

On-Shell Methods in Field Theory

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Lecture I

Tools for Computing Amplitudes

- Focus on gauge theories
...but they are useful for gravity too
- Motivations and connections
 - Particle physics
 - $\mathcal{N}=4$ supersymmetric gauge theories and AdS/CFT
 - Witten's twistor string

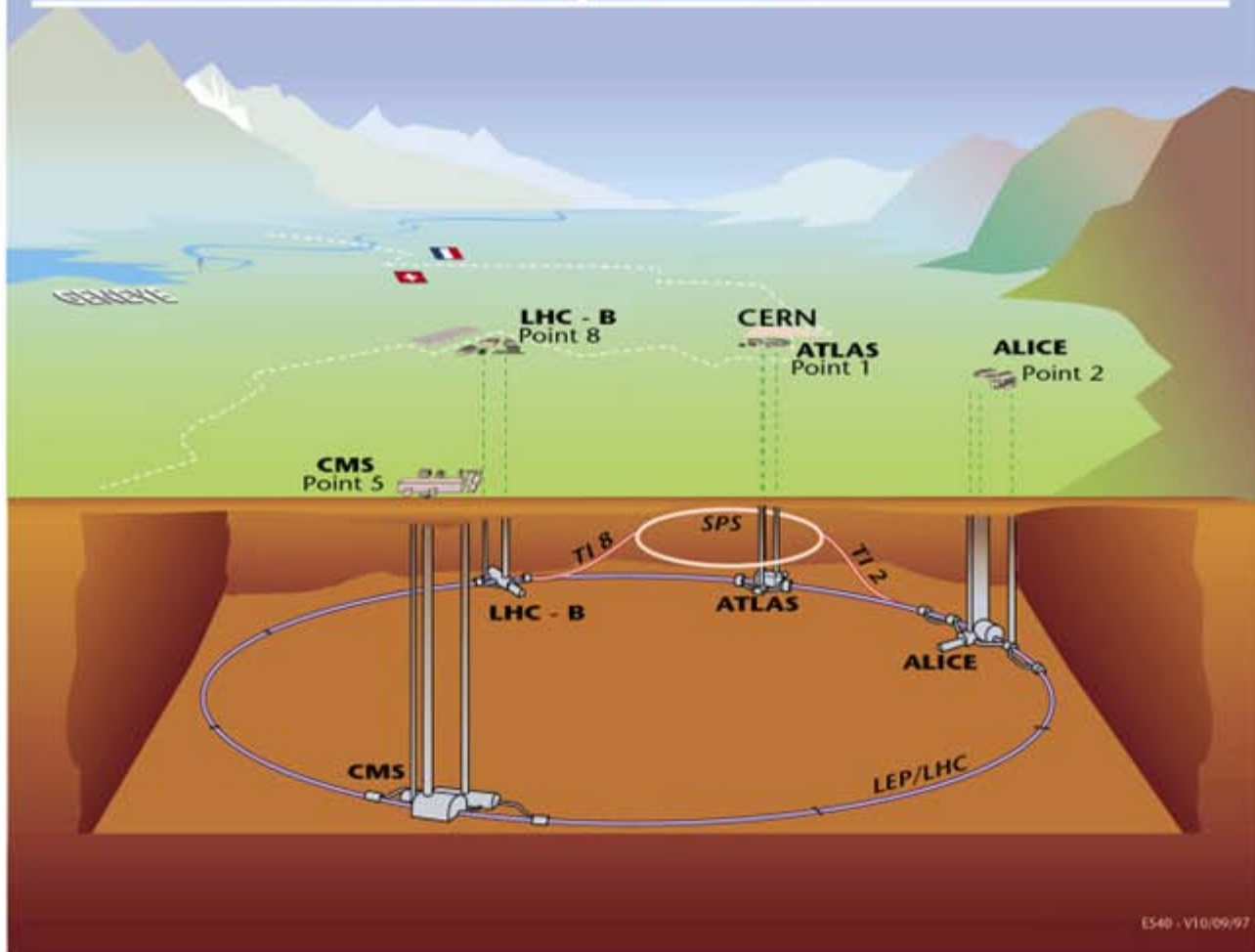
Particle Physics

- Why do we
- Why do we
- What quan



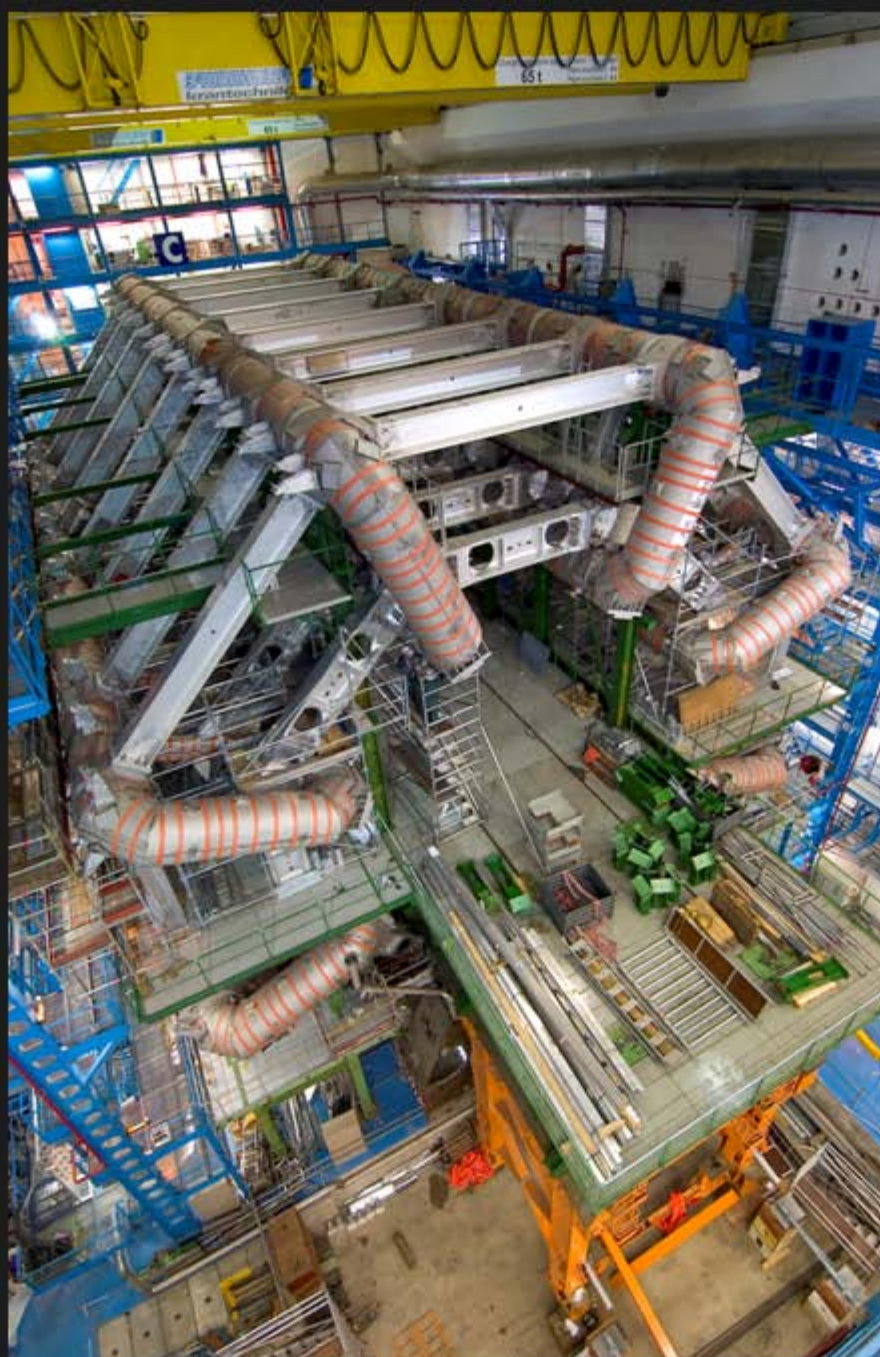
Now 450 to 600 days away...

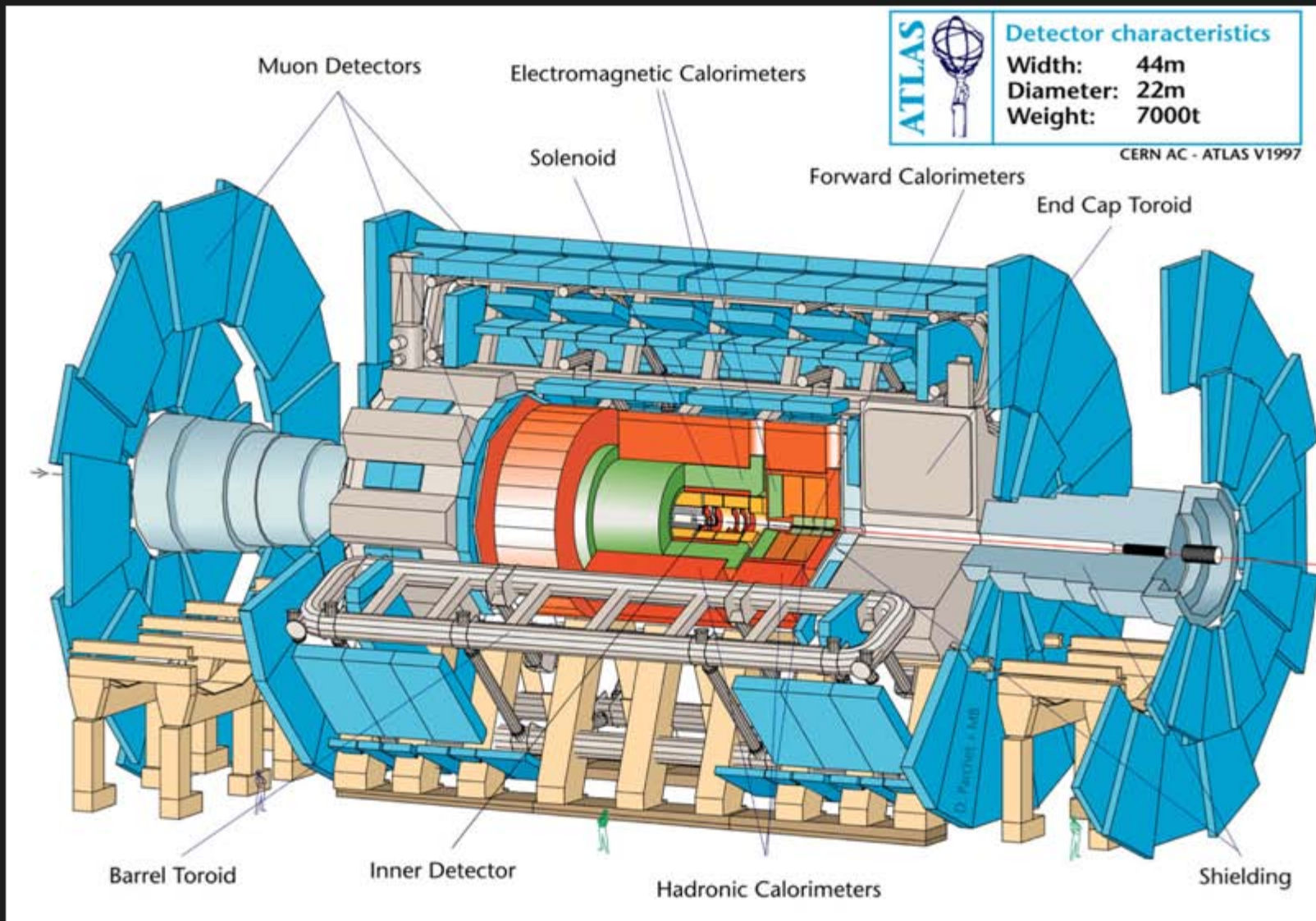
Overall view of the LHC experiments.



