FOLLOWING PINO - THROUGH THE CUSPS AND BEYOND THE PLANAR LANDS

Lorenzo Magnea

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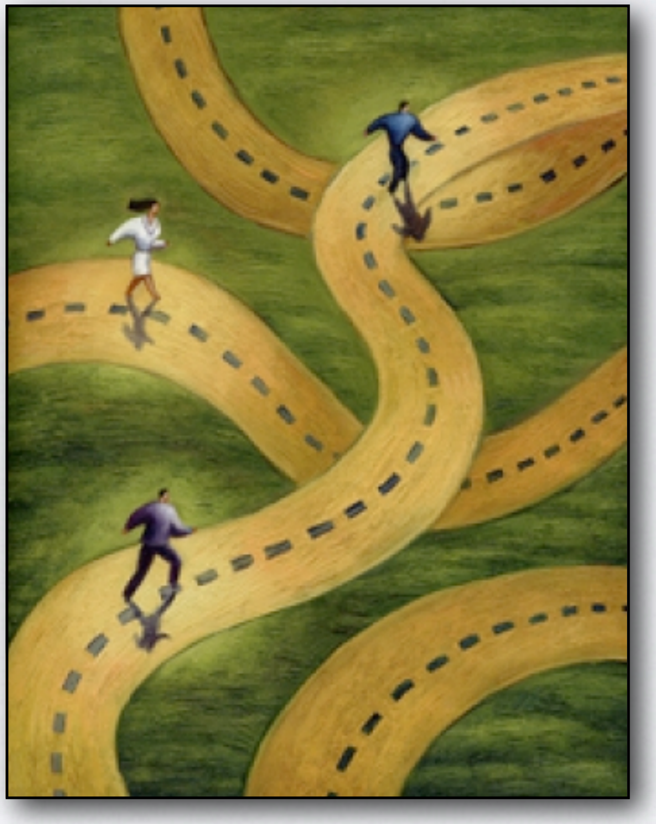
Pino Day, Cortona, 29/05/12

Outline

- Crossing paths with Pino
- Cusps, Wilson lines and Factorization
- The dipole formula
- Pino’s prophecies, Casimir conspiracies

CROSSING PATHS WITH PINO



1987 - Coherent States

Nuclear Physics B296 (1988) 546–556
North-Holland, Amsterdam

INFRARED FINITE *S*-MATRIX IN THE QCD COHERENT STATE BASIS*

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Received 29 June 1987

The infrared (IR) structure of the *S*-matrix in the basis of the QCD coherent states is studied. To any order perturbation theory it is shown that these matrix elements are IR finite in spite of the fact that they involve any number of soft gluons; moreover, these soft gluons do not even contribute to the IR singular part of the inclusive Bloch-Nordsieck distribution. Finally, the BRS charge related to the coherent state *S*-matrix elements is given and shown to be IR finite to any order perturbation theory.

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1. Introduction

During the last few years a considerable effort has been made [1] to analyze the perturbative infrared (IR) behaviour of QCD. This study has provided insights on many properties of perturbative QCD such as for instance: the non-factorization theorem and the non cancellation of IR singularities in Bloch-Nordsieck inclusive distributions [2]; the indications [3,4] that the perturbation theory is not consistent in the case that QCD is not confined; the phenomenological structure [1,5] of QCD radiation, and its consistency with present high energy hard scattering data.

A general method to study these IR properties is based on the construction of the QCD coherent states which describe the cloud of strongly correlated soft gluons surrounding a hard scattering process. These states are obtained [3,4,6,7] for instance by generalizing to QCD the method of asymptotic (or soft) dynamics introduced for QED by Faddeev and Kulish [8]. According to this method one introduces an arbitrary energy scale E separating soft and hard partons and one can show that the *S*-matrix can be factorized in the form

$$\boxed{S = \Omega^{\dagger}_{S,-}(E,\lambda)\, S_R(E)\, \Omega_{S,+}(E,\lambda),} \tag{1.1}$$

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G QCD 4

where $\Omega_{S,\pm}(E,\lambda)$ are the Møller operators for the soft or asymptotic dynamics. This means that the perturbative expansion of $\Omega_{S,\pm}(E,\lambda)$ corresponds to diagrams where all various transfer energies ν_i are soft, i.e. $\boxed{\nu_i < E}$. Since ν_i can be extended to the IR singular value $\nu_i = 0$ one introduces an IR cutoff in (1.1) by requiring $\boxed{\nu_i > \lambda}$. Consequently the perturbative expansion of $S_R(E)$ corresponds to diagrams where together with soft transfer energies one has at least one hard transfer every $\nu > E$.

Ω : all soft.

S_R : ∃ one hard.

In this paper we discuss the following two points arising from the factorization of the *S*-matrix in (1.1).

(i) In QED the operator $S_R(E)$ is IR regular [8], i.e. IR finite, in the Fock basis. Therefore even if in $S_R(E)$ there are soft photons, no singularities arise when their frequency vanishes, and the IR singularities of the *S*-matrix are fully described by the factorized soft Møller operators $\Omega_{S,\pm}(E,\lambda)$. As a consequence one can introduce the coherent states

$$\boxed{|h,\pm\rangle \equiv \Omega^{\dagger}_{S,\pm}(E,\lambda)|h\rangle,} \tag{1.2}$$

where $|h\rangle$ are hard Fock space states. The coherent state *S*-matrix reduces then to $S_R(E)$ and is IR regular.

In order to do the same formal operation in the case of QCD one has then to show that also in this case $S_R(E)$, although involving any number of soft gluons, is IR regular. There are various indications [7] that actually $S_R(E)$ is IR regular, but a detailed analysis of this important point is lacking. In this paper we analyze, within the framework of perturbation theory, the IR structure of $S_R(E)$ and show that is is actually IR regular. This is based on the following two points: (i) the possible IR singularity of each propagator is screened by the presence of at least one hard frequency $\nu > E$; (ii) the cancellation of IR singularities arising when some propagator goes on shell.

