Computer Physics Lab

www.labfiscomp.blog

Celestial Mechanics

on the personal computer

Here you will find some examples which illustrate how simple and easy it is to explore problems in solar system dynamics using `matlab`. We shall study the classical problem of Mercury's perihelion anomalous precession, the motion of the Moon and the stability of the Lagrange point L4 in the Earth-Moon system taking into account perturbations. You may find in the package CM.tar all the `matlab` codes which have been developed along the years of University teaching. The real fun however is to develop these codes by yourselves, it may turn out that they will be better than those present on the blog, in which case let us have them and they will be included in the package with due reference to the Authors.

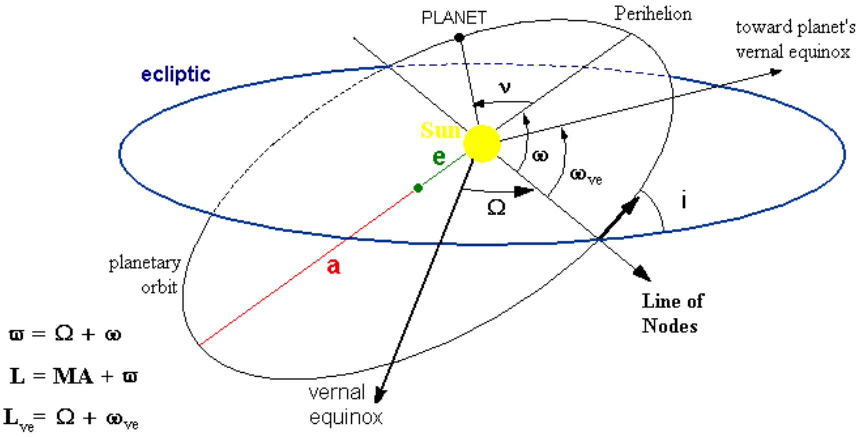
1. Mercury's perihelion precession

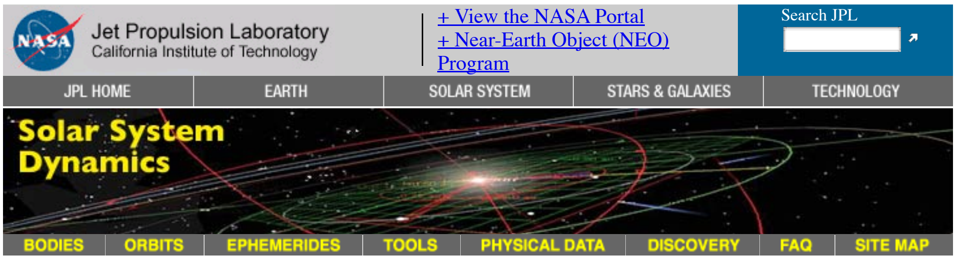
You find an authoritative account of this problem, which is one of the classic tests of General Relativity, on Weinberg's book on Cosmology[1] and on C.M.Will's book on Experimental Gravitation[2]. It suffices to recall here that the orbit of Mercury is affected by the attraction of other planets in a way that makes its Kepler ellipse periodically modify the direction of its main axis by an angle which is measured in ***seconds of arc per century -*** yes, it's a delicate effect, but astronomers are well able to measure such tiny effects with their instruments. The numbers are the following, $\Delta\varphi$ denoting the precession angle: $\Delta\varphi$ = 5600".7 /century (which makes less than two degrees) the main effect however is given by the optical illusion due to the fact that we observe Mercury's dynamics from an observatory (the Earth) whose rotation axis is subject itself to a secular precession which accounts for the major effect - 5025".6 - hence the real effect as seen by Andromeda (if possible) would be $\Delta\varphi_{corrected}$ = 575".

[1] S. Weinbberg, Gravitation and Cosmology, Wiley, 2015

[2] C.M. Will, Theory and Experiment in gravitational physics, rev.ed. Cambridge U.P. 1993

This is the number which we can try to reconstruct on the computer by solving Newton's equations for the main planets, provided we get all relevant orbital elements at a given epoch. So to do the calculation we need a computer running matlab (or your preferred language, python perhaps?), a list of the main objects in the Solar





Planets and Pluto: Physical Characteristics

This table contains selected physical characteristics of the planets and Pluto.