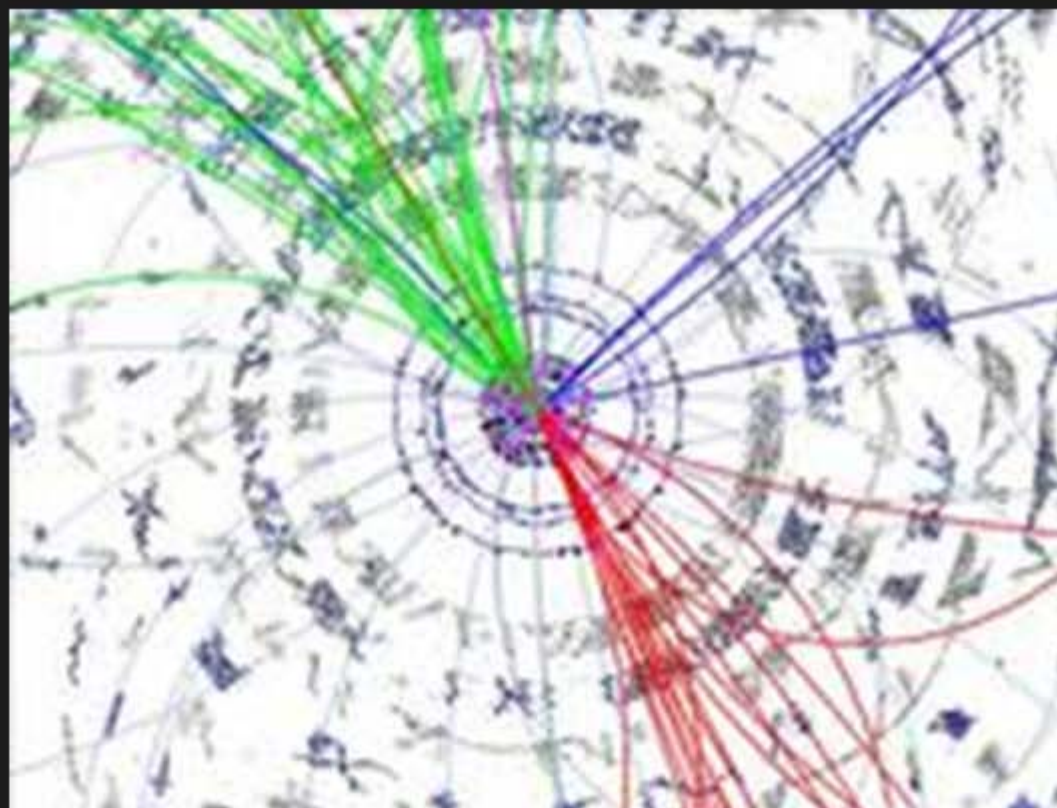


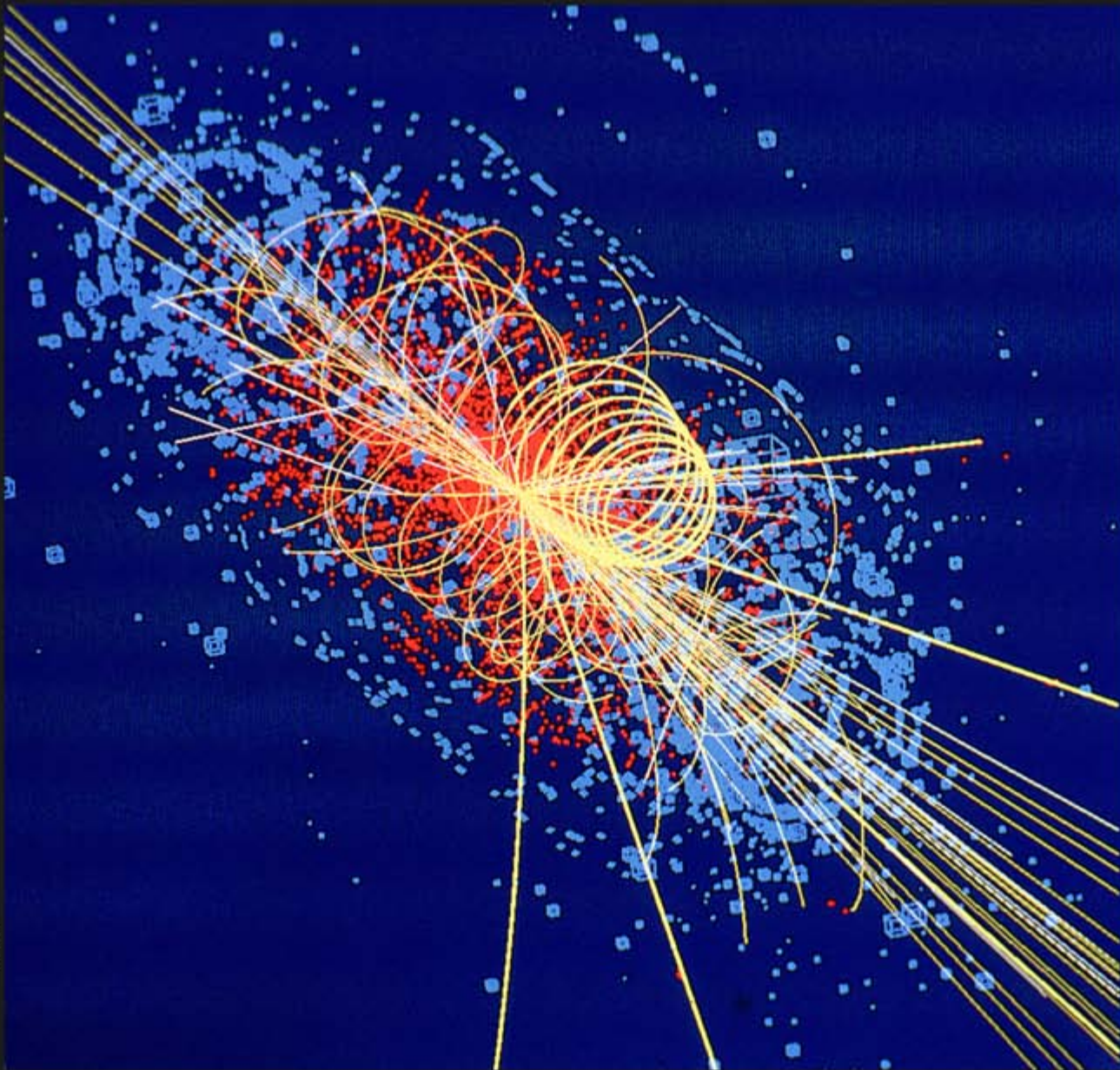




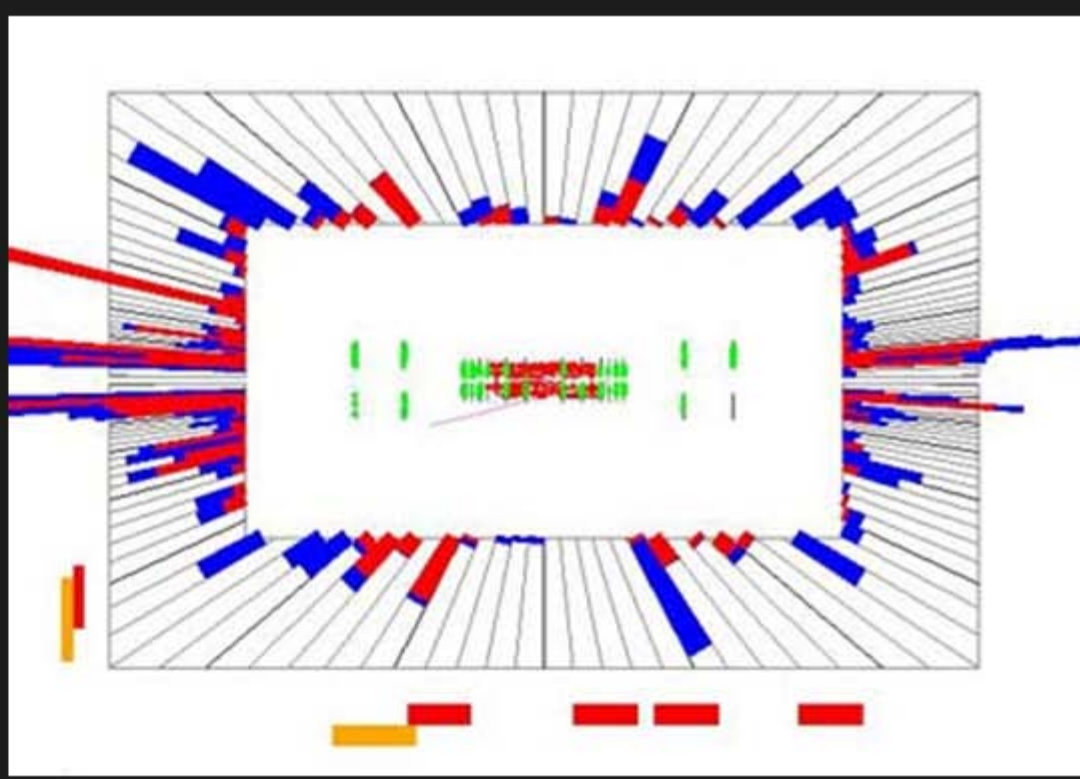
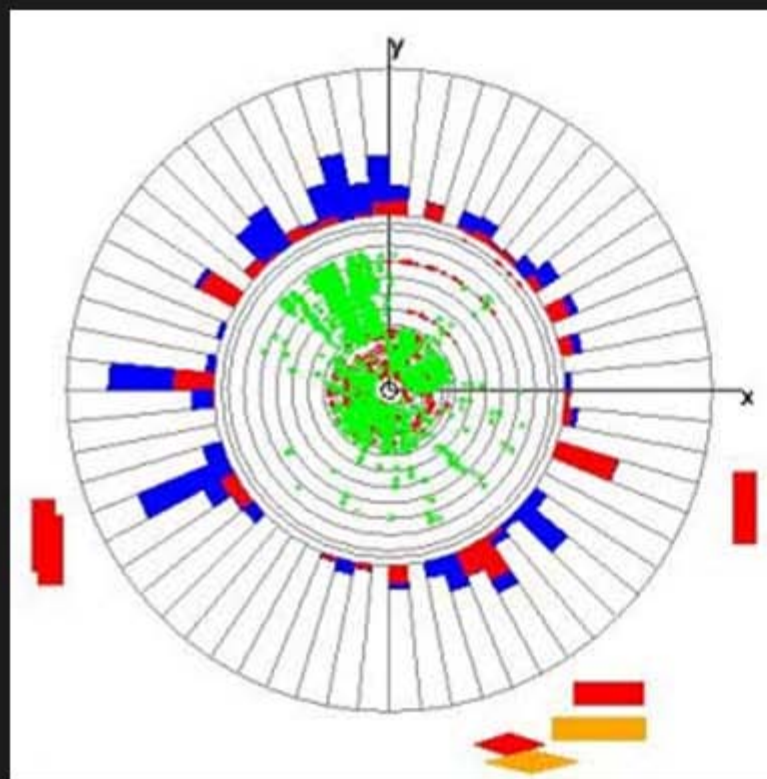


CDF event





CMS



$SU(3) \times SU(2) \times U(1)$ Standard Model

- Known physics, and background to new physics
- Hunting for new physics beyond the Standard Model
- Discovery of new physics
- Compare measurements to predictions — need to calculate signals
- Expect to confront backgrounds
- Backgrounds are large



Event rates

Event production rates at $L=10^{33} \text{ cm}^{-2} \text{ s}^{-1}$ and statistics to tape

Process	Events/s	Evts on tape, 10 fb^{-1}
$W \rightarrow e\nu$	15	10^8
$Z \rightarrow ee$	1	10^7
$t\bar{t}$	1	10^6
gluinos, $m=1 \text{ TeV}$	0.001	10^3
Higgs, $m=130 \text{ GeV}$	0.02	10^4
Minimum bias	10^8	10^7
$b\bar{b} \rightarrow \mu X$	10^3	10^7
QCD jets $p_T > 150 \text{ GeV}/c$	10^2	10^7

assuming 1%
of trigger
bandwidth

- ⇒ statistical error negligible after few days!
- ⇒ dominated by systematic errors (detector understanding, luminosity, theory)

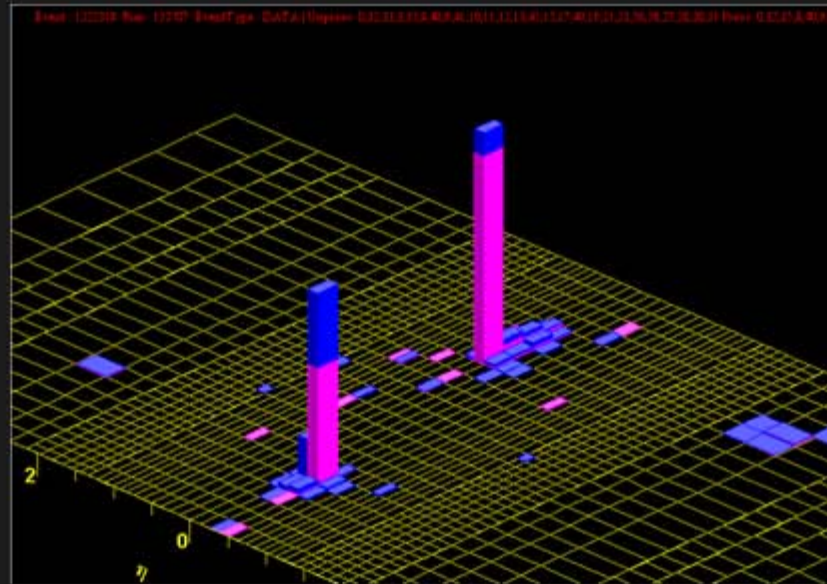
Hunting for New Physics

- Yesterday's new physics is tomorrow's background
- To measure new physics, need to understand backgrounds in detail
- Heavy particles decaying into SM or invisible states
 - Often high-multiplicity events
 - Low multiplicity signals overwhelmed by SM:
 $\text{Higgs} \rightarrow b\bar{b} \rightarrow 2 \text{ jets}$
- Predicting backgrounds requires precision calculations of known Standard Model physics

- Complexity is due to QCD
- Perturbative QCD:
Gluons & quarks \rightarrow gluons & quarks
- Real world:
Hadrons \rightarrow hadrons with **hard** physics described by pQCD
- Hadrons \rightarrow jets *narrow nearly collimated streams of hadrons*

Jets

- Defined by an experimental resolution parameter
 - invariant mass in e^+e^-
 - cone algorithm in hadron colliders: cone size in $R = \sqrt{(\Delta\eta)^2 + (\Delta\phi)^2}$ and minimum E_T
 - k_T algorithm: essentially by a relative transverse momentum



CDF (Lefevre 2004)
1374 GeV

In theory, theory and practice are the same. In practice, they are different — Yogi Berra

QCD-Improved Parton Model



$$\int dx_a dx_b d\text{Phase } f_a f_b \sigma_{ab} \delta(v - \text{Observable})$$

The Challenge

- Everything at a hadron collider (signals, backgrounds, luminosity measurement) involves QCD
- Strong coupling is **not** small: $\alpha_s(M_Z) \approx 0.12$ and running is important
 - ⇒ events have high multiplicity of hard clusters (jets)
 - ⇒ each jet has a high multiplicity of hadrons
 - ⇒ higher-order perturbative corrections are important
- Processes can involve multiple scales: $p_T(W)$ & M_W
 - ⇒ need resummation of logarithms
- Confinement introduces further issues of mapping partons to hadrons, but for suitably-averaged quantities (infrared-safe) avoiding small E scales, this is **not** a problem (power corrections)