1

a)

b)

We are able to show that each perturbative term for $S_R(E)$ is IR finite by using for the Møller operator of $S_R(E)$ not the standard time ordered expression, but a "frequency ordered" expression introduced in ref. [3]. For this expression in fact we are able to show that the kernel itself is IR finite. Moreover, by using this frequency ordered expansion we show that soft gluons exchanged between $S_R(E)$ and $\Omega_{S,\pm}(E,\lambda)$ do not produce IR singular contributions. This implies that the study of non-cancellation of IR singularities in inclusive Bloch-Nordsieck distributions can be done, to all orders, by disregarding the soft gluons in $S_R(E)$ thus treating $S_R(E)$ at the tree level as usually done [2–4].

(ii) The second point analyzed in this paper is related to the BRS invariance of the theory. As known [4] the *S*-matrix, in the interaction picture, commutes with the free BRS charge Q^0_{BRS}. As a consequence one can define BRS charges $Q_{BRS,\pm}(E)$ which provide the commutation property of $S_R(E)$ to preserve the gauge invariance properties of the theory. These charges are important also in the discussion [4] of the properties of the coherent states and the corresponding physicality condition. Their

2

Personal archeology: the way we worked ...

1997 - Power Corrections

ELSEVIER

Nuclear Physics B 511 (1998) 396–418

Universality of $1/Q$ corrections to jet-shape observables rescued ★

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Abstract

We address the problem of potential non-universality of the leading $1/Q$ power corrections to jet shapes emerging from the non-inclusive character of these observables. We consider the thrust distribution as an example and analyse the non-inclusive contributions which emerge at the two-loop level. Although formally subleading in α_s, they modify the existing naïve one-loop result for the expected magnitude of the power term by a factor of order unity. Such a promotion of a subleading correction into a numerical factor is natural since the non-perturbative power terms are explicitly proportional to powers of the QCD scale Λ which can be fixed precisely only at the two-loop level. The "jet-shape scaling factor" depends on the observable but remains perturbatively calculable. Therefore it does not undermine the universal nature of $1/Q$ power corrections, which remain expressible in terms of the universal running coupling and universal soft-gluon emission.

PACS: 12.38.Cy; 12.38.Lg; 13.65.+i

Keywords: $1/Q$ power corrections; Jet shape; Soft-gluon universality; Hard QCD processes

1990 - The Cusp as a Coupling

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QCD COHERENT BRANCHING AND SEMI-INCLUSIVE PROCESSES AT LARGE x*

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Using the QCD coherent branching algorithm, we compute the Deep Inelastic Scattering and Drell–Yan hard cross sections in the semi-inclusive region of large x. The calculation is done to next-to-leading logarithmic accuracy in the resummation of perturbative QCD. We compare the results with the known analytical expressions to the same accuracy in the $\overline{\mathrm{MS}}$ subtraction scheme. They coincide if one defines an improved branching algorithm suitable for Monte Carlo simulation, in which the two-loop running coupling constant and Altarelli–Parisi splitting function are used. Therefore such a simulation can be used to measure $\Lambda_{\overline{\mathrm{MS}}}$ from these semi-inclusive cross sections. Moreover we show that the same results can also be obtained using the usual one-loop splitting function, provided the scale parameter Λ_{MC} used in the Monte Carlo simulation is related to $\Lambda_{\overline{\mathrm{MS}}}$ by a computed factor: $\Lambda_{\mathrm{MC}} = 1.569\,\Lambda_{\overline{\mathrm{MS}}}$ (for five flavours).

The Punchline

branching algorithm provided one uses the two-loop expression for running α_s [eq. (10)] and the following expression for the Altarelli–Parisi splitting function at $z \to 1$:

$$P_i(z, \alpha_s) = \frac{A_i(\alpha_s)}{1-z}, \quad A_i(\alpha_s) = C_i \frac{\alpha_s}{\pi}\left(1 + K\frac{\alpha_s}{2\pi}\right), \tag{58}$$

with K given by eq. (9) and $C_i = C_F$ or C_A for a quark or a gluon respectively.

Since the Monte Carlo algorithm with these improvements is accurate to next-to-leading order in the large-x region, it can be used to determine the fundamental QCD scale $\Lambda_{\overline{MS}}$.

From eq. (58) we see that the next-to-leading correction to the splitting functions for $z \to 1$ is a universal factor associated with soft gluon emission [8]. Therefore it can be absorbed into the one-loop splitting functions used in existing Monte Carlo simulations with coherence [14, 15] (after conversion from the one-loop to the two-loop definition of α_s), simply by rescaling the value of Λ. Denoting by Λ_{MC} the rescaled value used in the simulation with one-loop splitting functions, the corresponding value of α_s should satisfy

$$\alpha_s^{(MC)} = \alpha_s^{(\overline{MS})}\left(1 + K\frac{\alpha_s^{(\overline{MS})}}{2\pi}\right), \tag{59}$$

and thus

$$\begin{aligned} \Lambda_{MC} &= \Lambda_{\overline{MS}} \exp(K/4\pi\beta_0) \\ &\simeq 1.569\,\Lambda_{\overline{MS}} \quad \text{for } N_f = 5. \end{aligned} \tag{60}$$

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The Cusp Anomalous Dimension

- Wilson lines meeting at a cusp develop new UV divergences depending on the cusp angle