Planet	Equatorial Radius	Mean Radius	Mass	Bulk Density	Sidereal Rotation Period	Sidereal Orbit Period	V(1,0)	Geometric Albedo	Equatorial Gravity	Escape Velocity
	(km)	(km)	(x 10^{24} kg)	(g cm^{-3})	(d)	(y)	(mag)		(m s^{-2})	(km s^{-1})
Mercury	2439.7 [D] ±1.0	2439.7 [D] ±1.0	0.330104 [E] ±.000036	5.427 [*] ±.007	58.6462 [D]	0.2408467 [B]	-0.60 [E] ±0.10	0.106 [B]	3.70 [*]	4.25 [*]
Venus	6051.8 [D] ±1.0	6051.8 [D] ±1.0	4.86732 [G] ±.00049	5.243 [*] ±.003	-243.018 [D]	0.61519726 [B]	-4.47 [E] ±0.07	0.65 [B]	8.87 [*]	10.36 [*]
Earth	6378.14 [D] ±.01	6371.00 [D] ±.01	5.97219 [H] ±.00060	5.5134 [*] ±.0006	0.99726968 [B]	1.0000174 [B]	-3.86 [B]	0.367 [B]	9.80 [*]	11.19 [*]
Mars	3396.19 [D] ±.1	3389.50 [D] ±.2	0.641693 [I] ±.000064	3.9340 [*] ±.0008	1.02595676 [D]	1.8808476 [B]	-1.52 [B]	0.150 [B]	3.71 [*]	5.03 [*]
Jupiter	71492 [D] ±4	69911 [D] ±6	1898.13 [J] ±.19	1.3262 [*] ±.0004	0.41354 [D]	11.862615 [B]	-9.40 [B]	0.52 [B]	24.79 [*]	60.20 [*]
Saturn	60268 [D] ±4	58232 [D] ±6	568.319 [K] ±.057	0.6871 [*] ±.0002	0.44401 [D]	29.447498 [B]	-8.88 [B]	0.47 [B]	10.44 [*]	36.09 [*]
Uranus	25559 [D] ±4	25362 [D] ±7	86.8103 [L] ±.0087	1.270 [*] ±.001	-0.71833 [D]	84.016846 [B]	-7.19 [B]	0.51 [B]	8.87 [*]	21.38 [*]
Neptune	24764 [D] ±15	24622 [D] ±19	102.410 [M] ±.010	1.638 [*] ±.004	0.67125 [D]	164.79132 [B]	-6.87 [B]	0.41 [B]	11.15 [*]	23.56 [*]
Pluto	1151 [C] ±6	1151 [C] ±6	.01309 [N] ±.00018	2.05 [*] ±.04	-6.3872 [D]	247.92065 [B]	-1.0 [B]	0.3 [B]	0.66 [*]	1.23 [*]

system with their so-called orbital elements; this is provided e.g. by JPL lab (http://ssd.jpl.nasa.gov).

There are several options at our disposal: we can use a routine in matlab's ODE suite and make all planets run their orbits under mutual Newton's attraction in the field of force of the Sun, which is the main engine. This is rather straightforward and can be found in the routine `mercury.m` in the package **CM.tar**; the program calls an auxiliary routine `orbel.m` to load orbital elements and then it invokes `ode113` as integration routine. The result is plotted in terms of the direction of Mercury's main axis along one or more centuries, together with its linear fit which aims at averaging the oscillating terms which are rather ample. The program allows to isolate the contribution of various planets, among these the more important effect is that of Venus, Earth and Jupiter. Running the program for a century takes up only a few seconds on a recent personal computer.

Another set up is suggested in Chapter 9.5 of Weinberg's book and applies the idea of studying the dynamics of the Laplace—Runge—Lenz vector whose direction is instantly along the major axis of the orbit. This vector

$$\boldsymbol{A} = \boldsymbol{v} \wedge \boldsymbol{h} - GM\frac{\boldsymbol{x}}{r}$$

represents an exact conserved quantity if the only force acting on the planet is given by the Sun attraction. Here $\boldsymbol{v}$ is the planet's velocity, $\boldsymbol{x}$ it's position with respect to the center of the Sun, G is Newton's constant and M the Solar mass. The importance of $\boldsymbol{A}$ is given by the fact the it's direction points to the perihelion and it length gives the eccentricity of the orbit. In presence of other forces beyond the attraction by the Sun