Approaches

- General parton-level fixed-order calculations
 - Numerical jet programs: general observables
 - Systematic to higher order/high multiplicity in perturbation theory
 - Parton-level, approximate jet algorithm; match detector events only statistically
- Parton showers
 - General observables
 - Leading- or next-to-leading logs only, approximate for higher order/high multiplicity
 - Can hadronize & look at detector response event-by-event
- Semi-analytic calculations/resummations
 - Specific observable, for high-value targets
 - Checks on general fixed-order calculations

Precision Perturbative QCD

- Predictions of signals, signals+jets
- Predictions of backgrounds
- Measurement of luminosity
- Measurement of fundamental parameters (α_s, m_t)
- Measurement of electroweak parameters
- Extraction of parton distributions — ingredients in any theoretical prediction

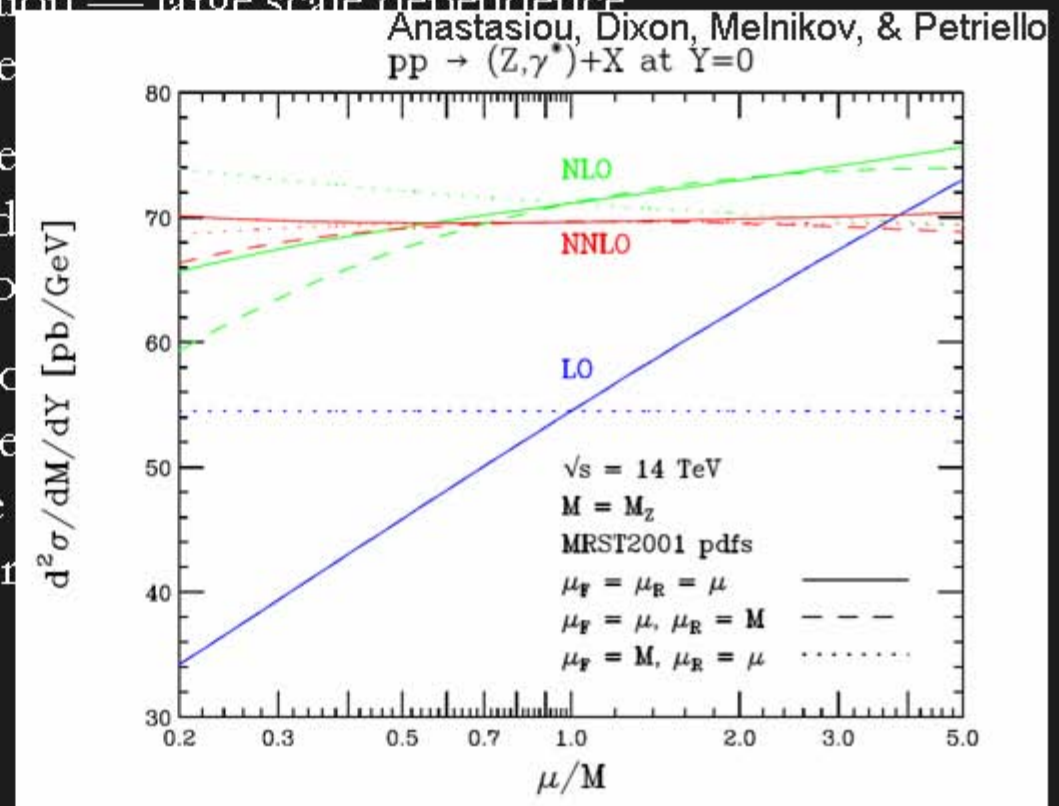
**Everything at a hadron
collider involves QCD**

Leading-Order, Next-to-Leading Order

- LO: Basic shapes of distributions
but: no quantitative prediction — large scale dependence
missing sensitivity to jet

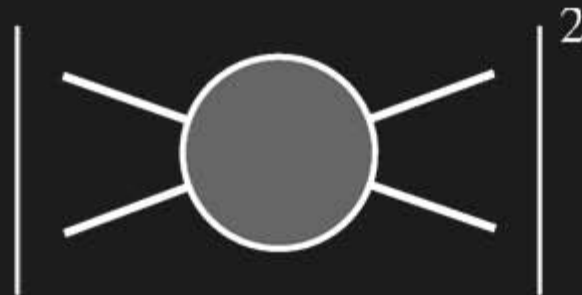
- NLO: First quantitative prediction
improved scale dependence
basic approximation to

- NNLO: Precision prediction
small scale dependence
better correspondence
understanding of theory

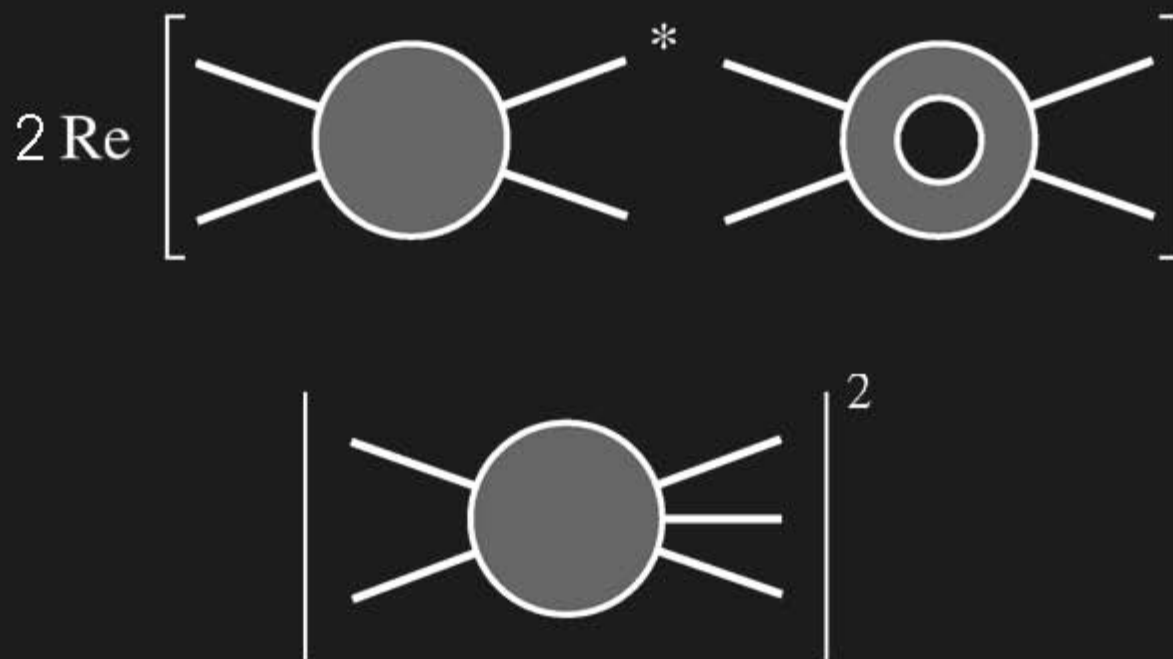


What Contributions Do We Need?

- Short-distance matrix elements to 2-jet production at leading order: tree level



- Short-distance matrix elements to 2-jet production at next-to-leading order: tree level + one loop + real emission



Real-Emission Singularities

$$e^+e^- \longrightarrow q\bar{q}g$$

Matrix element

$$|\mathcal{M}|^2 \propto \frac{(2 - y_{qg})^2 + (2 - y_{\bar{q}g})^2}{y_{qg}y_{\bar{q}g}}$$

Integrate

$$\int_{0 \leq y_{qg} + y_{\bar{q}g} \leq 1} dy_{qg} dy_{\bar{q}g} |\mathcal{M}|^2$$

$$y_{qg} = 0 = y_{\bar{q}g} \quad \text{soft}$$

$$y_{qg} = 0, y_{\bar{q}g} \neq 0$$

collinear

$$y_{qg} \neq 0, y_{\bar{q}g} = 0$$

- Physical quantities are finite
- Depend on resolution parameter
- Finiteness thanks to combination of Kinoshita–Lee–Nauenberg theorem and factorization

Scattering

Scattering matrix element

$$\text{out } \langle p_1 p_2 \cdots | k_a k_b \rangle_{\text{in}} \equiv \text{asympt } \langle p_1 p_2 \cdots | S | k_a k_b \rangle_{\text{asympt}}$$

Decompose it $S = 1 + iT$

Invariant matrix element \mathcal{M}

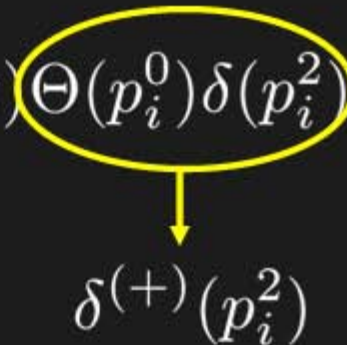
$$\langle p_1 p_2 \cdots | iT | k_a k_b \rangle = (2\pi)^4 \delta(k_a + k_b - P) i \mathcal{M}(k_a, k_b \rightarrow \{p_f\})$$

Differential cross section

$$d\sigma \stackrel{'}{=} \frac{1}{4E_a E_b |v_a - v_b|} \prod_f \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \\ \times (2\pi)^4 \delta^4(k_a + k_b - P) |\mathcal{M}(k_a, k_b \rightarrow \{p_f\})|^2$$

Lorentz-invariant phase-space measure

$$\int \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_i} F(p_i) = \int \frac{d^4 p_i}{(2\pi)^4} (2\pi) \Theta(p_i^0) \delta(p_i^2)$$



$$\delta^{(+)}(p_i^2)$$

Compute invariant matrix element by crossing


$$\begin{aligned} \mathcal{M}(k_a, k_b \rightarrow \{p_f\}) &= \mathcal{M}(0 \rightarrow -k_a, -k_b, \{p_f\}) \\ &= \mathcal{F} \langle \Omega | T \phi_a(x_a) \phi_b(x_b) \phi_1(x_1) \cdots | \Omega \rangle \end{aligned}$$

Lagrangian

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a) (\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu}) \\ & \sum_f \bar{q}(i\cancel{\partial})q + \frac{g}{\sqrt{2}} G_\mu^a \sum_f \bar{q}\gamma^\mu T^a q \\ & - \frac{g}{2} f^{abc} (\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu}) G_\mu^b G_\nu^c \\ & - \frac{g^2}{4} f^{abe} f^{cde} G^{a\mu} G^{b\nu} G_\mu^c G_\nu^d\end{aligned}$$

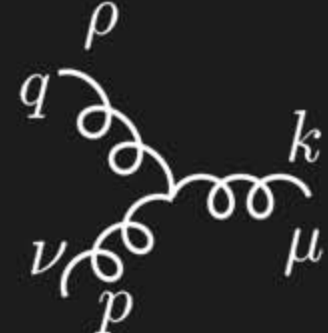
Feynman Rules

Propagator (like QED)



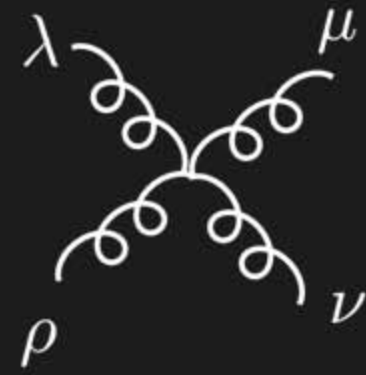
$$-\frac{i}{k^2 + i\epsilon} \delta^{ab} \left(g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right)$$

Three-gluon vertex (unlike QED)



$$g f^{abc} [g^{\mu\nu} (k-p)^\rho + g^{\nu\rho} (p-q)^\mu + g^{\rho\mu} (q-k)^\nu] ,$$

Four-gluon vertex (unlike QED)



$$-ig^2 [f^{abe} f^{dce} (g^{\mu\rho} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\lambda} - g^{\mu\lambda} g^{\rho\nu}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\lambda\rho} - g^{\mu\rho} g^{\lambda\nu})]$$

From the Faddeev–Popov functional determinant

$$\det \partial \cdot D = \int [Dc][D\bar{c}] \exp \left\{ -i \int \bar{c}(\partial^\mu D_\mu)c \right\}$$

anticommuting scalars or *ghosts*

Propagator $\frac{i}{k^2 + i\epsilon} \delta^{ab}$

coupling to gauge bosons $g f^{abc} k^\mu$

So What's Wrong with Feynman Diagrams?