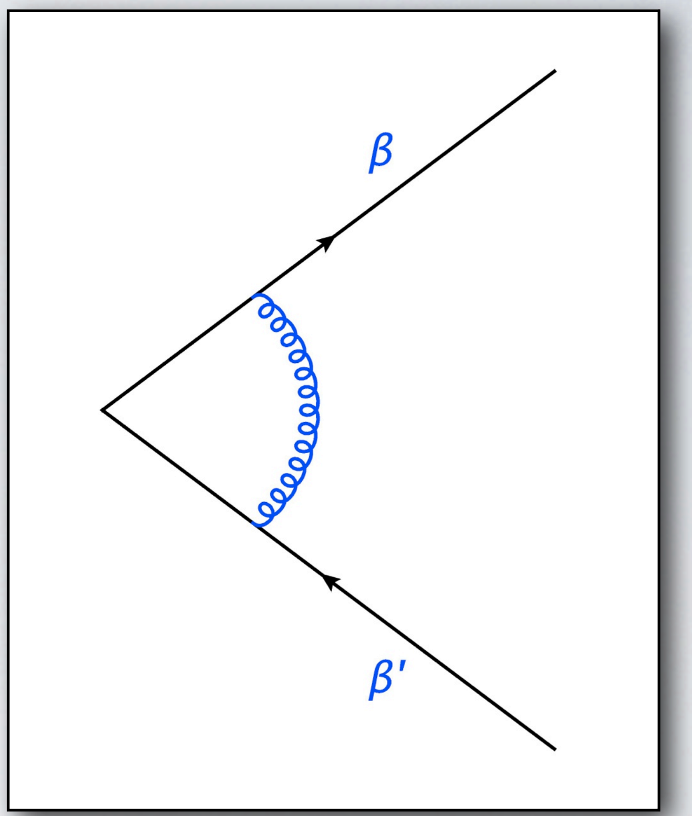
$$\cosh\theta \equiv \frac{\beta \cdot \beta'}{\sqrt{\beta^2\,\beta'^2}}$$

- The divergences are controlled by a new anomalous dimension

$$\Gamma_{\rm cusp}\left(\theta, \alpha_s\right) \; \equiv \; \mu \frac{\partial}{\partial \mu} \log \Big[W\left(C_\theta; \alpha_s, \mu\right) \Big]$$

- For light-like lines the cusp develops a collinear pole

$$\Gamma_{\rm cusp}\left(\theta, \alpha_s\right) \; \longrightarrow \; \gamma_K\left(\alpha_s\right) \, \log \left(\frac{\beta \cdot \beta'}{\sqrt{\beta^2 \beta'^2}} \right) \; \longrightarrow \; \frac{1}{\epsilon}\, \gamma_K\left(\alpha_s\right)$$



Wilson lines meeting at a cusp

- The cusp anomalous dimension γ_K (α_s) plays an increasingly fundamental role in massless gauge theories
  - It gives the soft limit of DGLAP splitting functions to all orders
  - It governs soft-gluon resummation for massless QCD cross sections
  - It controls soft singularities in planar massless gauge theory amplitudes
  - It is exactly known, from weak to strong coupling, in N=4 Super Yang-Mills theory
  - It is conjectured to control all soft singularities, including non-planar correlations, through the dipole formula.

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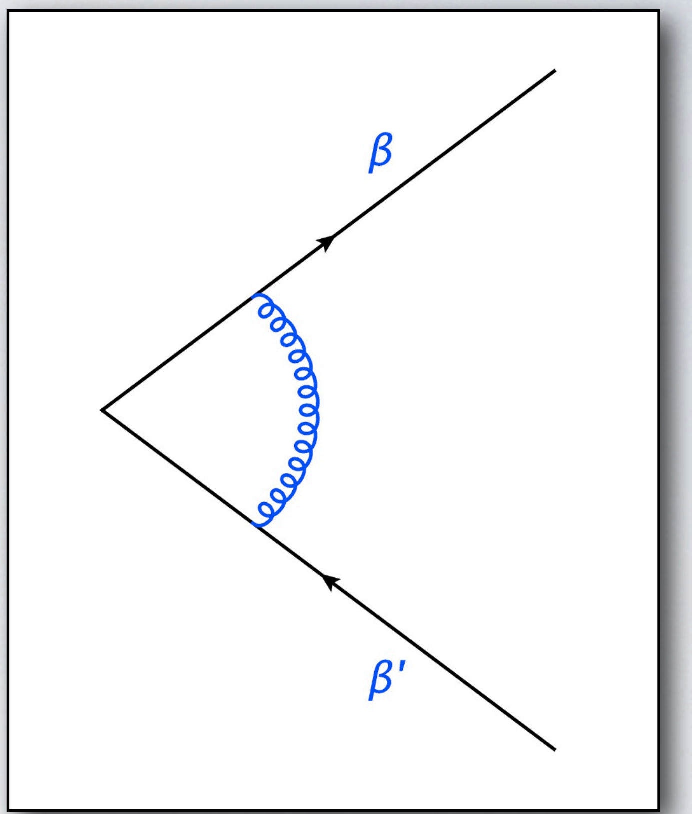

