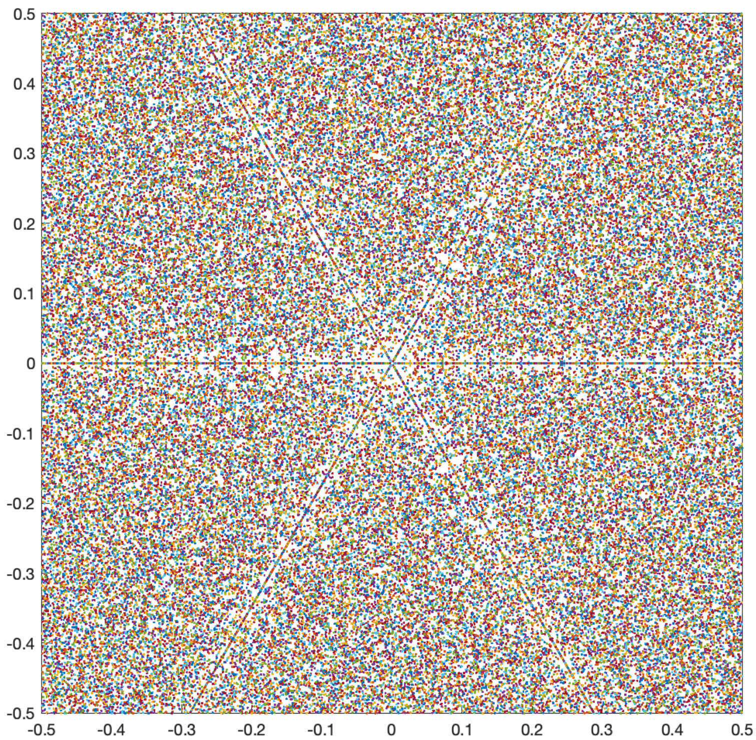


We did some experiments on a variant of this idea of random stochastic (Markov) matrices. Let's consider the class of positive matrices, column sum equal to one, with a block structure of the kind

$$\begin{pmatrix} 0 & 0 & 0 & \dots & M_{1k} \\ M_{21} & 0 & 0 & \dots & 0 \\ 0 & M_{32} & 0 & \dots & 0 \\ 0 & 0 & M_{43} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & M_{k,k-1} & 0 \end{pmatrix}$$

where each block is extracted from the ensemble of stochastic matrices as discussed above. Now the numerical evidence on large matrices (e.g. NxN blocks with up to 5 x5 block structure) suggests that the spectrum is concentrated in a disk of radius $1/\sqrt{N}$, independently of k. According to Perron-Frobenius there are k eigenvalues on the unit circle (k-roots of unity) and the pattern reflects a k-fold symmetry.

In the following picture k=3.



The challenge is then to prove or disprove the statements:

1) *for N-dimensional random stochastic matrices the spectrum is given by the union of a complex disk* $D = \{E \in \mathbb{C}, |E| \leq 1/\sqrt{N}\}$*, with the single eigenvalue E=1 and a discrete component on the real axis* $D_{exception} = \{E \in \mathbb{R}, |E| \leq 1/\sqrt{N}\}$.
2) *For a k-block structure and dimension N of each block the statement is the same (spectrum given by* $D \subset \{E \in \mathbb{C}, |E| \leq 1/\sqrt{N}\}$ *and a concentration of eigenvalues along the diameters at angle* π/k*).*

Notice that it is the dimension of the blocks, NOT of the whole matrix, which sets the size of the circle.

As for point 1 the result is already known, see e.g.

C.Bordenave, P. Caputo and D. Chafaï, "Circular law theorem for random Markov matrices, ArXiv:0808.1502v3[math.PR].