- Huge number of diagrams in calculations of interest
- But answers often turn out to be very simple
- Vertices and propagators involve gauge-variant off-shell states
- Each diagram is not gauge invariant — huge cancellations of gauge-noninvariant, redundant, parts in the sum over diagrams

Simple results should have a simple derivation — attr to Feynman

- Want approach in terms of physical states only

Light-Cone Gauge

Only physical (transverse) degrees of freedom propagate

$$-\frac{i}{k^2 + i\epsilon} \delta^{ab} \left(g_{\mu\nu} - \frac{n^\mu k^\nu + n^\nu k^\mu}{n \cdot k} \right)$$

$n^2 = 0$

physical projector — two degrees of freedom

Color Decomposition

Standard Feynman rules \Rightarrow function of momenta, polarization vectors ε , and color indices

Color structure is predictable. Use representation

$$f^{abc} = -\frac{i}{\sqrt{2}} \text{Tr}([T^a, T^b]T^c) \quad \text{Tr}(T^a T^b) = \delta^{ab}$$

to represent each term as a product of traces,

and the Fierz identity

$$(T^a)_{i_1 \bar{i}_1} (T^a)_{i_2 \bar{i}_2} = \delta_{i_1 \bar{i}_2} \delta_{i_2 \bar{i}_1} - \frac{1}{N} \delta_{i_1 \bar{i}_1} \delta_{i_2 \bar{i}_2}$$

To unwind traces

$$\begin{aligned} f^{abc} f^{cde} &= -\frac{1}{2} \text{Tr}([T^a, T^b] T^c) \text{Tr}(T^c [T^d, T^e]) \\ &= -\frac{1}{2} \text{Tr}([T^a, T^b] [T^d, T^e]) + \frac{1}{2N} \text{Tr}([T^a, T^b]) \text{Tr}([T^d, T^e]) \end{aligned}$$

Leads to tree-level representation in terms of single traces

$$\mathcal{A}_n^{\text{tree}}(\{k_i, \varepsilon_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n / Z_n} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}})$$

$\otimes A_n^{\text{tree}}(k_{\sigma(1)}, \varepsilon_{\sigma(1)}; k_{\sigma(2)}, \varepsilon_{\sigma(2)}; k_{\sigma(n)}, \varepsilon_{\sigma(n)})$

Color-ordered amplitude — function of momenta & polarizations alone; *not* Bose symmetric

Symmetry properties

- Cyclic symmetry

$$A_n^{\text{tree}}(1, \dots, n) = A_n(2, \dots, n, 1)$$

- Reflection identity

$$A_n^{\text{tree}}(n, \dots, 1) = (-1)^n A_n(1, \dots, n)$$

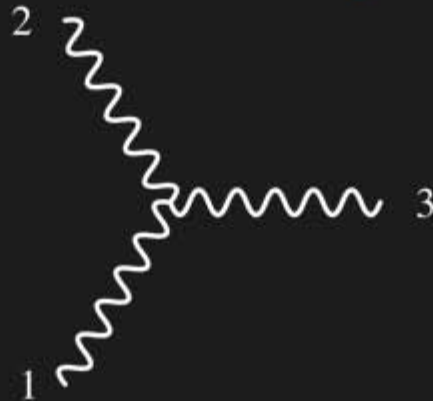
- Parity flips helicities

$$A_n^{\text{tree}}(1^{-\lambda_1}, \dots, n^{-\lambda_n}) = [A_n^{\text{tree}}(1^{\lambda_1}, \dots, n^{\lambda_n})]^\dagger$$

- Decoupling equation

$$A_{n+1}^{\text{tree}}(p, 1, 2, \dots, n) + A_{n+1}^{\text{tree}}(1, p, 2, \dots, n) + A_{n+1}^{\text{tree}}(1, 2, p, \dots, n) \\ + \dots + A_{n+1}^{\text{tree}}(1, 2, \dots, p, n) = 0$$

Color-Ordered Feynman Rules



$$\frac{i}{\sqrt{2}} [\varepsilon_1 \cdot \varepsilon_2 (k_1 - k_2) \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 (k_2 - k_3) \cdot \varepsilon_1 + \varepsilon_3 \cdot \varepsilon_1 (k_3 - k_1) \cdot \varepsilon_2]$$



$$i\varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot \varepsilon_4 - \frac{i}{2} (\varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 + \varepsilon_2 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_1)$$

Amplitudes

Functions of momenta k , polarization vectors ε for gluons;
momenta k , spinor wavefunctions u for fermions

Gauge invariance implies this is a redundant representation:

$$\varepsilon \rightarrow k: \mathcal{A} = 0$$

Spinor Helicity

Spinor wavefunctions $|j^\pm\rangle \equiv u_\pm(k_j), \quad \langle j^\pm| \equiv \bar{u}_\pm(k_j) .$

Introduce *spinor products*

$$\langle i j \rangle \equiv \langle i^- | j^+ \rangle = \bar{u}_-(k_i) u_+(k_j),$$

$$[i j] \equiv \langle i^+ | j^- \rangle = \bar{u}_+(k_i) u_-(k_j)$$

Explicit representation

$$u_+(k) = \begin{pmatrix} \sqrt{k_+} \\ \sqrt{k_-} e^{i\phi_k} \end{pmatrix}, \quad u_-(k) = \begin{pmatrix} \sqrt{k_-} e^{-i\phi_k} \\ -\sqrt{k_+} \end{pmatrix}$$

where

$$e^{\pm i\phi_k} = \frac{k^1 \pm ik^2}{\sqrt{k_+ k_-}}, \quad k_\pm = k^0 \pm k^3$$

We then obtain the explicit formulæ

$$\langle i j \rangle = \sqrt{k_{i-} k_{j+}} e^{i\phi_{k_i}} - \sqrt{k_{i+} k_{j-}} e^{i\phi_{k_j}} ,$$

$$[i j] = \langle j i \rangle^* = \sqrt{k_{i+} k_{j-}} e^{-i\phi_{k_j}} - \sqrt{k_{i-} k_{j+}} e^{-i\phi_{k_i}} \quad (k_{i,j}^0 > 0)$$

otherwise $[j i] = \text{sign}(k_i^0 k_j^0) \langle i j \rangle^*$

so that the identity $\langle i j \rangle [j i] = 2k_i \cdot k_j$ always holds

Introduce four-component representation

$$\begin{bmatrix} u_+(k) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ u_-(k) \end{bmatrix}$$

corresponding to γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

in order to define spinor strings

$$\langle i^\pm | \gamma^\mu | j^\pm \rangle \leftrightarrow \langle i^\pm | \sigma^\mu | j^\pm \rangle \quad \sigma^\mu = (1, \sigma^i)$$

Properties of the Spinor Product

- Antisymmetry $\langle j i \rangle = -\langle i j \rangle$, $[j i] = -[i j]$
- Gordon identity $\langle i^\pm | \gamma^\mu | i^\pm \rangle = 2k_i^\mu$
- Charge conjugation $\langle i^- | \gamma^\mu | j^- \rangle = \langle j^+ | \gamma^\mu | i^+ \rangle$,
- Fierz identity $\langle i^- | \gamma^\mu | j^- \rangle \langle p^+ | \gamma^\mu | q^+ \rangle = 2 \langle i q \rangle [p j]$
- Projector representation $|i^\pm\rangle\langle i^\pm| = \frac{1}{2}(1 \pm \gamma_5)\not{k}_i$
- Schouten identity $\langle i j \rangle \langle p q \rangle = \langle i q \rangle \langle p j \rangle + \langle i p \rangle \langle j q \rangle$.

Spinor-Helicity Representation for Gluons

Gauge bosons also have only \pm physical polarizations

Elegant — and covariant — generalization of circular polarization

$$\varepsilon_{\mu}^{+}(k, q) = \frac{\langle q^{-} | \gamma_{\mu} | k^{-} \rangle}{\sqrt{2} \langle q k \rangle}, \quad \varepsilon_{\mu}^{-}(k, q) = \frac{\langle q^{+} | \gamma_{\mu} | k^{+} \rangle}{\sqrt{2} [k q]}$$

Xu, Zhang, Chang (1984)

reference momentum q $q \cdot k \neq 0$

Transverse $k \cdot \varepsilon^{\pm}(k, q) = 0$

Normalized $\varepsilon^{+} \cdot \varepsilon^{-} = -1, \quad \varepsilon^{+} \cdot \varepsilon^{+} = 0$

What is the significance of q ?

$$\begin{aligned}
 \varepsilon_{\mu}^{+}(k, q') &= \frac{\langle q'^{-} | \gamma_{\mu} | k^{-} \rangle}{\sqrt{2} \langle q' k \rangle} = \frac{\langle q'^{-} | \gamma_{\mu} \not{k} | q^{+} \rangle}{\sqrt{2} \langle q' k \rangle \langle k q \rangle} \\
 &= -\frac{\langle q'^{-} | \not{k} \gamma_{\mu} | q^{+} \rangle}{\sqrt{2} \langle q' k \rangle \langle k q \rangle} + \frac{\sqrt{2} \langle q q' \rangle}{\langle q' k \rangle \langle k q \rangle} k_{\mu} \\
 &= \varepsilon_{\mu}^{+}(k, q) + \frac{\sqrt{2} \langle q q' \rangle}{\langle q' k \rangle \langle k q \rangle} k_{\mu}
 \end{aligned}$$

Properties of the Spinor-Helicity Basis

Physical-state projector

$$\sum_{\sigma=\pm} \varepsilon_{\mu}^{\sigma}(k, q) \varepsilon_{\nu}^{\sigma*}(k, q) = \sum_{\sigma=\pm} \varepsilon_{\mu}^{\sigma}(k, q) \varepsilon_{\nu}^{-\sigma}(k, q) = -g_{\mu\nu} + \frac{q_{\mu} k_{\nu} + k_{\mu} q_{\nu}}{q \cdot k}$$

Simplifications

$$q \cdot \varepsilon^{\pm}(k, q) = 0,$$

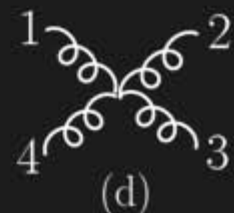
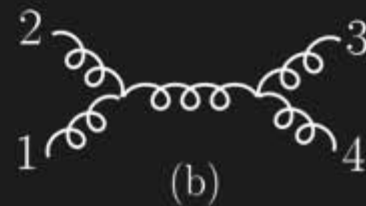
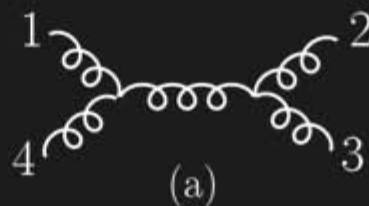
$$\varepsilon^{+}(k_1, q) \cdot \varepsilon^{+}(k_2, q) = \varepsilon^{-}(k_1, q) \cdot \varepsilon^{-}(k_2, q) = 0,$$

$$\varepsilon^{+}(k_1, q) \cdot \varepsilon^{-}(k_2, k_1) = 0$$

Examples

By explicit calculation (or other arguments), every term in the gluon tree-level amplitude has at least one factor of $\varepsilon_i \cdot \varepsilon_j$

Look at four-point amplitude



Recall three-point color-ordered vertex

$$\frac{i}{\sqrt{2}} [\varepsilon_1 \cdot \varepsilon_2 (k_1 - k_2) \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 (k_2 - k_3) \cdot \varepsilon_1 + \varepsilon_3 \cdot \varepsilon_1 (k_3 - k_1) \cdot \varepsilon_2]$$

Calculate $A_4^{\text{tree}}(1^+, 2^+, 3^+, 4^+)$

choose identical reference momenta for all legs \Rightarrow all $\varepsilon \cdot \varepsilon$ vanish

\Rightarrow amplitude vanishes

Calculate $A_4^{\text{tree}}(1^-, 2^+, 3^+, 4^+)$

choose reference momenta 4,1,1,1 \Rightarrow all $\varepsilon \cdot \varepsilon$ vanish

\Rightarrow amplitude vanishes

Calculate $A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$

choose reference momenta 3,3,2,2

\Rightarrow only nonvanishing $\varepsilon \cdot \varepsilon$ is $\varepsilon_1 \cdot \varepsilon_4$

\Rightarrow only s_{12} channel contributes

$$\begin{aligned}
& \left(\frac{i}{\sqrt{2}} \right)^2 \left(-\frac{ig^{\mu\nu}}{s_{12}} \right) [-2k_1 \cdot \varepsilon_2^- \varepsilon_{1\mu}^-] [2k_4 \cdot \varepsilon_3^+ \varepsilon_{4\nu}^+] \\
&= -\frac{2i}{s_{12}} k_1 \cdot \varepsilon_2^- k_4 \cdot \varepsilon_3^+ \varepsilon_1^- \cdot \varepsilon_4^+ \\
&= -\frac{i}{s_{12}} \left(\frac{[3\ 1] \langle 1\ 2 \rangle}{[3\ 2]} \right) \left(\frac{\langle 2\ 4 \rangle [4\ 3]}{\langle 2\ 3 \rangle} \right) \left(\frac{\langle 2\ 1 \rangle [3\ 4]}{\langle 2\ 4 \rangle [3\ 1]} \right) \\
&= -i \frac{\langle 1\ 2 \rangle^2 [3\ 4]^2}{s_{12} s_{23}} \\
&= i \frac{\langle 1\ 2 \rangle^3}{\langle 2\ 3 \rangle \langle 3\ 4 \rangle \langle 4\ 1 \rangle}
\end{aligned}$$

No diagrammatic calculation required for the last helicity amplitude,

$$A_4^{\text{tree}}(1^-, 2^+, 3^-, 4^+)$$

Obtain it from the decoupling identity

$$\begin{aligned} & -A_4^{\text{tree}}(3^-, 1^-, 2^+, 4^+) - A_4^{\text{tree}}(1^-, 3^-, 2^+, 4^+) \\ &= i \frac{\langle 13 \rangle^3}{\langle 24 \rangle} \left(-\frac{1}{\langle 12 \rangle \langle 43 \rangle} + \frac{1}{\langle 32 \rangle \langle 41 \rangle} \right) \\ &= i \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \end{aligned}$$

These forms hold more generally, for larger numbers of external legs:

$$A_n^{\text{tree}}(1^+, 2^+, \dots, n^+) = 0, \quad \text{Parke-Taylor equations}$$

$$A_n^{\text{tree}}(1^-, 2^+, \dots, n^+) = 0 \quad \text{Mangano, Xu, Parke (1986)}$$

Maximally helicity-violating or ‘MHV’

$$A_n^{\text{tree}}(1^+, \dots, m_1^-, (m_1 + 1)^+, \dots, m_2^-, (m_2 + 1)^+, \dots, n^+) =$$

$$i \frac{\langle m_1 m_2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \cdots \langle (n-1) n \rangle \langle n 1 \rangle}$$

Proven using the Berends–Giele recurrence relations

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Lecture II

Review of Lecture I

Color-ordered amplitude

$$\mathcal{A}_n^{\text{tree}}(\{k_i, \varepsilon_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}}) \\ \times A_n^{\text{tree}}(k_{\sigma(1)}, \varepsilon_{\sigma(1)}; k_{\sigma(2)}, \varepsilon_{\sigma(2)}; k_{\sigma(n)}, \varepsilon_{\sigma(n)})$$

Color-ordered amplitude — function of momenta & polarizations alone; *not* Bose symmetric