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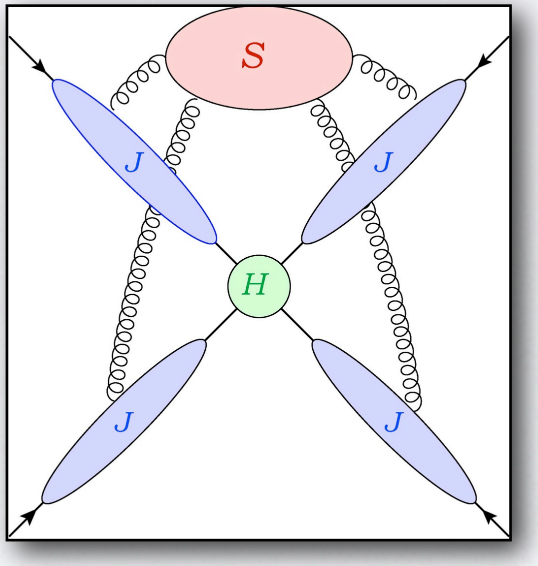
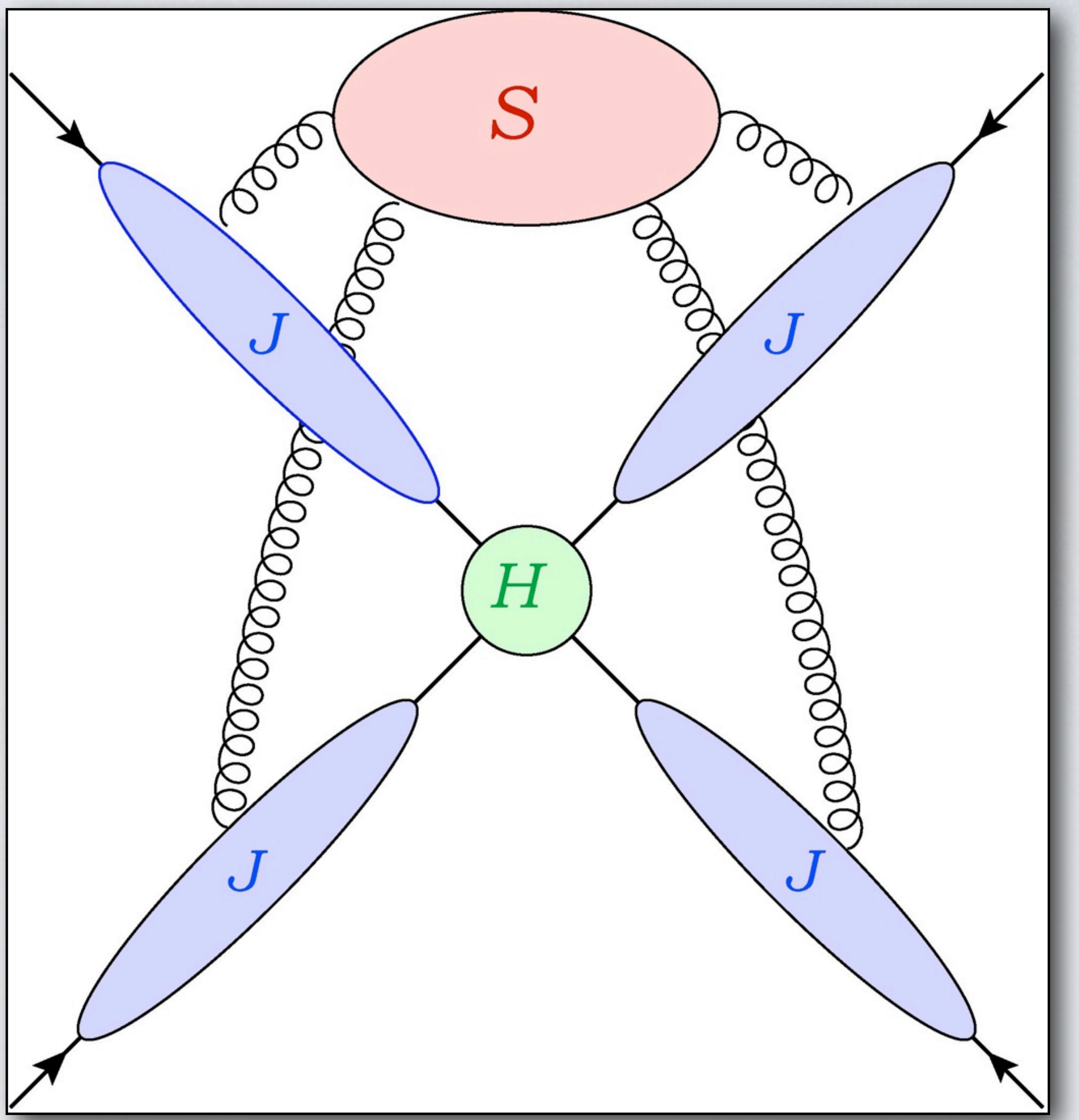
SOFT-COLLINEAR FACTORIZATION