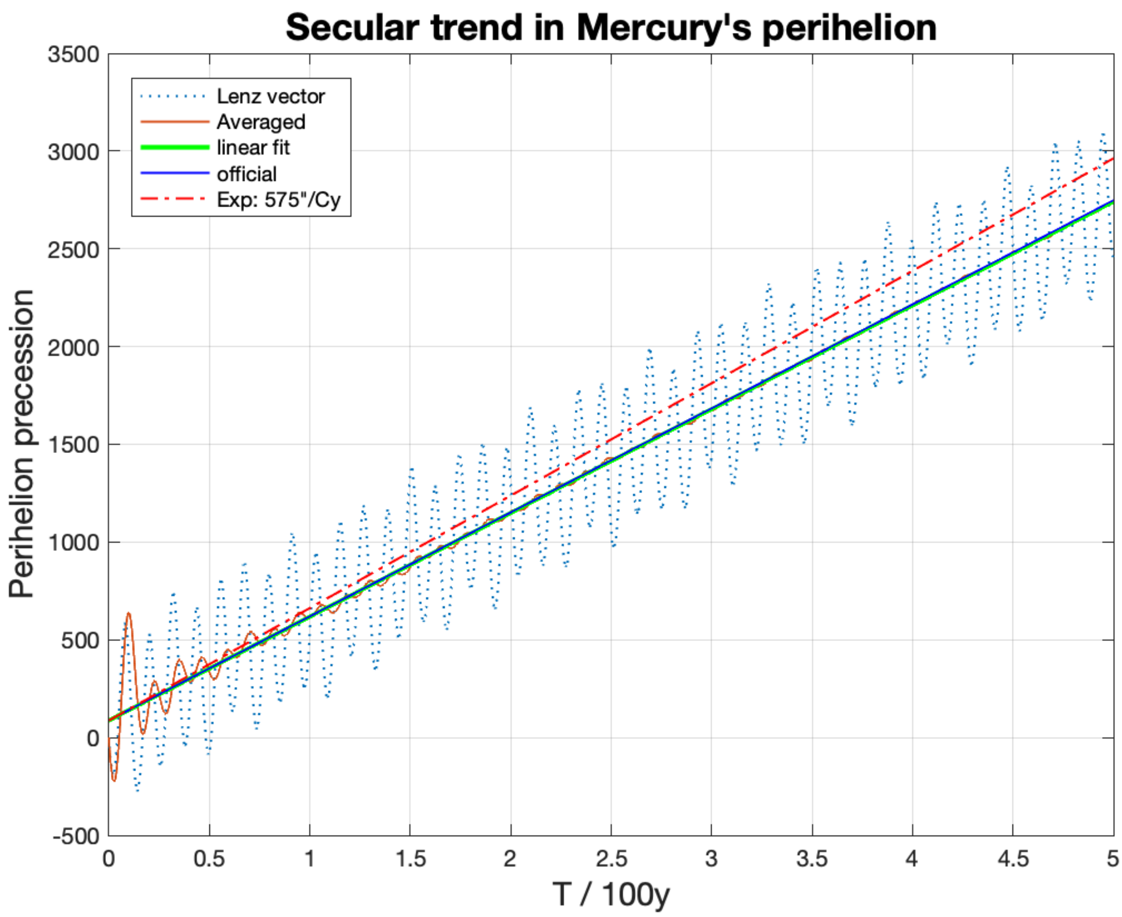
$\boldsymbol{A}$ are acting on the planet then it is not constant any more and tracking its direction gives the precession of the perihelion. The interesting thing is that $\boldsymbol{A}$ satisfies a first order differential equation which can be found in Weinberg's book ***provided we know the position of all relevant celestial objects.*** Here is the catch: if we calculate the position of the planets by solving the equations of motion then the approach is not going to give us any gain. However we can imagine that a good approximation consists in putting all planets on their Kepler ellipses since their mutual interaction is not going to have a big impact on the motion of Mercury. We try to do so and apply Weinberg's equation

$$\frac{d\boldsymbol{A}}{dt} = \boldsymbol{\eta} \wedge \boldsymbol{h} + \boldsymbol{v} \wedge (\boldsymbol{x} \wedge \boldsymbol{\eta})$$