Spinor-Helicity Representation for Gluons

Gauge bosons also have only \pm physical polarizations

Elegant — and covariant — generalization of circular polarization

$$\varepsilon_{\mu}^{+}(k, q) = \frac{\langle q^{-} | \gamma_{\mu} | k^{-} \rangle}{\sqrt{2} \langle q k \rangle}, \quad \varepsilon_{\mu}^{-}(k, q) = \frac{\langle q^{+} | \gamma_{\mu} | k^{+} \rangle}{\sqrt{2} [k q]}$$

Xu, Zhang, Chang (preprint 1984); NPB291:392 (1987)

reference momentum q $q \cdot k \neq 0$

Transverse $k \cdot \varepsilon^{\pm}(k, q) = 0$

Normalized $\varepsilon^{+} \cdot \varepsilon^{-} = -1, \quad \varepsilon^{+} \cdot \varepsilon^{+} = 0$

Parke-Taylor Equations

For any number of external legs:

$$A_n^{\text{tree}}(1^+, 2^+, \dots, n^+) = 0, \quad \text{Parke \& Taylor, PRL 56:2459 (1986)}$$

$$A_n^{\text{tree}}(1^-, 2^+, \dots, n^+) = 0$$

Mangano, Xu, & Parke, NPB298:653 (1986)

Maximally helicity-violating or ‘MHV’

$$A_n^{\text{tree}}(1^+, \dots, m_1^-, (m_1 + 1)^+, \dots, m_2^-, (m_2 + 1)^+, \dots, n^+) = \\ i \frac{\langle m_1 m_2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \cdots \langle (n-1) n \rangle \langle n 1 \rangle}$$

Proven using the Berends–Giele recurrence relations

Berends & Giele, NPB294:700 (1987)

Gauge-theory amplitude

↓ Color decomposition & stripping

Color-ordered amplitude: function of k_i and ε_i

↓ Spinor-helicity basis

Helicity amplitude: function of spinor products and helicities ± 1

Spinor products \rightarrow spinors

Spinor Variables

From Lorentz vectors to bi-spinors

$$p_\mu \quad \longleftrightarrow \quad p_{a\dot{a}} \equiv p \cdot \sigma = \begin{pmatrix} p^0 + p^3 & p^1 + ip^2 \\ p^1 - ip^2 & p^0 - p^3 \end{pmatrix}$$

$$p^2 \quad \longleftrightarrow \quad \det(p)$$

$$p' = \Lambda p \quad \longleftrightarrow \quad p' = upu^\dagger, \quad u \in SL(2, C)$$

2×2 complex matrices
with $\det = 1$

Null momenta $p^2 = 0 \implies \det(p) = 0$

can write it as a bispinor $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$

phase ambiguity in $\lambda_a, \tilde{\lambda}_{\dot{a}}$ (same as seen in spinor products)

For real Minkowski p , take $\tilde{\lambda} = \text{sign}(p^0) \bar{\lambda}$

Invariant tensor ϵ_{ab}

$$u_{aa'} u_{bb'} \epsilon_{a'b'} = \det(u) = 1$$

gives spinor products

$$\langle \lambda_1, \lambda_2 \rangle = \epsilon_{ab} \lambda_1^a \lambda_2^b$$

$$[\tilde{\lambda}_1, \tilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$$

Connection to earlier spinor products

$$\langle \tilde{\lambda}_1, \lambda_2 \rangle = \langle 1 \ 2 \rangle$$

$$[\tilde{\lambda}_1, \tilde{\lambda}_2] = - [1 \ 2]$$

and spinor-helicity basis

$$+1 : \quad \varepsilon_{a\dot{a}} = \frac{\eta_a \tilde{\lambda}_{\dot{a}}}{\langle \eta, \lambda \rangle}$$

$$-1 : \quad \varepsilon_{a\dot{a}} = \frac{\lambda_a \tilde{\eta}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\eta}]}$$

\Rightarrow Amplitudes as functions of spinor variables $\lambda_a, \tilde{\lambda}_{\dot{a}}$ and helicities ± 1

Scaling of Amplitudes

Suppose we scale the spinors

also called 'phase weight'

$$\lambda_i \mapsto \alpha_i \lambda_i,$$

$$\tilde{\lambda}_i \mapsto \alpha_i^{-1} \tilde{\lambda}_i,$$

then by explicit computation we see that the MHV amplitude

$$A^{\text{MHV}} \mapsto i \frac{\alpha_{m_1}^2 \alpha_{m_2}^2}{\prod_{j \neq m_1, m_2} \alpha_j^2} \frac{\langle m_1 m_2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \cdots \langle (n-1) n \rangle \langle n 1 \rangle}$$

and that more generally

$$A \mapsto \prod_j \alpha_j^{-2h_j} A$$

For the non-trivial parts of the amplitude, we might as well use uniformly rescaled spinors $\Rightarrow \mathbb{CP}^1$ ‘complex projective space’

Start with \mathbb{C}^2 , and rescale all vectors by a common scale

$$\begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \equiv \tau \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}$$

the spinors are then ‘homogeneous’ coordinates on \mathbb{CP}^1

If we look at each factor in the MHV amplitude,

$$\frac{1}{\langle \lambda_1, \lambda_2 \rangle} = \frac{1}{\lambda_1^1 \lambda_2^1 (w_1 - w_2)} \quad w_i = \lambda_i^2 / \lambda_i^1$$

we see that it is just a free-field correlator (Green function) on \mathbb{CP}^1

This is the essence of Nair’s construction of MHV amplitudes as correlation functions on the ‘line’ = \mathbb{CP}^1

Gauge-theory amplitude

↓ Color decomposition & stripping

Color-ordered amplitude: function of k_i and ε_i

↓ Spinor-helicity basis

Helicity amplitude: function of spinor products and helicities ± 1

↓

Function of spinor variables and helicities ± 1

↓ Half-Fourier transform

Conjectured support on simple curves in twistor space

Let's Travel to Twistor Space!

It turns out that the natural setting for amplitudes is not exactly spinor space, but something similar. The motivation comes from studying the representation of the conformal algebra.

Half-Fourier transform of spinors: transform $\tilde{\lambda}_{\dot{a}}$, leave alone λ_a
 \Rightarrow Penrose's original twistor space, real or complex

$$\tilde{\lambda}_{\dot{a}} \rightarrow i \frac{\partial}{\partial \mu^{\dot{a}}}, \quad -i \frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \rightarrow \mu_{\dot{a}}$$

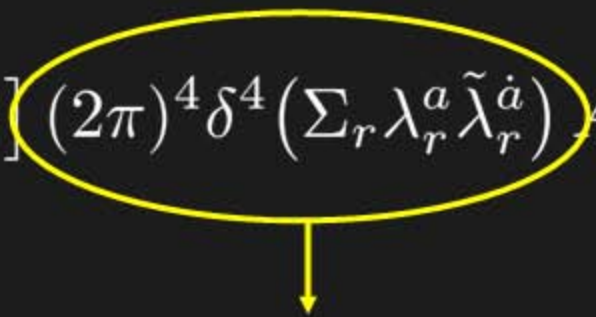
Study amplitudes of definite helicity: introduce homogeneous coordinates $Z_I = (\lambda_a, \mu_{\dot{a}})$

\Rightarrow \mathbb{CP}^3 or \mathbb{RP}^3 (projective) twistor space

Back to momentum space by Fourier-transforming μ

MHV Amplitudes in Twistor Space

Write out the half-Fourier transform including the energy-momentum conserving δ function

$$\tilde{A}(Z) = \int \prod_j \frac{d^2 \tilde{\lambda}_j}{(2\pi)^2} \exp[i\mu_{j\dot{a}} \tilde{\lambda}_j^{\dot{a}}] (2\pi)^4 \delta^4(\sum_r \lambda_r^a \tilde{\lambda}_r^{\dot{a}}) A^{\text{MHV}}(\lambda_j)$$


$$\int d^4 x \exp[\sum_j x_{a\dot{a}} \lambda_j^a \tilde{\lambda}_j^{\dot{a}}]$$

$$\tilde{A}(Z) = \int d^4 x \prod_j \frac{d^2 \tilde{\lambda}_j}{(2\pi)^2} \exp[\sum_j i(\mu_{j\dot{a}} + x_{a\dot{a}} \lambda_j^a) \tilde{\lambda}_j^{\dot{a}}] A^{\text{MHV}}(\lambda_j)$$

Result

$$\tilde{A}(Z) = \int d^4x \prod_j \delta^2(\mu_{j\dot{a}} + x_{a\dot{a}}\lambda_j^a) A^{\text{MHV}}(\lambda_j)$$

equation for a line

MHV amplitudes live on lines in twistor space

Value of the twistor-space amplitude is given by a correlation function on the line

Analyzing Amplitudes in Twistor Space

Amplitudes in twistor space turn out to be hard to compute directly. Even with computations in momentum space, the Fourier transforms are hard to compute explicitly.

We need other tools to analyze the amplitudes.

Simple ‘algebraic’ properties in twistor space — support on \mathbb{CP}^1 s or \mathbb{CP}^2 s — become differential properties in momentum space.

Construct differential operators.

Equation for a line (\mathbb{CP}^1): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K = 0$

gives us a differential ('line') operator in terms of momentum-space spinors

$$F_{123} = \langle \lambda_1 \lambda_2 \rangle \frac{\partial}{\partial \tilde{\lambda}_3} + \langle \lambda_2 \lambda_3 \rangle \frac{\partial}{\partial \tilde{\lambda}_1} + \langle \lambda_3 \lambda_1 \rangle \frac{\partial}{\partial \tilde{\lambda}_2}.$$

Equation for a plane (\mathbb{CP}^2): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K Z_4^L = 0$

also gives us a differential ('plane') operator

$$K_{1234} = \langle \lambda_1 \lambda_2 \rangle \frac{\partial}{\partial \tilde{\lambda}_{3\dot{a}}} \frac{\partial}{\partial \tilde{\lambda}_4^{\dot{a}}} + \text{perms}$$

Properties

$$F_{ijl} f(p_i + p_j + p_l) = 0$$

$$F_{ijl} f(\{\lambda_r\}) = 0$$

$$K_{ijklm} f(\{\lambda_r\}) = 0$$

Thus for example

$$F_{ijl} \frac{\langle m_1 m_2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \cdots \langle (n-1) n \rangle \langle n 1 \rangle} = 0$$

Beyond MHV

Witten's proposal:

ph/0312171

- Each external particle represented by a point in twistor space
- Amplitudes non-vanishing only when points lie on a curve of degree d and genus g , where
 - $d = \# \text{ negative helicities} - 1 + \# \text{ loops}$
 - $g \leq \# \text{ loops}; g = 0$ for tree amplitudes
- Integrand on curve supplied by a topological string theory
- Obtain amplitudes by integrating over all possible curves \Rightarrow moduli space of curves
- Can be interpreted as D_1 -instantons

Strings in Twistor Space

- String theory can be defined by a two-dimensional field theory whose fields take values in target space:
 - n -dimensional flat space
 - 5-dimensional Anti-de Sitter \times 5-sphere
 - twistor space: intrinsically four-dimensional \Rightarrow Topological String Theory
- Spectrum in Twistor space is $\mathcal{N} = 4$ supersymmetric multiplet (gluon, four fermions, six real scalars)
- Gluons and fermions each have two helicity states

A New Duality

- String Theory
Topological B -model on $CP^{3|4}$
- ←→ Gauge Theory
 $\mathcal{N} = 4$ SUSY

‘Twistor space’

Witten (2003); Berkovits & Motl; Neitzke & Vafa; Siegel (2004)

weak–weak

Simple Cases

Amplitudes with all helicities ‘+’ \Rightarrow degree -1 curves.

No such curves exist, so the amplitudes should vanish.

Corresponds to the first Parke–Taylor equation.

Amplitudes with one ‘ $-$ ’ helicity \Rightarrow degree-0 curves: points.

Generic external momenta, all external points won’t coincide (singular configuration, all collinear), \Rightarrow amplitudes must vanish.

Corresponds to the second Parke–Taylor equation.

Amplitudes with two ‘ $-$ ’ helicities (MHV) \Rightarrow degree-1 curves: lines.

All F operators should annihilate them, and they do.

Other Cases

Amplitudes with three negative helicities (next-to-MHV) live on conic sections (quadratic curves)

Amplitudes with four negative helicities (next-to-next-to-MHV) live on twisted cubics

Fourier transform back to spinors \Rightarrow differential equations in conjugate spinors

Even String Theorists Can Do Experiments

- Apply F operators to NMHV (3 –) amplitudes: products annihilate them! K annihilates them;
- Apply F operators to N^2 MHV (4 –) amplitudes: longer products annihilate them! Products of K annihilate them;

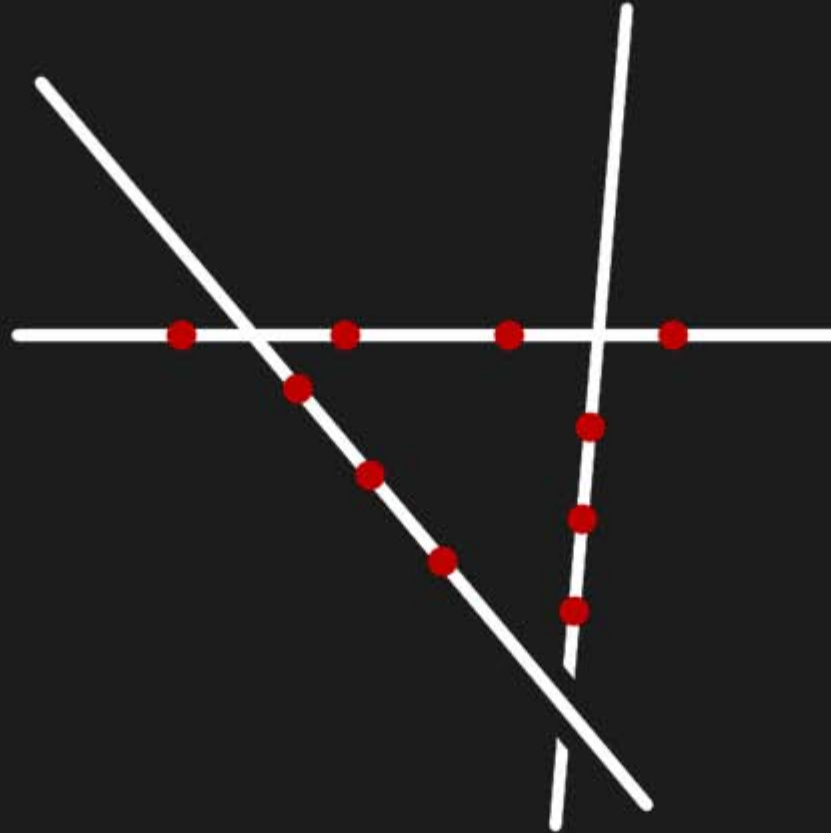
$$F_{512}F_{234}F_{345}F_{451}A_5(1^-, 2^-, 3^-, 4^+, 5^+) =$$
$$F_{512}F_{234}F_{345}F_{451} \frac{[45]^4}{[12][23][34][45][51]} = 0$$

A more involved example

$$F_{612}F_{234}F_{345}F_{561}A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = 0$$

Don't try this at home!