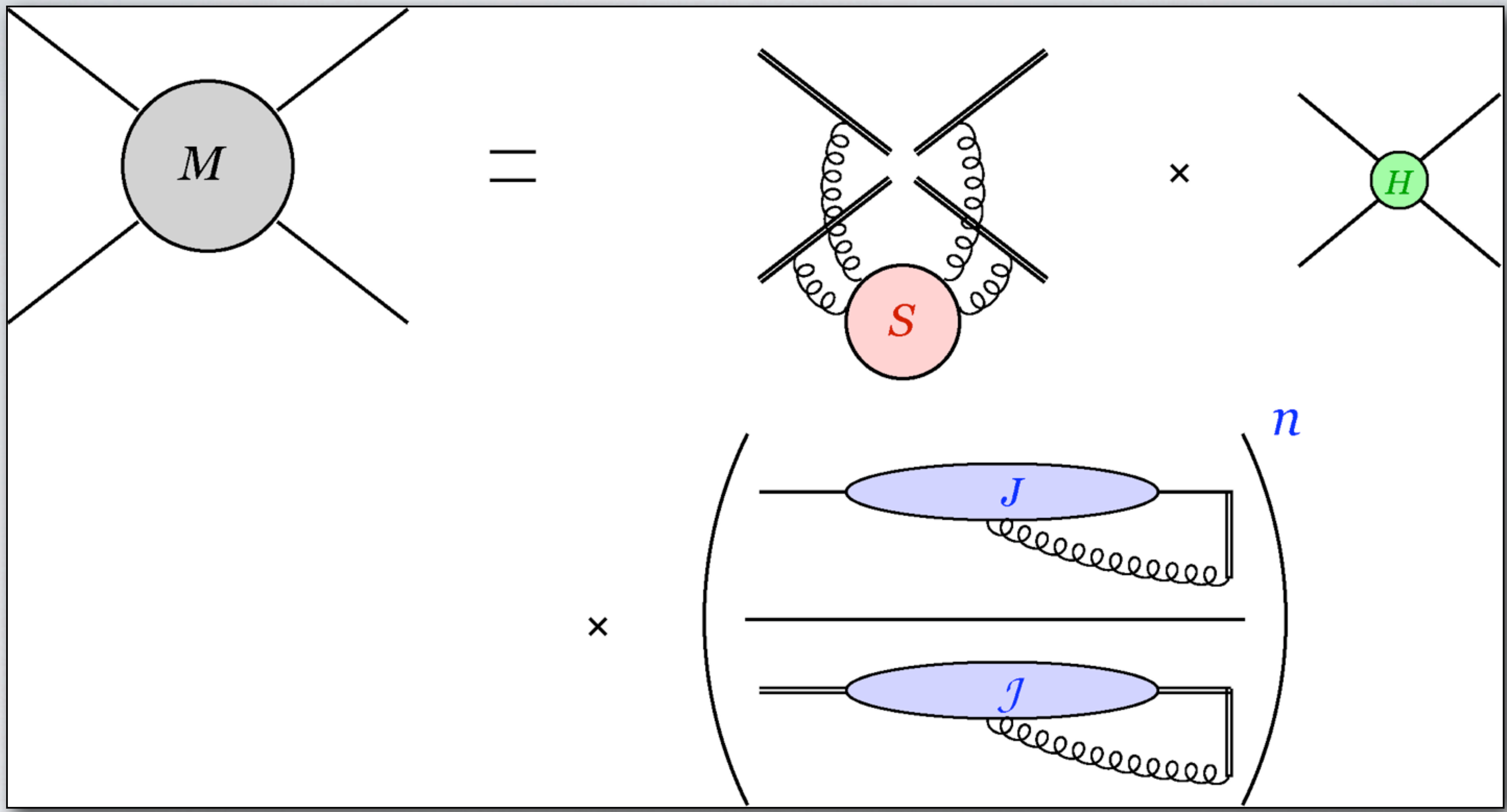
Soft-collinear factorization

- Divergences arise in scattering amplitudes from leading regions in loop momentum space.
- Power-counting arguments show that soft gluons decouple from the hard subgraph.
- Ward identities decouple soft gluons from jets and restrict color transfer to the hard part.
- Jet functions J represent color singlet evolution of external hard partons.
- The soft function S is a matrix mixing the available color representations.
- In the planar limit soft exchanges are confined to wedges: S is proportional to the identity.
- Beyond the planar limit S is determined by an anomalous dimension matrix Γ_S.
- The matrix Γ_S correlates color exchange with kinematic dependence.



Leading integration regions in loop momentum space for Sudakov factorization

Soft-collinear factorization: pictorial



A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes

Operator Definitions

The precise functional form of this graphical factorization is

$$\mathcal{M}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) \;=\; \mathcal{S}_{LK}\left(\beta_i\cdot\beta_j, \alpha_s(\mu^2), \epsilon\right)\, H_K\left(\frac{p_i\cdot p_j}{\mu^2}, \frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2)\right)$$
$$\times \prod_{i=1}^{n}\left[J_i\left(\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)\Big/\,\mathcal{J}_i\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)\right] \,,$$

We introduced factorization vectors $n_i^\mu \,, \quad n_i^2 \neq 0$ to define the jets,

$$J\left(\frac{(p\cdot n)^2}{n^2\mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) \;=\; \langle 0\,|\Phi_n(\infty,0)\,\psi(0)\,|p\rangle \,.$$

where Φ_n is the Wilson line operator along the direction n^μ ,

$$\Phi_n(\lambda_2,\lambda_1) = P\exp\left[\mathrm{i}g\int_{\lambda_1}^{\lambda_2} d\lambda\, n\cdot A(\lambda n)\right] \,.$$

The vectors n^μ :

- Ensure gauge invariance of the jets.
- Separate collinear gluons from wide-angle soft ones.
- Replace other hard partons with a collinear-safe absorber.

Soft anomalous dimensions

The soft function S obeys a matrix RG evolution equation

$$\mu \frac{d}{d\mu} \mathcal{S}_{IK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) = - \mathcal{S}_{IJ} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) \Gamma^{\mathcal{S}}_{JK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right)$$

- Γ^S is singular due to overlapping UV and collinear poles.

In dimensional regularization, using $\alpha_s(\mu^2 = 0, \varepsilon < 0) = 0$, one finds

$$\mathcal{S} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) = P \exp \left[- \frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \Gamma^{\mathcal{S}} \left(\beta_i \cdot \beta_j, \alpha_s(\xi^2, \epsilon), \epsilon \right) \right] .$$

Double poles cancel in the reduced soft function

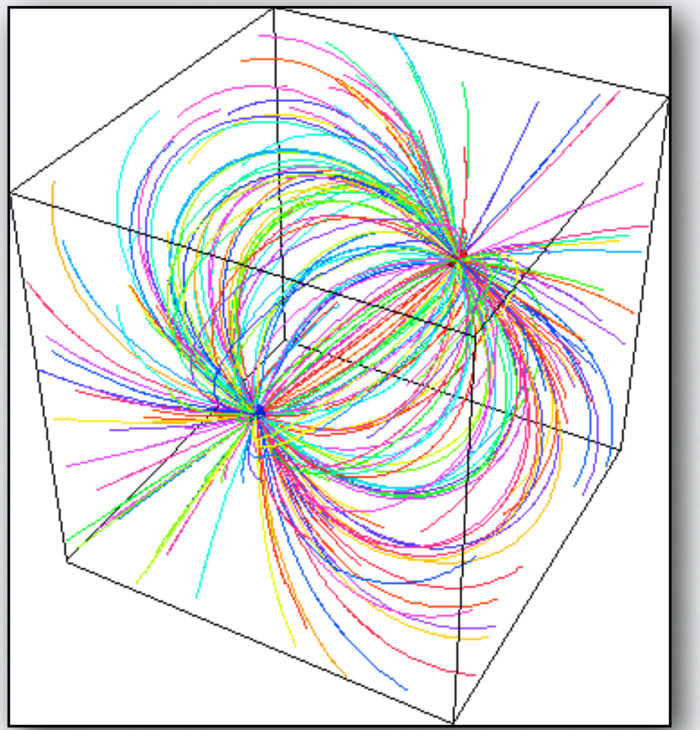
$$\overline{\mathcal{S}}_{LK} \left(\rho_{ij}, \alpha_s(\mu^2), \epsilon \right) = \frac{\mathcal{S}_{LK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right)}{\prod_{i=1}^{n} \mathcal{J}_i \left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right)}$$

- The matrix $\overline{\mathcal{S}}$ must depend on rescaling invariant variables

$$\rho_{ij} \equiv \frac{n_i^2 \, n_j^2 \, (\beta_i \cdot \beta_j)^2}{(\beta_i \cdot n_i)^2 (\beta_j \cdot n_j)^2} .$$

- The anomalous dimension $\Gamma^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s)$ for the evolution of $\overline{\mathcal{S}}$ is finite.

