Dedicated to Carlo Maria Becchi

the Ricci flow
and
the Sausage

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MISSED OPPORTUNITIES[1]

BY FREEMAN J. DYSON

It is important for him who wants to discover not to confine himself to one chapter of science, but to keep in touch with various others.
JACQUES HADAMARD

- A tale of two scientific quests - integrable QFT models and the classification of Riemannian manifolds, another chapter in Dyson's "MISSED OPPORTUNITIES" ?

- The renormalization of two-dim nonlinear "Sigma-model", integrable deformations and TBA *(Fateev and Al.B.Zamolodchikov)*

- The proof of Poincaré's conjecture after 100 years *(Grisha Perelman)*

- At the end, a personal recollection involving *Carlo Maria Becchi*.

The study of the "nonlinear sigma model" started in the '70s thanks to *Polyakov*, *Esker-Honenkamp*, and the Saclay school, *Brezin*, *Zinn-Justin*, *Le Grillou*.

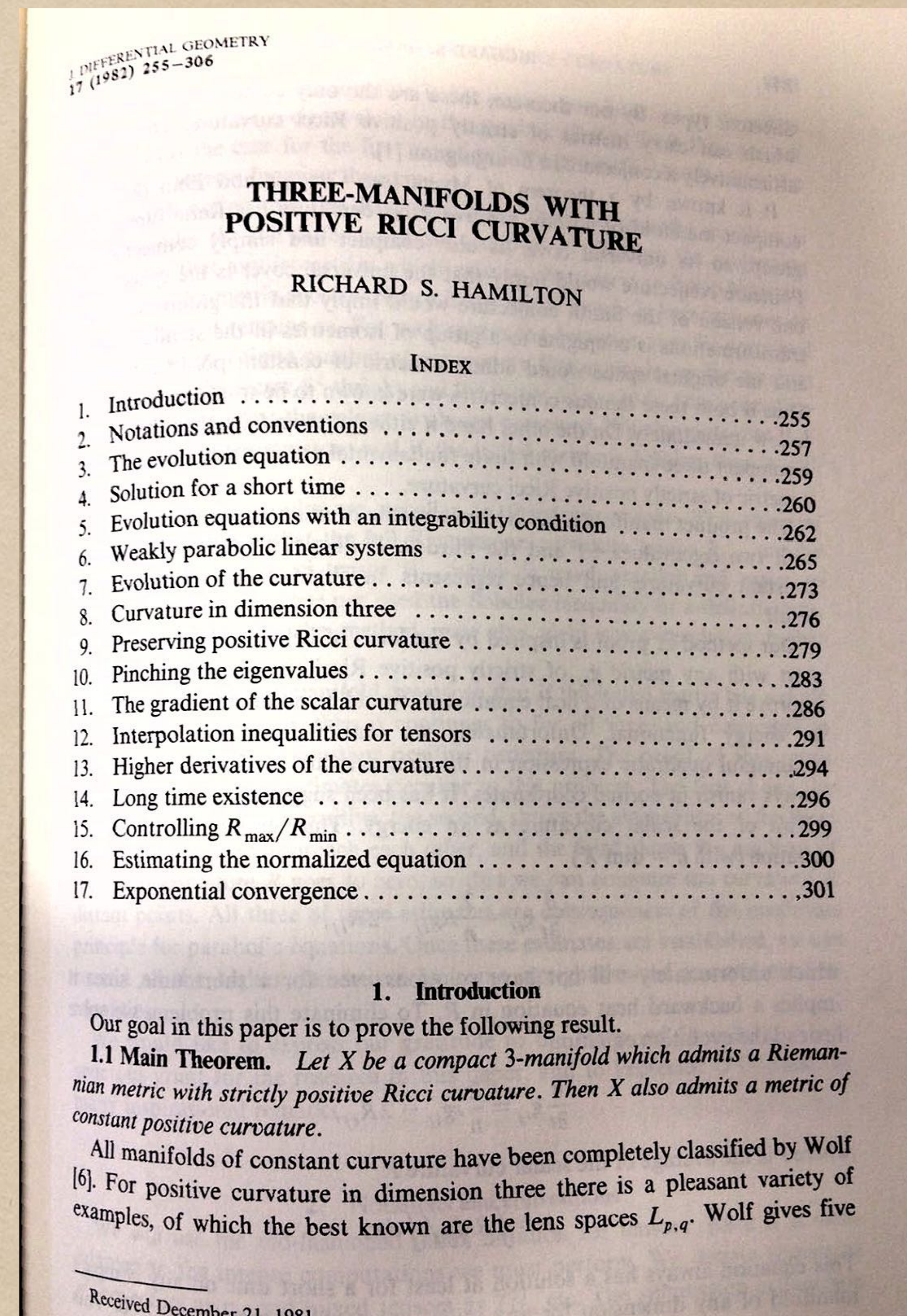
Our story however starts with **Dan Friedan, 1980**: "*Nonlinear Models in 2+ε Dimensions*", *PRL 45 (1980)*