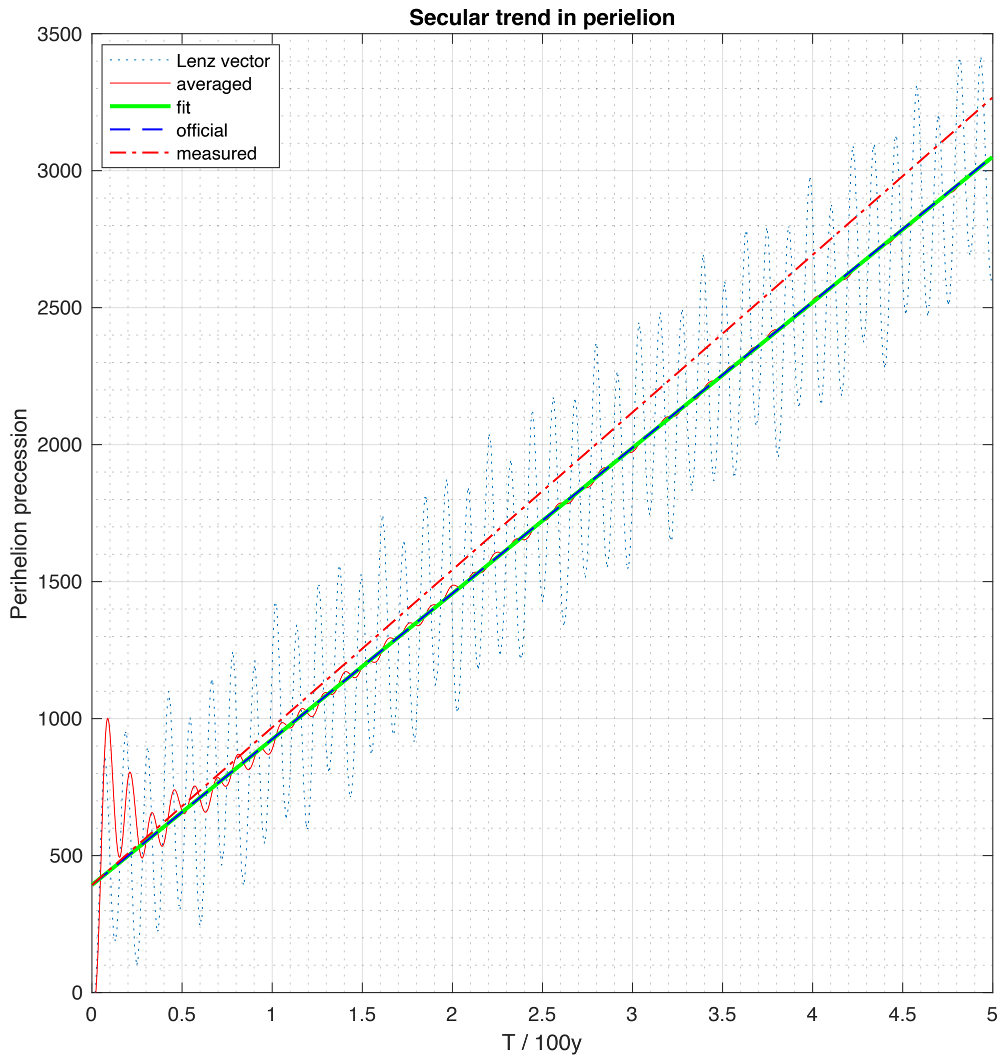
where η represents all forces acting on Mercury except Solar Newtonian attraction.
η may contain also the contribution of Solar oblateness o the attraction of the asteroid

clouds. For simplicity the program `~/firstorder/lenz.m` only contains the attraction of other planets.

You can make experiments with `mercury.m` and/or `lenz.m;` results will vary from one experiment to the other, we must keep in mind that this number 575”/ *century* is not a constant of Nature, on the opposite it depends on many accidental data, the initial positions of the Planets along the orbit, the inclination of the orbits with respect to the ecliptic, etc. One surprising aspect of this historic experiment of late XIX century is that it took many decades to collect data AND it took many decades to perform the Newton perturbation calculation without the aid of a computer - what is now a matter of few seconds took many years of deeply intricate calculations based of Hamilton-Jacobi equation which allows to set up a perturbative calculation. The very complex results were recently found to contain some slight mistakes when the calculations were done again using a symbolic code on the computer, luckily enough



those were not catastrophic mistakes!