THE DIPOLE FORMULA



The Dipole Formula

For massless partons, the soft anomalous dimension matrix obeys a set of exact equations that correlate color exchange with kinematics.

The simplest solution to these equations is a sum over color dipoles (Becher, Neubert; Gardi, LM, 09). It gives an ansatz for the all-order singularity structure of all multiparton fixed-angle massless scattering amplitudes: the dipole formula.

- All soft and collinear singularities can be collected in a multiplicative operator Z

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = Z\left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon\right) \, \mathcal{H}\left(\frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon\right) ,$$

- Z contains both soft singularities from S, and collinear ones from the jet functions. It must satisfy its own matrix RG equation

$$\frac{d}{d \ln \mu} Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = - Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) .$$

The matrix Γ inherits the dipole structure from the soft matrix. It reads

$$\boxed{\Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = -\frac{1}{4} \widehat{\gamma}_K\left(\alpha_s(\mu^2)\right) \sum_{j \neq i} \ln\left(\frac{-2\, p_i \cdot p_j}{\mu^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j + \sum_{i=1}^{n} \gamma_{J_i}\left(\alpha_s(\mu^2)\right) .}$$

Note that all singularities are generated by integration over the scale of the coupling.

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Features of the dipole formula

- All known results for IR divergences of massless gauge theory amplitudes are recovered.
- The absence of multiparton correlations implies remarkable diagrammatic cancellations.
  - First observed at two loops by Aybat, Dixon and Sterman (2006).
- The color matrix structure is fixed at one loop: path-ordering is not needed.
- All divergences are determined by a handful of anomalous dimensions.
- The cusp anomalous dimension plays a very special role: a universal IR coupling.
  - All correlations between color and kinematics are governed by the cusp.
- A simple generalization of the planar solution: sum over all dipoles, not just color-adjacent ones
- Massive partons spoil the simplicity: non-vanishing tripole correlations already at two loops (Neubert et al., Sterman et al. 2010).

Can this be the definitive answer for IR divergences in massless non-abelian gauge theories?

- There are precisely two sources of possible corrections.
  - Quadrupole correlations may enter starting at three loops: they must be tightly constrained functions of conformal cross ratios of parton momenta.
  - The cusp anomalous dimension may violate Casimir scaling beyond three loops.

The dipole formula at high energy

- Introducing `Mandelstam' color operators, and using color and momentum conservation

$$\begin{aligned} \mathbf{T}_s &= \mathbf{T}_1 + \mathbf{T}_2 = -(\mathbf{T}_3 + \mathbf{T}_4)\,, \\ \mathbf{T}_t &= \mathbf{T}_1 + \mathbf{T}_3 = -(\mathbf{T}_2 + \mathbf{T}_4)\,, \\ \mathbf{T}_u &= \mathbf{T}_1 + \mathbf{T}_4 = -(\mathbf{T}_2 + \mathbf{T}_3) \end{aligned}$$

$$s + t + u = 0$$

$$\mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = \sum_{i=1}^{4} C_i$$

it is easy to see that the infrared dipole operator Z factorizes in the high-energy limit

$$Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) Z_{\mathbf{1}}\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$

- The operator Z_1 is s-independent and proportional to the unit matrix in color space.
- Color dependence and s dependence are collected in the factor

$$\widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K\Big(\alpha_s(\mu^2), \epsilon\Big)\left[\ln\left(\frac{s}{-t}\right)\mathbf{T}_t^2 + i\pi\,\mathbf{T}_s^2\right]\right\}\,,$$

where the coupling dependence is (once again!) completely determined by the cusp anomalous dimension and by the β function, through the function (Korchemsky 94-96)

$$K\Big(\alpha_s(\mu^2), \epsilon\Big) \equiv -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\widehat{\gamma}_K\left(\alpha_s(\lambda^2, \epsilon)\right)$$

- The simple structure of the high-energy operator governs Reggeization and its breaking.

The dipole formula at high energy

- Introducing \`Mandelstam' color operators, and using color and momentum conservation

$$\begin{aligned} \mathbf{T}_s &= \mathbf{T}_1 + \mathbf{T}_2 = -\left(\mathbf{T}_3 + \mathbf{T}_4\right) , \\ \mathbf{T}_t &= \mathbf{T}_1 + \mathbf{T}_3 = -\left(\mathbf{T}_2 + \mathbf{T}_4\right) , \\ \mathbf{T}_u &= \mathbf{T}_1 + \mathbf{T}_4 = -\left(\mathbf{T}_2 + \mathbf{T}_3\right) \end{aligned}$$

$$s + t + u = 0$$

$$\mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = \sum_{i=1}^{4} C_i$$

it is easy to see that the infrared dipole operator Z factorizes in the high-energy limit

$$Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) Z_{\mathbf{1}}\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$

- The operator Z_1 is s-independent and proportional to the unit matrix in color space.
- Color dependence and s dependence are collected in the factor

$$\widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K\Big(\alpha_s(\mu^2), \epsilon\Big)\left[\ln\left(\frac{s}{-t}\right)\mathbf{T}_t^2 + \mathrm{i}\pi\,\mathbf{T}_s^2\right]\right\} ,$$

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- The simple structure of the high-energy operator governs Reggeization and its breaking.

Reggeization of leading logarithms

- At leading logarithmic accuracy, the (imaginary) s-channel contribution can be dropped, and the dipole operator becomes diagonal in a t-channel basis.