$$S(\varphi) = \Lambda^{\epsilon} \int dx \, \tfrac{1}{2} T^{-1} g_{ij}(\varphi(x)) \, \partial_\mu \varphi^i(x) \, \partial_\mu \varphi^j(x)$$

The field ϕ takes values in the "target space M", a compact 2-dim Riemannian manifold. Path integral quantization leads to a regularized QFT. The "coupling" g_{ij}, under renormalizazion at two-loops evolves according to

$$\mu \frac{\partial}{\partial \mu} g_{ij} = -R_{ij} - \tfrac{1}{2} T \, R_{ik\ell n} R_{jk\ell n} + O(T^2)$$

Full detail is found in Friedan's PhD Thesis, which appeared only in 1985 on Annals of Physics; this is the favorite quotation in the Math literature,probably because it appeared after Hamilton's paper? More later on…



J. DIFFERENTIAL GEOMETRY
17 (1982) 255–306

THREE-MANIFOLDS WITH POSITIVE RICCI CURVATURE

RICHARD S. HAMILTON

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1. Introduction

Our goal in this paper is to prove the following result.

1.1 Main Theorem. *Let X be a compact 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature. Then X also admits a metric of constant positive curvature.*

All manifolds of constant curvature have been completely classified by Wolf [6]. For positive curvature in dimension three there is a pleasant variety of examples, of which the best known are the lens spaces $L_{p,q}$. Wolf gives five

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- Developments: Fateev and Zamolodchikov analized the problem of quantizing the nonlinear sigma-model identifying it with a class of "Factorized Scattering Amplitudes" via Bethe Ansatz techniques. They introduced a class of exact solutions to the RG equation, the so-called "Sausage solutions" which are conjectured to act as "attractors" in the RG flow.
- Belardinelli, Destri and myself studied the RG equation numerically to find that the sausage solutions actually act as attractors in the space of metrics.

ELSEVIER Nuclear Physics B 443 [FS] (1995) 607–619

A numerical study of the RG equation for the deformed O(3) nonlinear sigma model

L. Belardinelli [a,1], C. Destri [a,2], E. Onofri [b,3]

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The numerical integration applies to a simplified form of the RG equation, obtained by specializing the metric to a conformal one, $g_{ij} = e^{\lambda(\varphi)}\,\delta_{ij}$ which yields ($x \in S^2$) a sort of "dynamical Liouville equation"

$$\frac{\partial \lambda(x,t)}{\partial t} = -e^{-\lambda(x,t)}\,\Delta\lambda(x,t)$$