Here are two experiments, T=500y, the first with `mercury.m`, full Newtonian dynamics, the second with `lenz.m` using Weinberg's equation for ***A*** and exploiting the known Kepler orbits given their realistic orbit elements.

Results are indistinguishable within fluctuations of initial conditions. This means that the perturbation of Mercury's perihelion does not depend on the detailed Newtonian dynamics, it can be estimated by using Kepler ellipses.

Of course the interesting thing is the fact that computed and experimental value for the perihelion precession **do not match!** Experimental value is ~575" while the number laboriously derived by astronomers and reproduced on our computer today in a few seconds is ~532". It seems a tiny discrepancy, but on the plots it is rather evident. A difference of ~43" was difficult to justify. This was the puzzle at the end of XIX century for Astronomers. Several hypotheses were formulated to account for the mismatch, including a bigger Solar oblateness, the existence of a misteryous planet invisible from Earth… The state of the Art was expressed in the final notes to his authoritative book[3] by the Astronomer F. R. Moulton (I quote)

" *At the present time Celestial Mechanics is entitled to be regarded as the most perfect science and one of the most splendid achievements of the human kind … the only unexplained irregularities (probably due to unknown forces) are a very few small ones in the motion of the Moon and the motion of the perihelion of the orbit of Mercury."* Dated 1914! The following year Albert Einstein published his new theory of Gravitation in terms of which the 43" anomaly took a precise analytic form

$$\Delta\varphi = 6\pi \frac{G\,M}{R\,c^2\,(1-\varepsilon^2)}$$

astonishingly enough, the theory did not need any new natural constants to account for the anomaly.

2. Moon's dynamics - the Saros cycle

The dynamics of the Moon can be studied in its simplest form by considering the three body motion of Sun-Earth-Moon; this has been a problem central to Celestial Mechanics and its Astronomical implications were clear even before the advent of modern Science, i.e. Newtonian mechanics. The purely phenomenological study of the motion of the Moon gave remarkably precise results already at the time of Babylonian astronomy[4]. Thanks to their powerful arithmetic, Astronomers had a deep control on the motion of the Moon, and this meant the possibility of foreseeing

[3] An Introduction to Celestial Mechanics, The MacMillan Company, 1914, pag.430, Dover, 1970

[4] O. Neugebauer, "The Exact Sciences in Antiquity", Dover, second ed., 1969.

eclipses. Observations were performed along many decades, passing the information from one generation to the following, and a marvellous result was the identification of a fundamental period in the motion of the Moon which is called “the Saros cycle”: this cycle is 18 years, 11 days +1/3 long and could have not be identified without a patient data taking along at least a century. This is the fundamental cycle which characterises the periodic repetition in the sequence of eclipses. Even Isaac Newton had a hard interaction with the problem of Moon dynamics, since the three-body problem does not allow for easy analytical solutions not even approximate. Only at the end of XIX century a full grown perturbative solution was formulated. A modern presentation can be found in Gutzwiller’s book[5].

Now in this blog we are going to show how the Moon’s dynamics can be easily studied by numerical analysis using modern ODE codes, such as they are implemented in matlab. Our setup of the computer experiment has been described in a paper on Comp.Sci.Eng. [6] and will not be repeated here. The matlab codes are included in the package: you may use `newton3D.m` to run Newton’s equations for the coupled three body system Sun-Earth-Moon; the routine ode113 appears to be the most efficient in terms of speed and accuracy. When the run is completed along a time T sufficiently long (i.e. 100 years or more), Moon’s motion can be analysed in the frequency domain using the Fourier transform. The spectrum turns out to be highly complex, typical of a multi-periodic phenomenon. As a convenient observable to be analysed it has been chosen the so called Laplace-Runge-Lenz vector (see Mercury’s perihelion), which would be just a conserved constant in the two-body problem Earth-Moon, but its behaviour is strongly modified by the Sun’s influence. As a result its spectrum shows clusters of peaks around a frequency month^{-1}, other clusters around the frequency year^{-1} and finally the interesting fact is to be found by blowing up the low frequency spectrum at the scale saros^{-1}, i.e. approximately 20 times smaller: here we find very clean peaks at multiples of the frequency saros^{-1}. By a linear fit we get the fundamental frequency .05545 nicely corresponds to the Saros cycle.