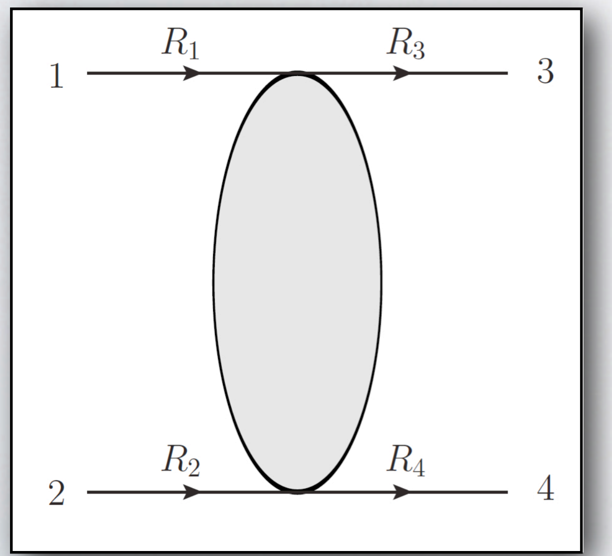
$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K\left(\alpha_s(\mu^2), \epsilon\right) \ln\left(\frac{s}{-t}\right) \mathbf{T}_t^2\right\} Z_\mathbf{1} \mathcal{H}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right)$$

- If, at LO and at leading power in t/s, the scattering is dominated by t-channel exchange, then the hard function is an eigenstate of the color operator $\mathbf{T}_t^2$

$$\mathbf{T}_t^2 \mathcal{H}^{gg\to gg} \xrightarrow{|t/s|\to 0} C_t \mathcal{H}_t^{gg\to gg}$$

- Leading-logarithmic Reggeization for arbitrary t-channel color representations follows

$$\mathcal{M}^{gg\to gg} = \left(\frac{s}{-t}\right)^{C_A K\left(\alpha_s(\mu^2),\epsilon\right)} Z_\mathbf{1} \mathcal{H}_t^{gg\to gg}$$



Scattering for generic color exchange

- The LL Regge trajectory is universal and obeys Casimir scaling.
- Scattering of arbitrary color representations can be analyzed
  Example: let 1 and 2 be antiquarks, 4 a gluon and 3 a sextet; use

$$\overline{\mathbf{3}} \otimes \mathbf{6} = \mathbf{3} \oplus \mathbf{15} \qquad \overline{\mathbf{3}} \otimes \mathbf{8}_a = \overline{\mathbf{3}} \oplus \mathbf{6} \oplus \overline{\mathbf{15}}$$

LL Reggeization of the $\mathbf{3}$ and $\mathbf{15}$ t-channel exchanges follows.

Beyond leading logarithms

- The high-energy infrared operator can be systematically expanded beyond LL, using the Baker-Campbell-Hausdorff formula. At NLL one finds a series of commutators

$$\begin{aligned}\widetilde{Z}\left(\frac{s}{t},\alpha_s,\epsilon\right)\Big|_{\rm NLL} \; = \; & \left(\frac{s}{-t}\right)^{K(\alpha_s,\epsilon)\,\mathbf{T}_t^2}\Bigg\{1+\mathrm{i}\,\pi K\left(\alpha_s,\epsilon\right)\bigg[\mathbf{T}_s^2-\frac{K\left(\alpha_s,\epsilon\right)}{2!}\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right] \\ & + \frac{K^2\left(\alpha_s,\epsilon\right)}{3!}\ln^2\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]+\ldots\bigg]\Bigg\}\end{aligned}$$

- The real part of the amplitude Reggeizes also at NLL for arbitrary t-channel exchanges.
- At NNLL Reggeization generically breaks down also for the real part of the amplitude.
  - At two loops, terms that are non-logarithmic and non-diagonal in a t-channel basis arise

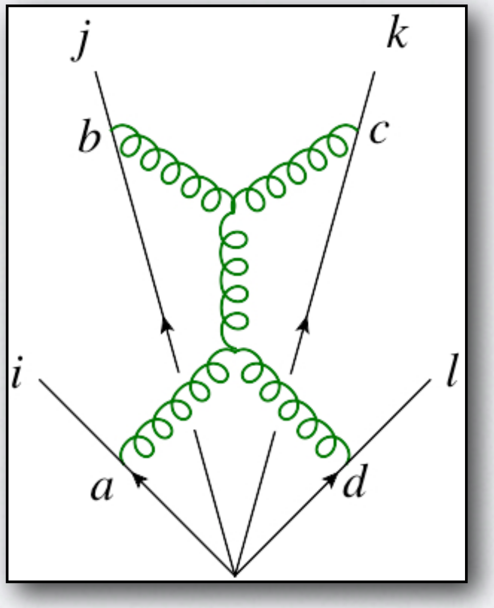
$$\mathcal{E}_0\left(\alpha_s,\epsilon\right) \equiv -\frac{1}{2}\pi^2\,K^2\left(\alpha_s,\epsilon\right)\,\left(\mathbf{T}_s^2\right)^2$$

  - At three loops, the first Reggeization-breaking logarithms of s/t arise, generated by

$$\mathcal{E}_1\left(\frac{s}{t},\alpha_s,\epsilon\right) \equiv -\frac{\pi^2}{3}\,K^3(\alpha_s,\epsilon)\,\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_s^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]$$

- **NOTE**
  - In the planar limit ($N_C \to \infty$) all commutators vanish and Reggeization holds also beyond NLL (as perhaps expected from string theory).
  - Possible quadrupole corrections to the dipole formula cannot come to the rescue.

BEYOND DIPOLES

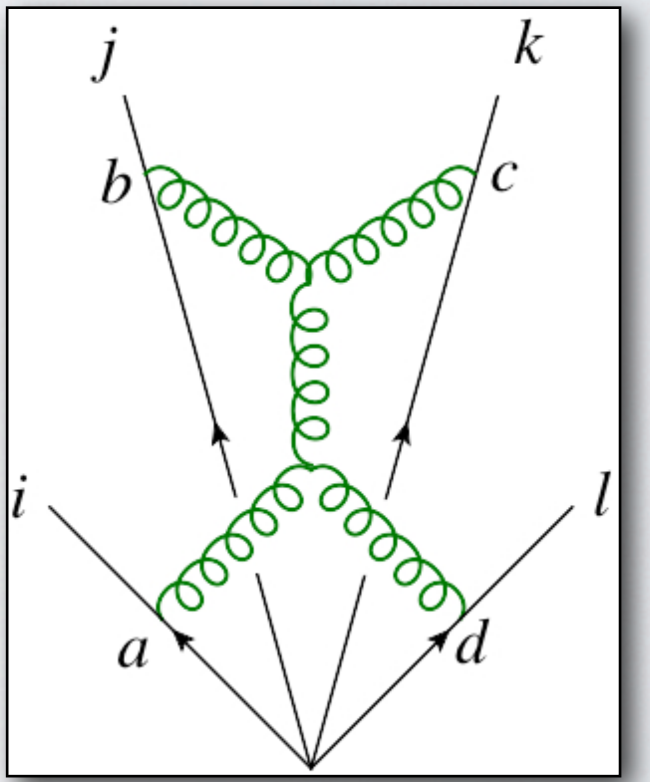


Conformal cross-ratios

The dipole formula is a solution to an exact inhomogeneous equation for Γ. It may be corrected by adding a solution to the corresponding homogeneous equation.