Interpretation: twistor-string amplitudes are supported on intersecting line segments

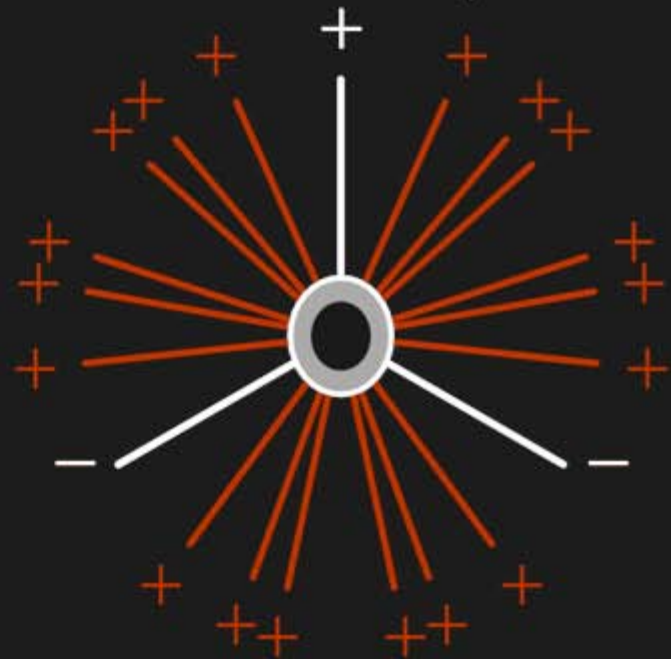
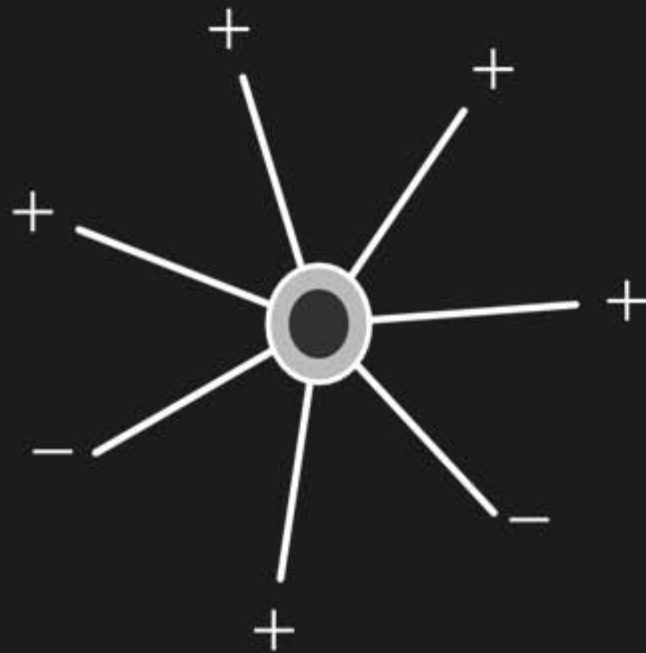
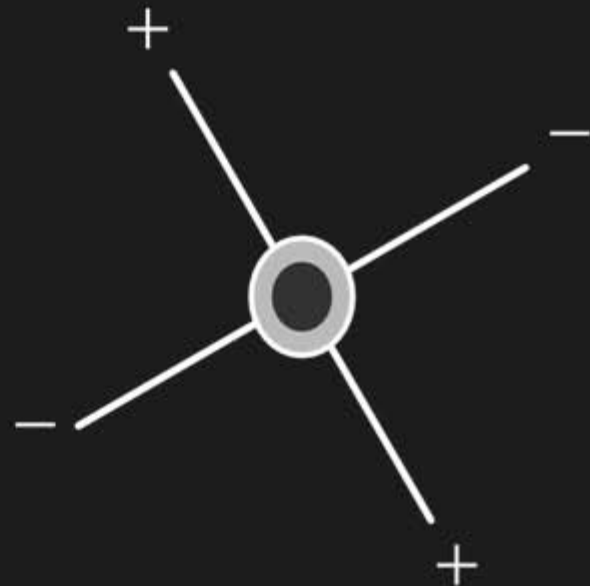
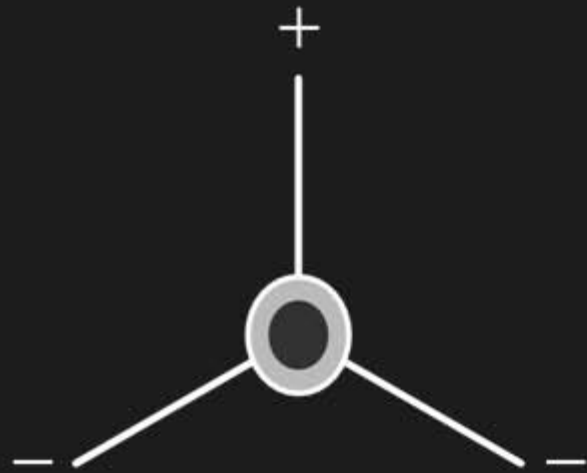


Simpler than expected: what does this mean in field theory?

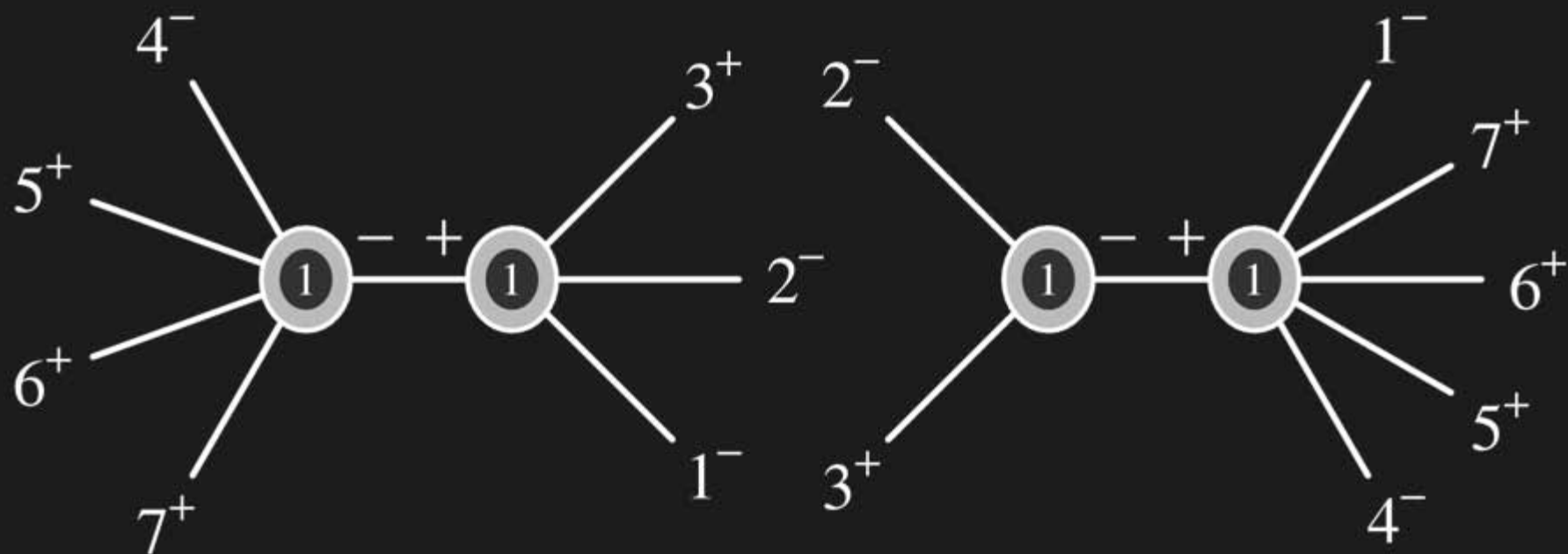
Cachazo–Svrček–Witten Construction

Cachazo, Svrček, & Witten, th/0403047

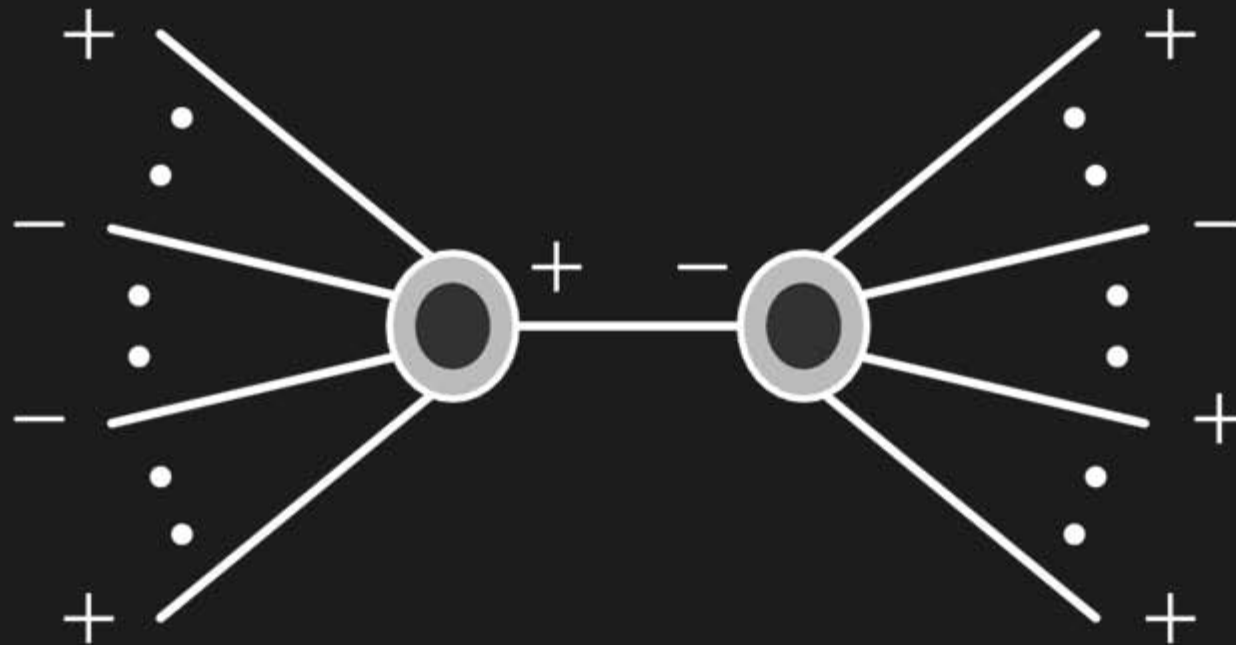
- Vertices are off-shell continuations of MHV amplitudes: every vertex has two ‘−’ helicities, and one or more ‘+’ helicities
- Includes a three-point vertex
- Propagators are scalar ones: i/K^2 ; helicity projector is in the vertices
- Draw all tree diagrams with these vertices and propagator
- Different sets of diagrams for different helicity configurations
- Corresponds to all multiparticle factorizations



- Seven-point example with three negative helicities



Next-to-MHV



Factorization Properties of Amplitudes

- As sums of external momenta approach poles,

$$p^2 = (k_1 + k_2)^2 \rightarrow m_X^2$$

- amplitudes factorize

$$A(1+2 \rightarrow \dots) \rightarrow A_L(1+2 \rightarrow X) \frac{i}{p^2 - m^2} A_R(X \rightarrow \dots)$$

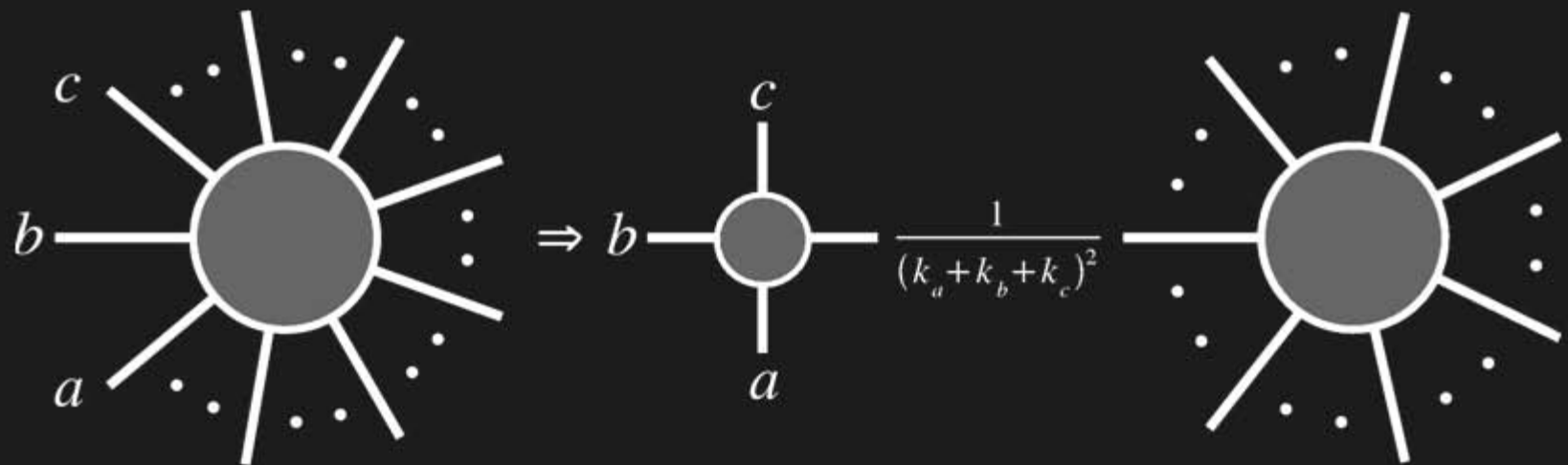
- More generally as

$$p^2 = (k_1 + \dots + k_n)^2 \rightarrow m_X^2$$

$$A(1+\dots+n \rightarrow \dots) \rightarrow A_L(1+\dots+n \rightarrow X) \frac{i}{p^2 - m^2} A_R(X \rightarrow \dots)$$

Factorization in Gauge Theories

Tree level



As $(k_a + k_b + k_c)^2 \rightarrow 0$ but $s_{ab}, s_{bc}, s_{ac} \not\rightarrow 0$

Sum over helicities of intermediate leg

In massless theories beyond tree level, the situation is more complicated but at tree level it works in a standard way

What Happens in the Two-Particle Case?