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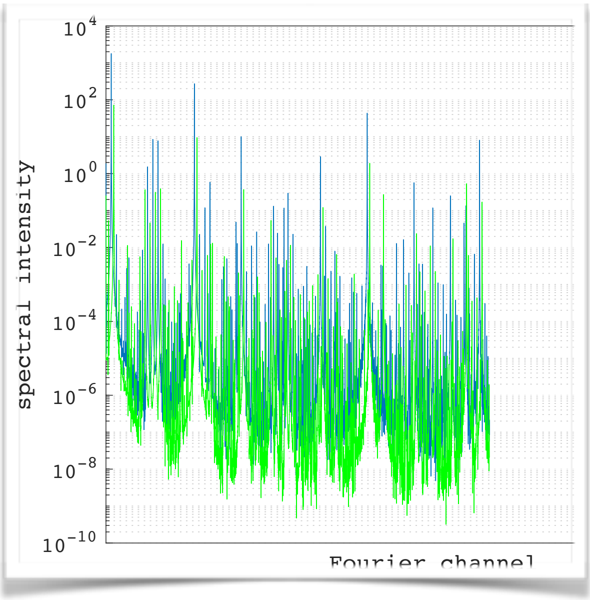
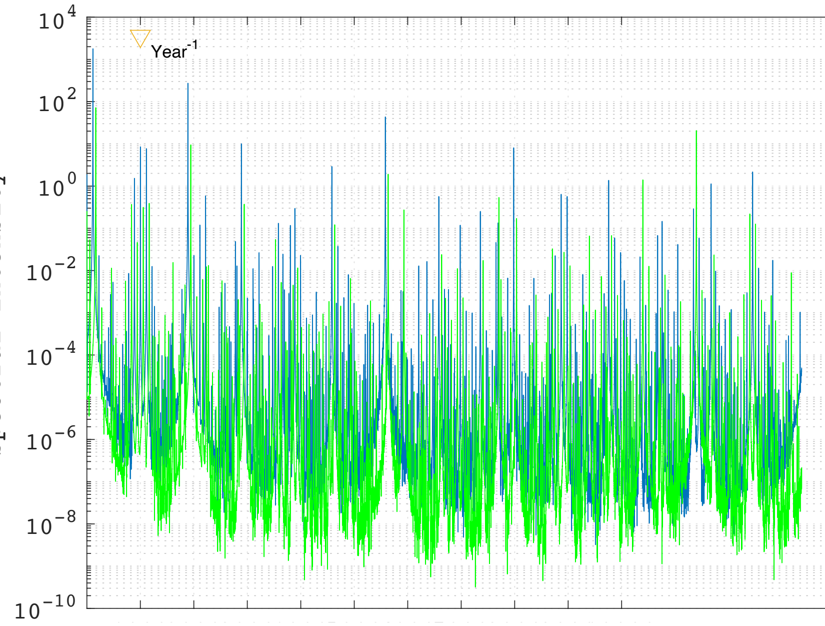
[5] M. C. Gutzwiller, “Chaos in Classical and Quantum Mechanics”, Springer, N.Y.,1990.

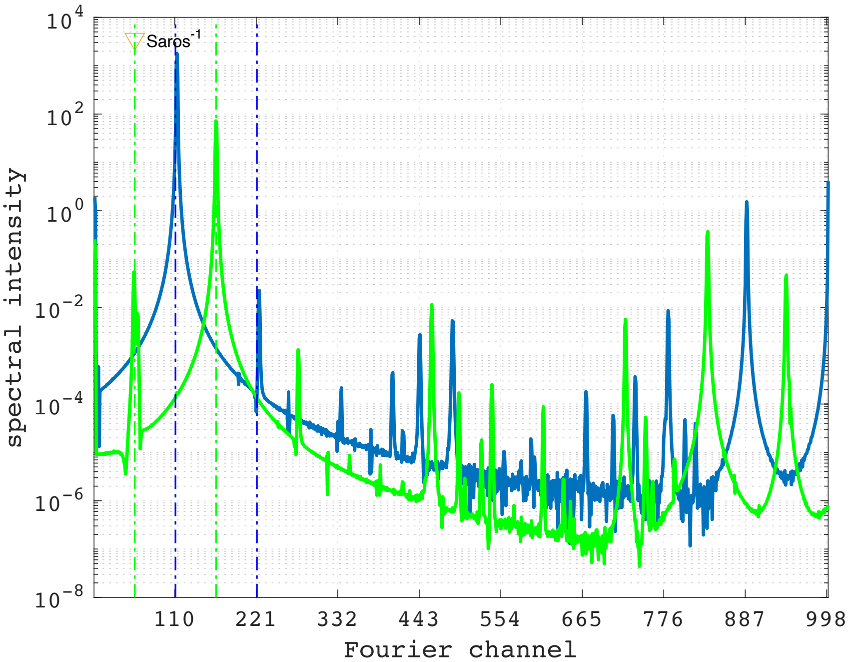
[6] E. Onofri, “Elementary Celestial Mechanics using matlab”, Computing in Science and Engineering, Nov.2001, 48.