$$\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) + \Delta\left(\rho_{ijkl}, \alpha_s(\mu^2)\right) \,, \qquad \rho_{ijkl} = \frac{p_i \cdot p_j \, p_k \cdot p_l}{p_i \cdot p_k \, p_j \cdot p_l}$$

- The function Δ can only depend on conformal invariant cross ratio of parton momenta.
- The function Δ must correlate at least four partons: it an arise starting at three loops.
- The function Δ is tightly constrained:
  - It must vanish in all non-trivial collinear limits.
  - Its degree of transcendentality is bounded from above (and must be $\tau = 5$ at three loops).
  - It must be a Bose symmetric gluon correlator.
  - It must not generate super-leading Regge logarithms.
- No examples satisfying all constraints are known.
- Work is in progress to compute Δ directly, both via amplitudes and Wilson lines: a non-trivial, four-point, three-loop non-planar calculation. Symbol technology may help.



A three-loop diagram for Δ

Casimir Conspiracies

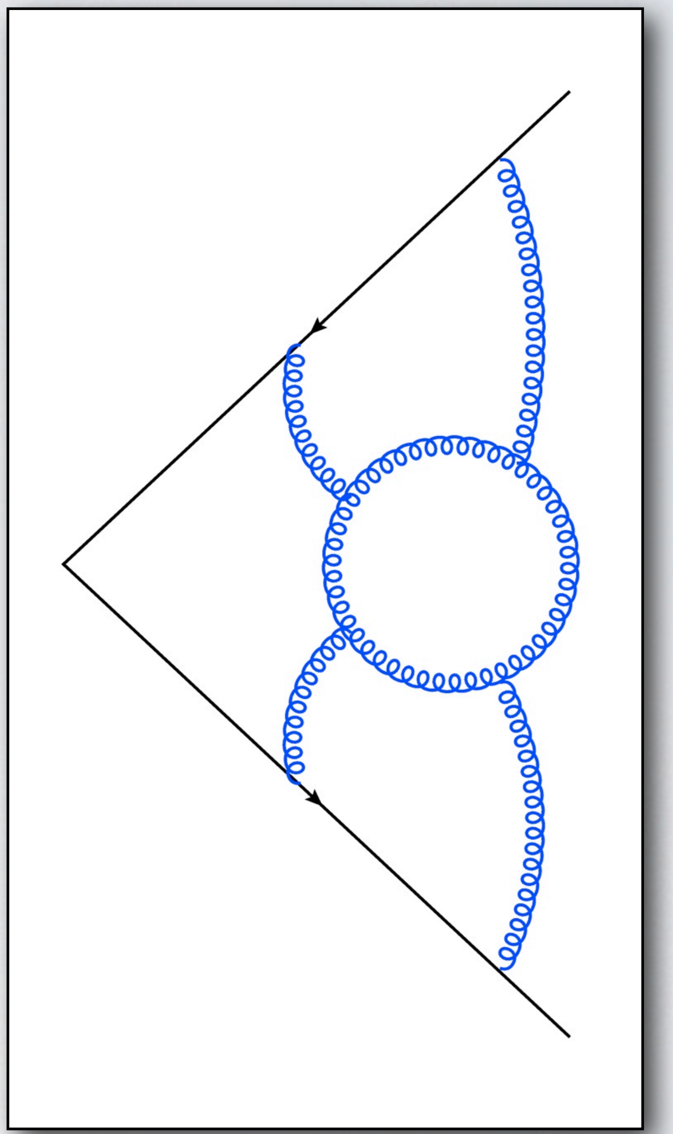
The dipole formula was derived assuming that the cusp anomalous dimension in a given color representation satisfies (quadratic) Casimir scaling

$$\gamma_K^{(i)}(\alpha_s) = C^{(i)}\,\widehat{\gamma}_K(\alpha_s) \qquad\qquad C^{(i)} = \mathbf{T}_i \cdot \mathbf{T}_i$$

- Casimir scaling holds to three loops but can be violated starting at four loops, when quartic Casimirs can appear

$$C_4^{(r)} = d_{abcd}\, Tr\left[T^a T^b T^c T^d\right]_r$$

- An indirect argument (Becher, Neubert 2009) shows that quartic Casimirs at four loops would be inconsistent with factorization and collinear constraints
- Strong coupling results in planar N=4 Super Yang-Mills theory (Armoni, Maldacena 2006-2007) suggest that Casimir scaling should not hold to all orders
- The cusp anomalous dimension is known exactly in the planar limit of N=4 SYM: not enough to disentangle C_4.
- A direct calculation is just outside feasibility with current technology



A possible contribution involving quartic Casimirs

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- A simple dipole formula may encode all infrared singularites for any massless gauge theory, a natural generalization of the planar limit.
- The study of possible corrections to the dipole formula is under way.
- The high-energy limit of the dipole formula provides insights into Reggeization and beyond, at least for divergent contributions to the amplitude.
- Regge factorization generically breaks down at NNLL, with computable corrections which may be related to Regge cuts in the angular momentum plane.

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- The high-energy limit of the dipole formula provides insights into Reggeization and beyond, at least for divergent contributions to the amplitude.
- Regge factorization generically breaks down at NNLL, with computable corrections which may be related to Regge cuts in the angular momentum plane.
- QCD is a theory of great beauty, and it's a privilege to study it.

  Thank you Pino for teaching us a lot about it!

Tanti Auguri Pino!