We would get a three-gluon amplitude on the left-hand side

$$\text{But } k_3^2 = 0 = (k_1 + k_2)^2 = 2k_1 \cdot k_2$$

so all invariants vanish,

$$k_1 \cdot k_2 = k_2 \cdot k_3 = k_3 \cdot k_1 = 0$$

hence all spinor products vanish

$$\langle 12 \rangle = 0, \quad \langle 23 \rangle = 0, \quad \langle 31 \rangle = 0$$

$$[12] = 0, \quad [23] = 0, \quad [31] = 0$$

hence the three-point amplitude vanishes

$$A_3(1, 2, 3) = 0$$

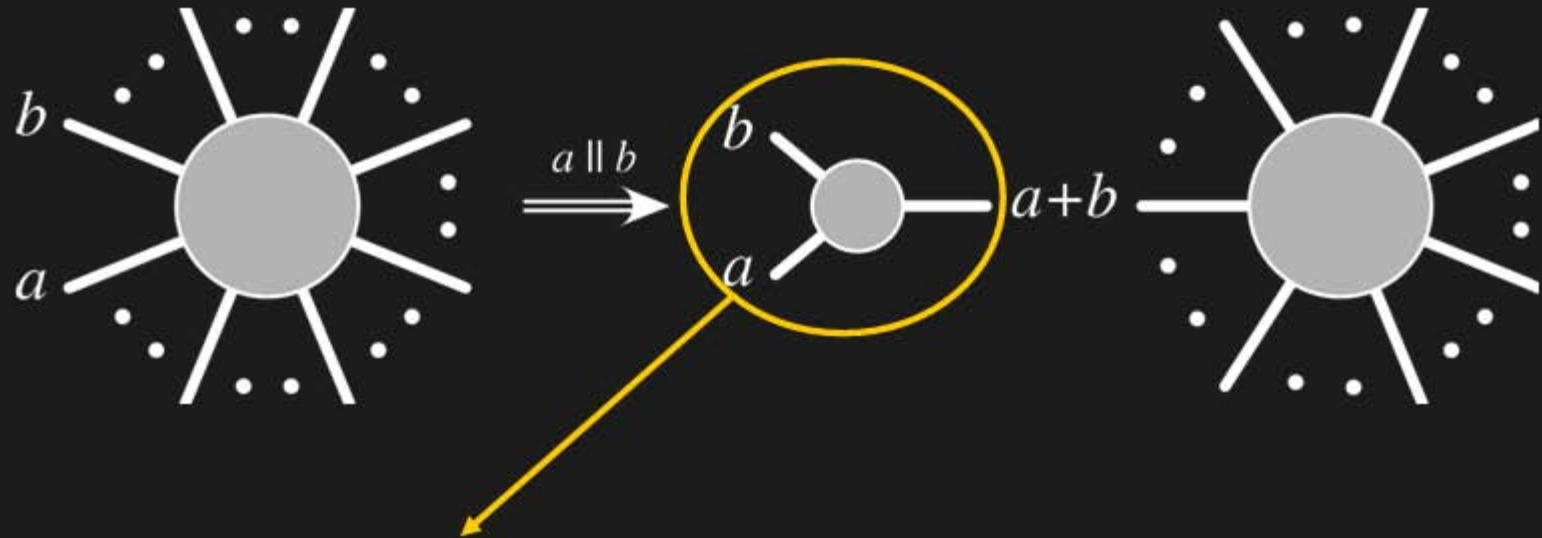
- In gauge theories, it holds (at tree level) for $n \geq 3$ but breaks down for $n = 2$: $\mathcal{A}_3 = 0$ so we get $0/0$
- However \mathcal{A}_3 only vanishes linearly, so the amplitude is not finite in this limit, but should $\sim 1/k$, that is $1/\sqrt{s_{12}}$
- This is a *collinear* limit

$$(k_1 + k_2)^2 = 2k_1 \cdot k_2 \rightarrow 0$$

$$\implies k_1 \propto k_2, \text{ i.e., } k_1 \parallel k_2$$

- Combine amplitude with propagator to get a non-vanishing object

Two-Particle Case



Collinear limit: *splitting* amplitude

Universal Factorization

- Amplitudes have a universal behavior in this limit

$$A_n^{\text{tree}}(\dots, a^{h_a}, b^{h_b}, \dots) \xrightarrow{k_a \parallel k_b} \sum_{h=\pm} \text{Split}_{-h}^{\text{tree}}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{tree}}(\dots, (k_a + k_b)^h, \dots) + \text{non-singular}$$

- Depend on a collinear momentum fraction z

$$k_a = z(k_a + k_b), \quad k_b = (1 - z)(k_a + k_b)$$

- In this form, a powerful tool for checking calculations
- As expressed in on-shell recursion relations, a powerful tool for computing amplitudes

Example: Three-Particle Factorization

Consider

$$\begin{aligned}
 -iA_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = & \\
 & \frac{(\langle 23 \rangle [56] \langle 1^- | 1 + 2 + 3 | 4^- \rangle)^2}{s_{234} s_{23} s_{34} s_{56} s_{61}} \\
 & + \frac{(\langle 12 \rangle [45] \langle 3^- | 1 + 2 + 3 | 6^- \rangle)^2}{s_{345} s_{34} s_{45} s_{61} s_{12}} \\
 & + \frac{s_{123} \langle 23 \rangle [56] \langle 1^- | 1 + 2 + 3 | 4^- \rangle \langle 12 \rangle [45] \langle 3^- | 1 + 2 + 3 | 6^- \rangle}{s_{12} s_{23} s_{34} s_{45} s_{56} s_{61}}
 \end{aligned}$$

As $s_{123} \rightarrow 0$, it's finite: expected because $A_4(1^-, 2^-, 3^-, X^\pm) = 0$

As $s_{234} \rightarrow 0$, pick up the first term; with $K = k_2 + k_3 + k_4$

$$\begin{aligned}
 & \frac{\langle 23 \rangle [K4]^2}{[23] s_{34}} \times \frac{1}{s_{234}} \times \frac{[56] \langle 1K \rangle^2}{\langle 56 \rangle s_{61}} = \\
 & \frac{[4K]^2 [K2] \langle 23 \rangle}{[23] [34] \langle 43 \rangle [K2]} \times \frac{1}{s_{234}} \times \frac{\langle 1K \rangle^2 \langle K5 \rangle [56]}{\langle K5 \rangle \langle 56 \rangle \langle 61 \rangle [16]} \\
 & = \frac{[4K]^3}{[23] [34] [K2]} \times \frac{1}{s_{234}} \times \frac{\langle 1K \rangle^3}{\langle K5 \rangle \langle 56 \rangle \langle 61 \rangle} \\
 & = A_4(2^-, 3^-, 4^+, K^+) \times \frac{1}{s_{234}} \times A_4(1^-, (-K)^-, 5^+, 6^+)
 \end{aligned}$$

Splitting Amplitudes

Compute it from the three-point vertex

$$\begin{aligned}
 \text{Split}_{-}^{\text{tree}}(a^{+}, b^{+}) &= -\frac{\sqrt{2}}{s_{ab}} [k_b \cdot \varepsilon_a \varepsilon_b \cdot \varepsilon_{a+b} - k_a \cdot \varepsilon_b \varepsilon_a \cdot \varepsilon_{a+b}] \\
 &= -\frac{1}{s_{ab}} \left[\frac{\langle q b \rangle [b a]}{\langle q a \rangle} \frac{\langle q (a+b) \rangle [q b]}{\langle q b \rangle [(a+b) q]} \right. \\
 &\quad \left. - \frac{\langle q a \rangle [a b]}{\langle q b \rangle} \frac{\langle q (a+b) \rangle [q a]}{\langle q a \rangle [(a+b) q]} \right] \\
 &= \frac{1}{\langle a b \rangle} \left[\sqrt{\frac{1-z}{z}} + \sqrt{\frac{z}{1-z}} \right] \\
 &= \frac{1}{\sqrt{z(1-z)} \langle a b \rangle} .
 \end{aligned}$$

Explicit Values

$$\text{Split}_{-}^{\text{tree}}(a^{-}, b^{-}) = 0$$

$$\text{Split}_{-}^{\text{tree}}(a^{+}, b^{+}) = \frac{1}{\sqrt{z(1-z)} \langle a b \rangle}$$

$$\text{Split}_{-}^{\text{tree}}(a^{+}, b^{-}) = -\frac{z^2}{\sqrt{z(1-z)} [a b]}$$

$$\text{Split}_{-}^{\text{tree}}(a^{-}, b^{+}) = -\frac{(1-z)^2}{\sqrt{z(1-z)} [a b]}$$

Collinear Factorization at One Loop

$$A_n^{1\text{-loop}; \text{LC}}(\dots, a^{h_a}, b^{h_b}, \dots) \xrightarrow{k_a \parallel k_b} \sum_{h=\pm} \left(\text{Split}_{-h}^{\text{tree}}(a^{h_a}, b^{h_b}) A_{n-1}^{1\text{-loop}; \text{LC}}(\dots, (k_a + k_b)^h, \dots) + \text{Split}_{-h}^{1\text{-loop}}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{tree}}(\dots, (k_a + k_b)^h, \dots) \right) + \text{non-singular}$$

Anomalous Dimensions & Amplitudes

- In QCD, one-loop anomalous dimensions of twist-2 operators in the OPE are related to the tree-level Altarelli-Parisi function,

$$\begin{array}{ccccc} \text{Twist-2} & & \text{Mellin} & & \text{Helicity-} \\ \text{Anomalous} & & \text{Transform} & & \text{summed} \\ \text{Dimension} & \Leftrightarrow & & = & \text{splitting} \\ & & & & \text{amplitude} \end{array}$$

- Relation understood between two-loop anomalous dimensions & one-loop splitting amplitudes

DAK & Uwer (2003)

Recursion Relations

Considered color-ordered amplitude with **one** leg off-shell, amputate its polarization vector

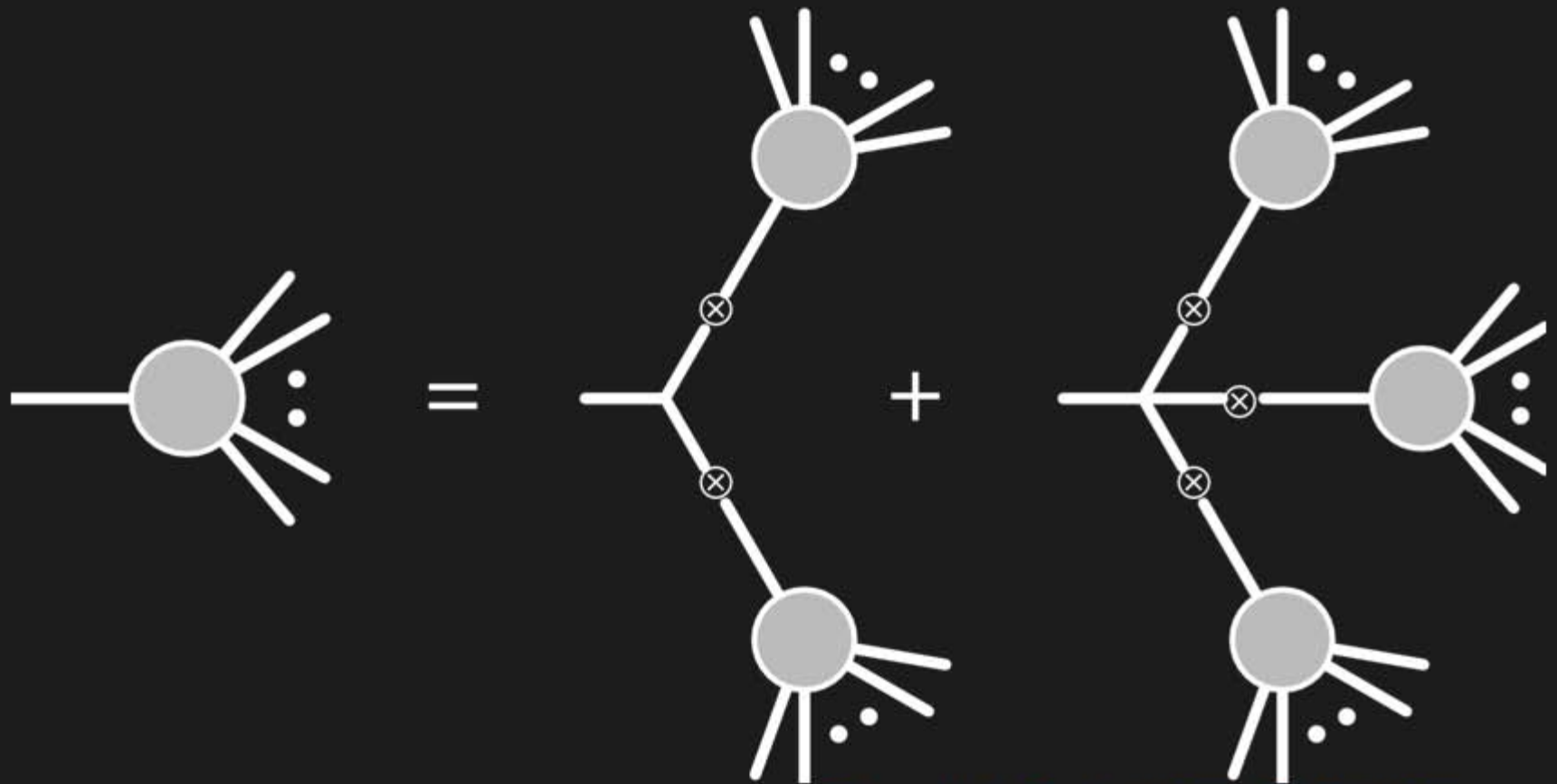
This is the Berends–Giele current $J^\mu(1, \dots, n)$

Given by the sum of all $(n+1)$ -point color-ordered diagrams with legs $1 \dots n$ on shell

Follow the off-shell line into the sum of diagrams. It is attached to either a three- or four-point vertex.