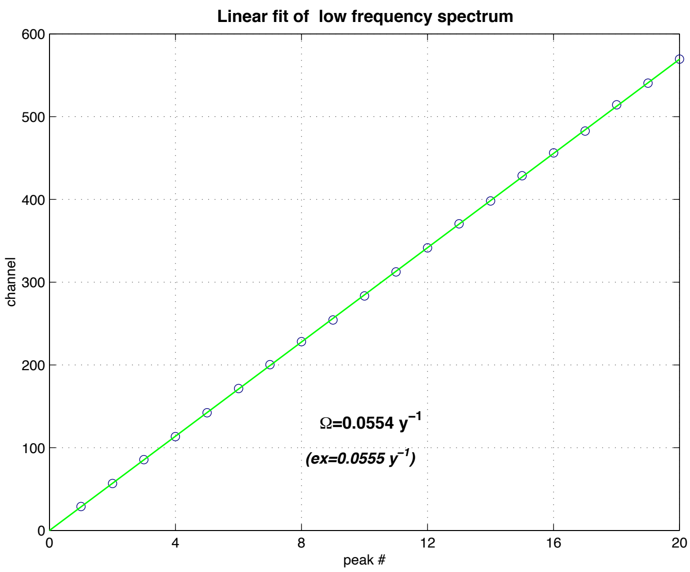
10^4
10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
spectral intensity
Fourier channel



Year^{-1}
10^4
10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
spectral intensity



Saros^{-1}
10^4
10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
spectral intensity
110 221 332 443 554 665 776 887 998
Fourier channel