Deforming the sausage in an arbitrary way

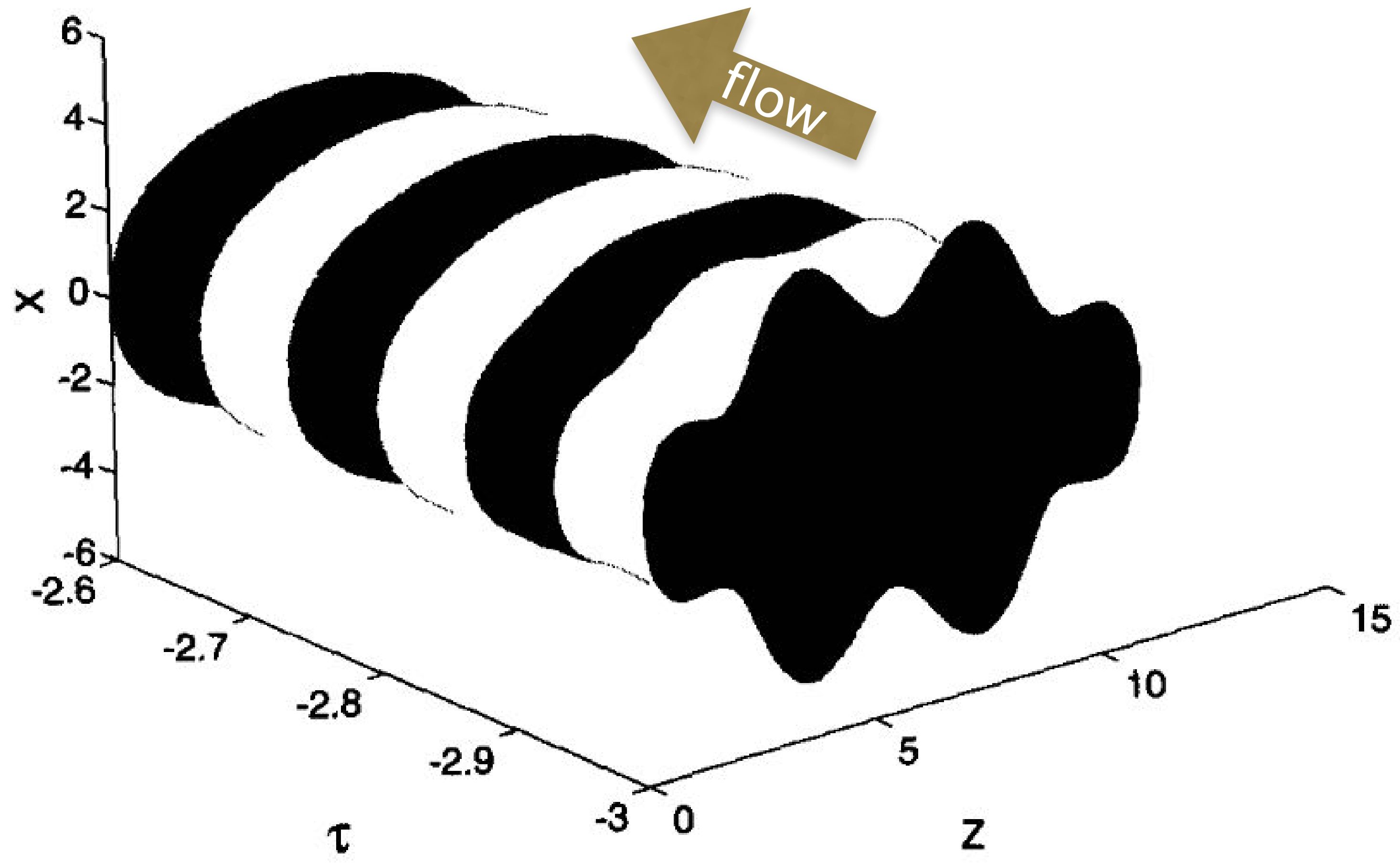


Fig. 4. A pictorial way of illustrating the attractive nature of the sausage solution.

O(2) symmetric geometry evolves toward O(3) symmetry through the sausage.