Other lines attaching to that vertex are also sums of diagrams with one leg off-shell and other on shell, that is currents

Recursion Relations



Berends & Giele (1988); DAK (1989)

⇒ Polynomial complexity per helicity

$$\begin{aligned}
J^\mu(1, \dots, n) = & \\
& - \frac{i}{K_{1,n}^2} \left[\sum_{j=1}^{n-1} V_3^{\mu\nu\lambda} J_\nu(1, \dots, j) J_\lambda(j+1, \dots, n) \right. \\
& + \sum_{j=1}^{n-2} \sum_{l=j+1}^{n-1} V_4^{\mu\nu\lambda\rho} J_\nu(1, \dots, j) \\
& \quad \left. \times J_\lambda(j+1, \dots, l) J_\rho(l+1, \dots, n) \right]
\end{aligned}$$

Properties of the Current

- Decoupling identity
- Reflection identity
- Conservation $K_{1,n}^\mu J_\mu(1, \dots, n) = 0$

Complex Momenta

For real momenta, $|k^+\rangle = \pm |k^-\rangle^*$

but we can choose these two spinors independently and still have $k^2 = 0$

Recall the polarization vector: $\varepsilon^+ \propto \langle q^- | \gamma^\mu | k^- \rangle$

but $\varepsilon \cdot \varepsilon = 0$

Now when two momenta are collinear $k \cdot k' = 0$

only one of the spinors has to be collinear

$\langle k k' \rangle = 0$ or $[k k'] = 0$ but **not** necessarily both

On-Shell Recursion Relations

Britto, Cachazo, Feng [th/0412308](#); & Witten [th/0501052](#)

- Ingredients
 - Structure of factorization
 - Cauchy's theorem

Introducing Complex Momenta

- Define a shift $[j, l]$ of spinors by a complex parameter z

$$\begin{aligned} |j^-\rangle &\rightarrow |j^-\rangle - z|l^-\rangle, \\ |l^+\rangle &\rightarrow |l^+\rangle + z|j^+\rangle \end{aligned}$$

- which induces a shift of the external momenta

$$\begin{aligned} k_j^\mu &\rightarrow k_j^\mu(z) = k_j^\mu - \frac{z}{2} \langle j^- | \gamma^\mu | l^- \rangle, \\ k_l^\mu &\rightarrow k_l^\mu(z) = k_l^\mu + \frac{z}{2} \langle j^- | \gamma^\mu | l^- \rangle \end{aligned}$$

- and defines a z -dependent continuation of the amplitude $A(z)$
- Assume that $A(z) \rightarrow 0$ as $z \rightarrow \infty$

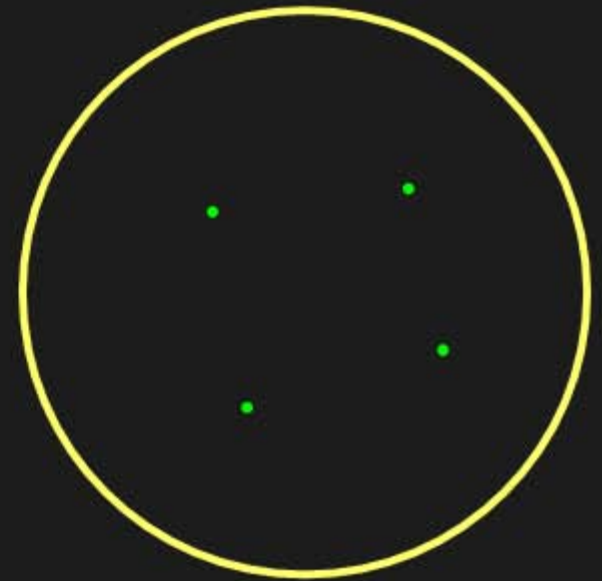
A Contour Integral

Consider the contour integral

$$\frac{1}{2\pi i} \oint_C \frac{dz}{z} A(z)$$

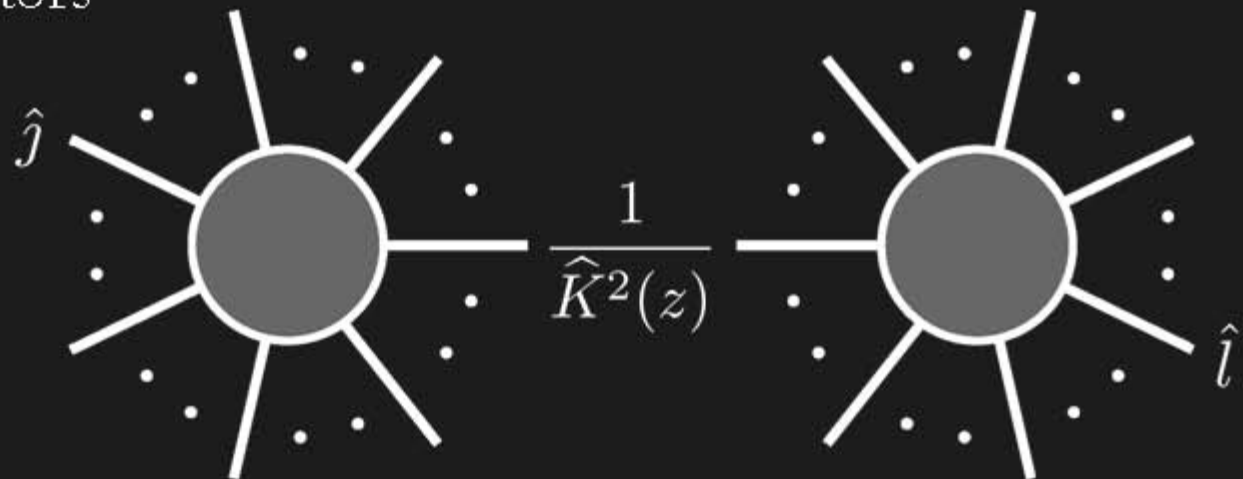
Determine $A(0)$ in terms of other residues

$$A(0) = - \sum_{\text{poles } \alpha} \text{Res}_{z=z_\alpha} \frac{A(z)}{z}$$



Using Factorization

Other poles in z come from zeros of z -shifted propagator denominators



Splits diagram into two parts with z -dependent momentum flow

→ $\sum_{\text{partitions}}$ shifted legs on opposite sides

Exactly factorization limit of z -dependent amplitude
poles from zeros of

$$K_{a\dots j\dots b}^2(z) = K_{a\dots b}^2 - z \langle j^- | K_{a\dots b} | l^- \rangle$$

That is, a pole at

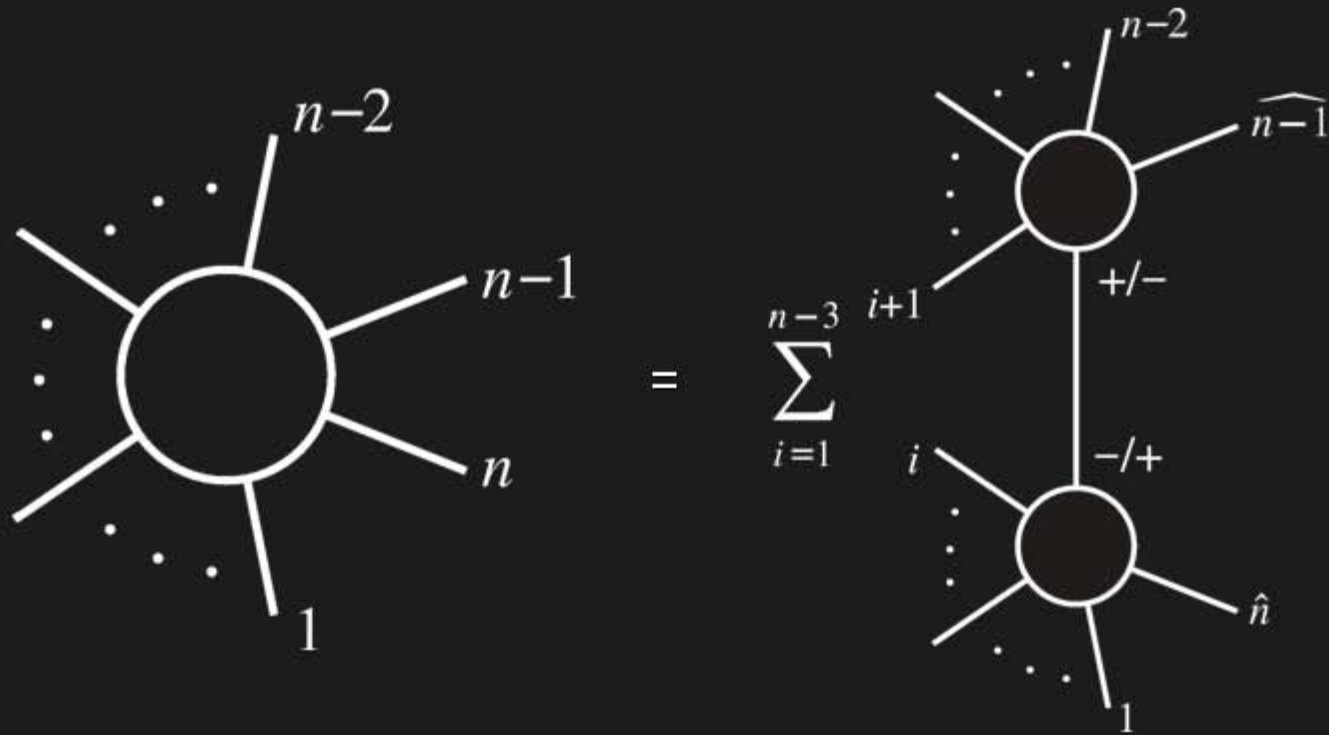
$$z_{ab} = \frac{K_{a\dots b}^2}{\langle j^- | K_{a\dots b} | l^- \rangle}$$

Residue

$$\text{Res}_{z=z_{ab}} \frac{f(z)}{z K_{a\dots b}^2(z)} = A_L(z_{ab}) \times \frac{i}{K_{a\dots b}^2} \times A_R(z_{ab})$$

Notation $\hat{k} = k(z_{ab})$

On-Shell Recursion Relation



- Partition P : two or more cyclicly-consecutive momenta containing j , such that complementary set \bar{P} contains l ,

$$\begin{aligned}
 P &\equiv \{P_1, P_2, \dots, j, \dots, P_{-1}\}, \\
 \bar{P} &\equiv \{\bar{P}_1, \bar{P}_2, \dots, l, \dots, \bar{P}_{-1}\}, \\
 P \cup \bar{P} &= \{1, 2, \dots, n\}
 \end{aligned}$$

- The recursion relations are then

$$\begin{aligned}
 A_n(1, \dots, n) &= \sum_{\substack{\text{partitions } P \\ h=\pm}} A_{\#P+1}(k_{P_1}, \dots, \hat{j}, \dots, k_{P_{-1}}, -\hat{P}^h) \\
 &\quad \times \frac{i}{P^2} \times A_{\#\bar{P}+1}(k_{\bar{P}_1}, \dots, \hat{l}, \dots, k_{\bar{P}_{-1}}, \hat{P}^{-h})
 \end{aligned}$$

On shell

Number of terms $\sim |l-j| \times (n-3)$

so best to choose l and j nearby

Complexity still exponential, because shift changes as we descend the recursion

Applications

- Very general: relies only on complex analysis + factorization
- Fermionic amplitudes
- Applied to gravity

Bedford, Brandhuber, Spence, & Travaglini (2/2005)

Cachazo & Svrček (2/2005)

- Massive amplitudes

Badger, Glover, Khoze, Svrček (4/2005, 7/2005)

Forde & DAK (7/2005)

- Other rational functions

Bern, Bjerrum-Bohr, Dunbar, & Ita (7/2005)

- Connection to Cachazo–Svrček–Witten construction

Risager (8/2005)

- CSW construction for gravity

Bjerrum-Bohr, Dunbar, Ita, Perkins, & Risager (9/2005)