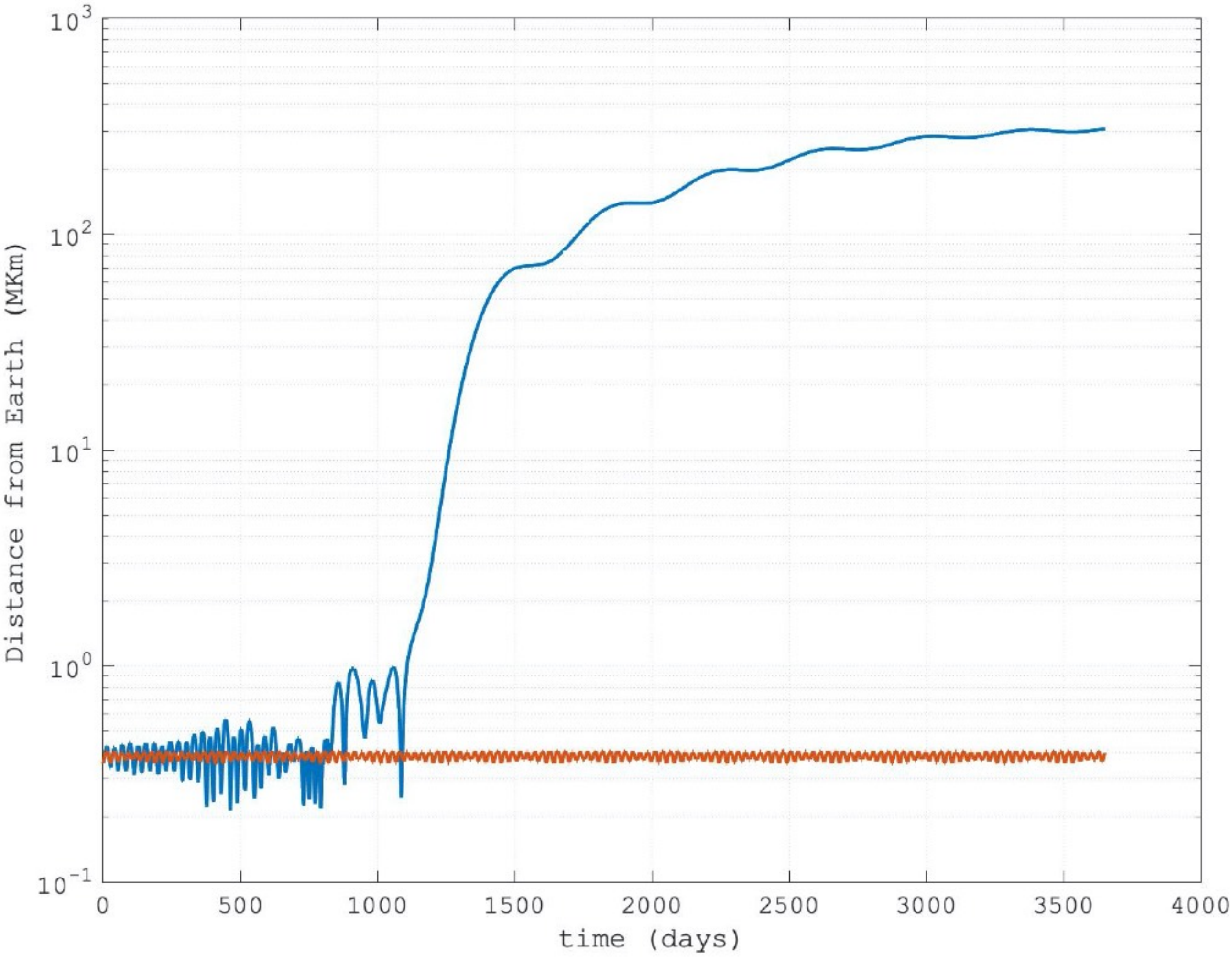
Linear fit of low frequency spectrum
600
500
400
300
200
100
0
channel
0 4 8 12 16 20
peak #
Ω=0.0554 y^{-1}
(ex=0.0555 y^{-1})

The main effort in writing the code to compute the Moon's dynamics consists in 1) identifying the more convenient coordinates (a simple minded approach risks to go into trouble because of the different scale distance Sun-Earth w.r.t. distance Moon-Earth. Hence a good choice has been presented in ref.[6].

2) Then one has to collect the relevant astronomical constants which can easily be found e.g. in the Particle Physics Data Group web site pdg.lbl.gov.

3. Moon's dynamics - Lagrange points and their stability

In the same paper on Comp.Sci.and Engin. a numerical experiment was described regarding the stability of the motion of so-called Lagrange point L4. Lagrange discovered an interesting special solution in the dynamics of three gravitating bodies, in the special case that the third object is very light: an example is given by the Trojan asteroids who run on the same orbit of Jupiter located with the Sun and Jupiter at the vertices of a regular triangle. We can imagine that there could be a practical interest in placing a satellite in such a position with respect to the Earth and the Moon - it could be used for instance for telecommunications. The question is: since the Sun exerts a strong attraction on all celestial objects such an orbit would be stable taking into account also the interaction with the Sun, hence forming a four-body system? With the same setup used to explore the motion of the Moon, we can add the artificial satellite, study its orbit and verify its stability properties. The matlab code can be found in the folder Moon-L4. You can modify, in the simulation, the Solar mass hence checking that the code works fine giving a periodic orbit for zero Solar mass. But the Sun is there and its effect is going to disrupt the Lagrange L4 orbit very soon. In the folloing picture we see the satellite that after oscillating around L4 for almost a year, starts a series of wild oscillations and after two years leaves the Moon-Earth system and goes to a Solar orbit at a distance of ~300 MKm from earth. The matlab codes let you make experiments by varying various parameters, the eccentricity of the orbits, the inclination of Moon's orbit on the ecliptic, the mass of the Sun and the ratio of masses Earth/Moon. You may also try to start the satellite not exactly at L4 but slightly off. At the end you will recognise that the main difficulty in keeping a stable orbit is the Sun's attraction.



10^3
10^2
10^1
10^0
10^{-1}
Distance from Earth (MKm)
0
500
1000
1500
2000
2500
3000
3500
4000
time (days)