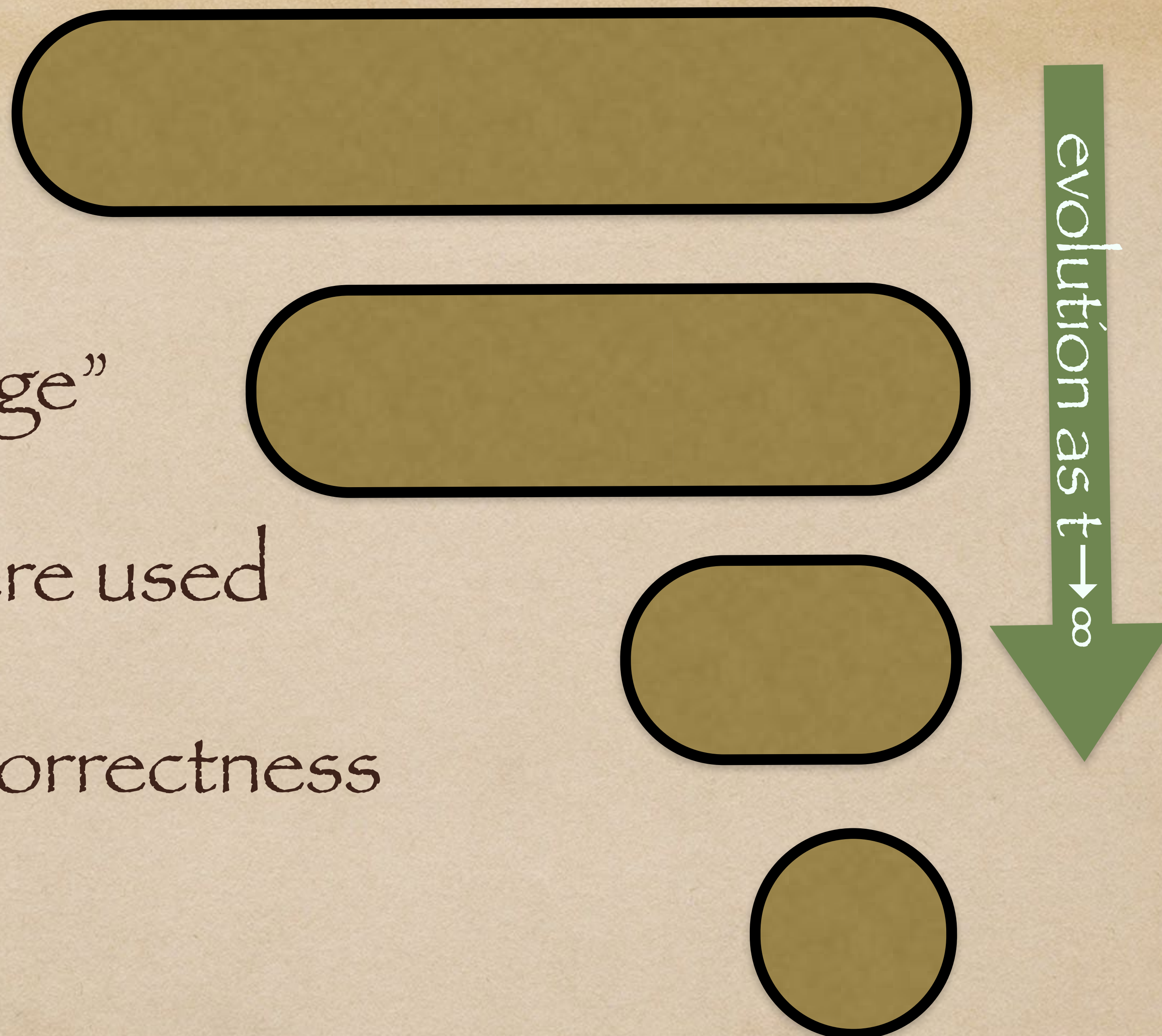
Our numerical code was inspired to the general idea of "spectral methods" in partial differential equations: the idea is to represent partial differential operators using spectral transforms, like Fourier, in such a way that one never uses finite differences approximations. This is particularly useful working on surfaces where Cartesian coordinates are not appropriate. The Laplace operator on S^2 is then represented algebraically by using a spherical harmonics transform. This made feasible also exploring two-loop RG equation with terms $(\Delta\lambda)^2$. The program was entirely realized in Matlab and it's still applicable.

FZ's "Sausage"

solutions were used

to test the correctness

of the code

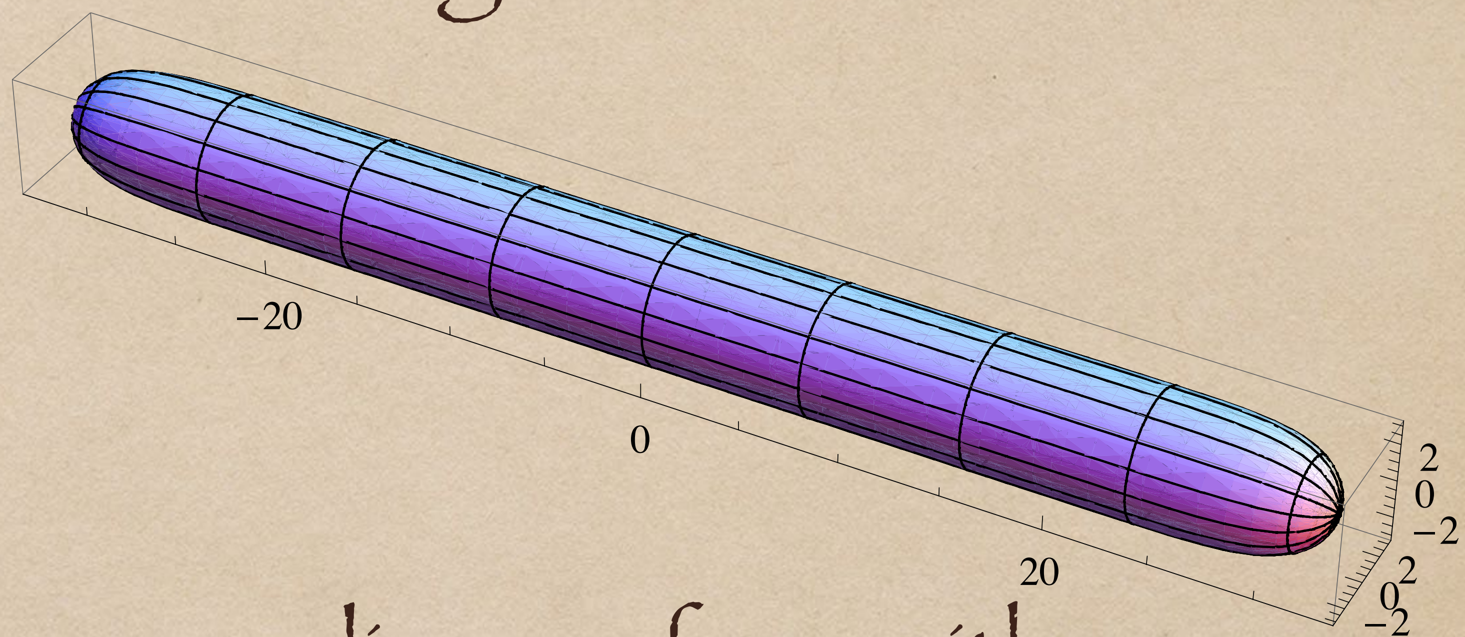


$$\lambda(y,\phi) = -\log\left(a(t) + b(t)\ \cosh(2y)\right)$$

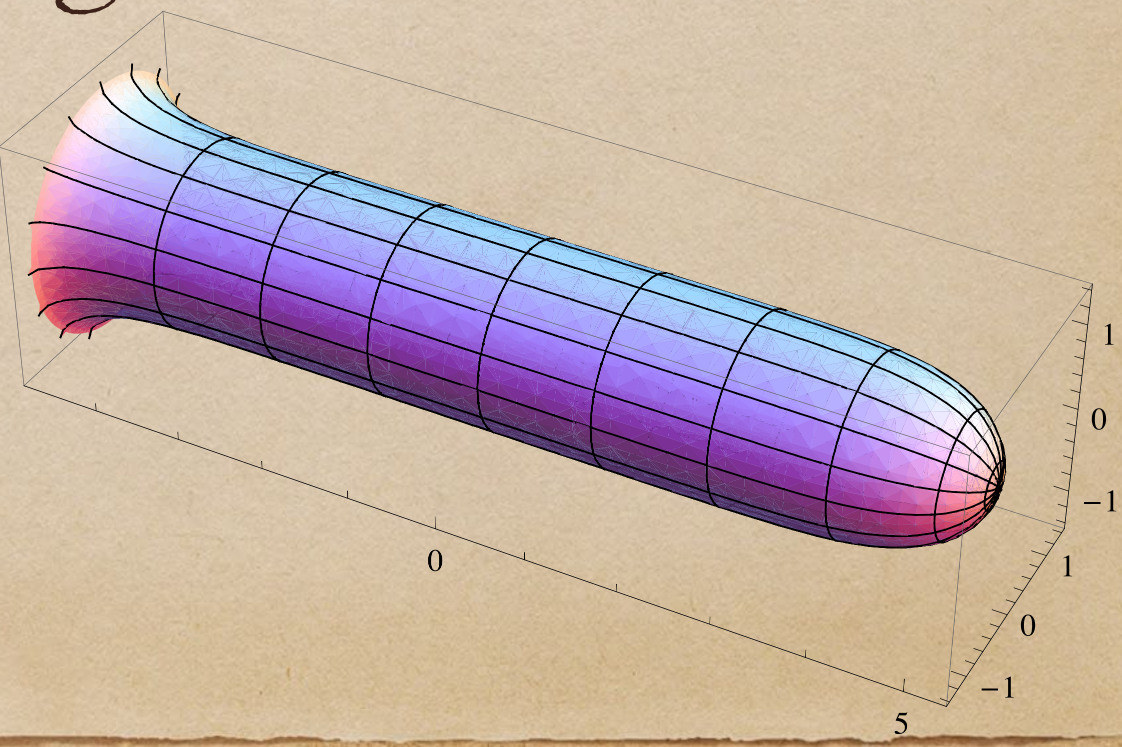
$$a(t) = \tfrac{1}{2}\nu \coth\left(\tfrac{1}{2\pi}\nu(t_0 - t)\right)$$

$$b(t) = \tfrac{1}{2}\nu / \sinh\left(\tfrac{1}{2\pi}\nu(t_0 - t)\right)$$

- another view of sausages



- and the corresponding surface with Minkowski metric:



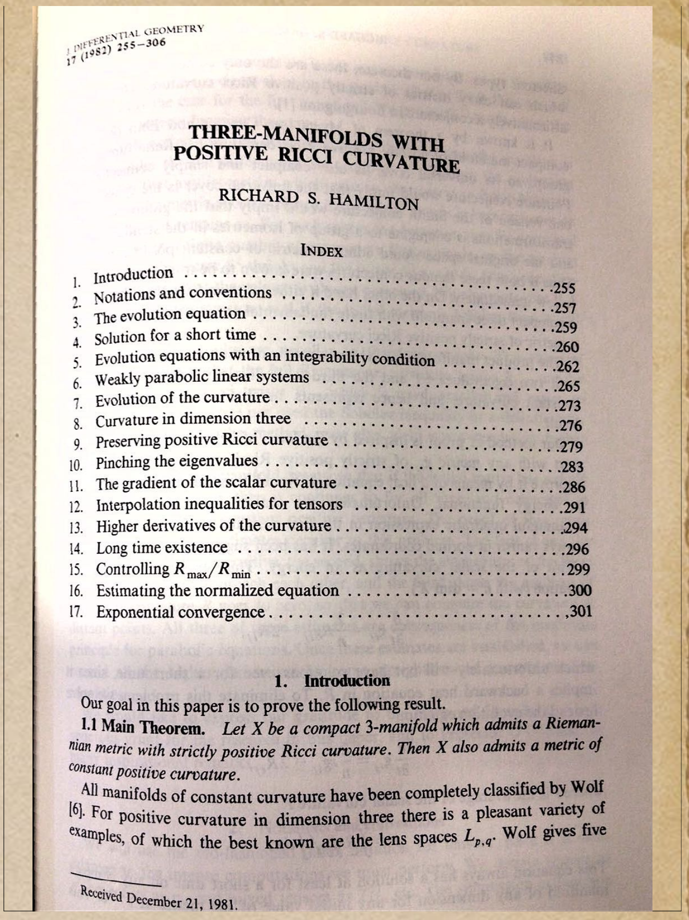
...and meanwhile

- Hamilton's Ricci equation (1982)

Richard Hamilton starts a new chapter in Differential Geometry and Topology by introducing a flow in the space of all metrics generated by the equation

$$\frac{\mathrm{d}\, g_{ij}}{\mathrm{d}\, t} = -2R_{ij}$$

the so-called "Ricci equation"; this can be made a rigorous tool to explore the classification problem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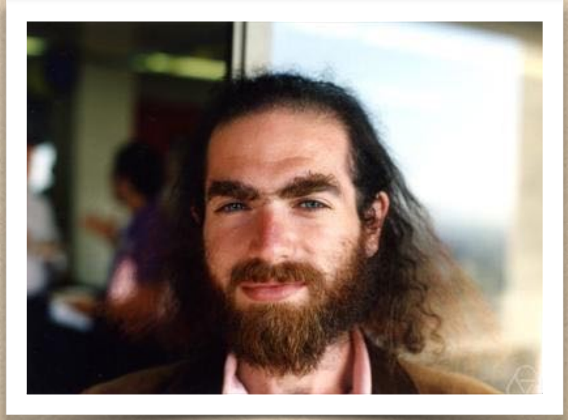
There followed an explosion of activities, documented in the mathematical literature - just try google, asking for “Ricci flow”, you will be overwhelmed by a massive amount of papers, difficult to master for a non-mathematician. This movement culminated in a rather unexpected way and in full glory with Perelman’s proof of Poincaré’s conjecture. Review papers, books both technical and expository appeared in recent times. Just see

B.Chow and D.Knopf, “The Ricci flow, an introduction”, AMS, 2004

D. O’Shea, “The Poincaré conjecture”, W&Co, 2007

Masha Gessen, “Perfect Rigor”, 2018 Carbonio Ed. srl, Milano



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- What I find rather astonishing is that in the era of internet with communication running at the speed of light Theoretical Physicists and Mathematicians have remained unaware for so long of each other efforts. For instance the convergence of any metric on S^2 to the constant curvature metric is contained as a special case in one of the theorems which you may find in Chow's papers. It's true, Mathematics produces very general results often too general for the sake of physics. FZ went a long way in deriving any single detail of the QFT of the nonlinear sigma-model, spectral results which are only qualitatively contained in Perelman's papers. This is what I regard as a candiate for a sequel of Dyson's "Missed Opportunities".

Now we can see some bridges laid down - for instance papers by I. Bakas and G. Carfora, to which I refer you for greater detail. In Chow-Knopf book reference is made to "Witten's black hole" or "cigar solution" to the **RE.** So some barriers are falling.

Let me end now with a personal recollection of a conversation I had long time ago with Carlo Maria.

- Carlo spent a couple of days in Trento, where I served as a newly appointed professor and spent some fruitful years in the late '80s. In a private conversation he expressed the idea that one should *study the evolution of the metric in the sigma model starting from deformed geometries, "something like potatoes"*… I could not benefit much from Carlo suggestion, since I was totally ignorant at the time (and still now…) but I remembered his observation when I was involved in the "sausage" dynamics by Volodya Fateev. So, young men (and madams) listen to Carlo!

- "Potato physics" appeared in C. *Becchi and C.Imbimbo, Nucl.Phys.**B462**, 1996, 571, but beware: it's not light reading after dinner…*

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Thank you
for your patience