AN INTRODUCTION TO INTEGRABLE MODELS AND CONFORMAL FIELD THEORY*

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ABSTRACT

We review first the steps which lead to solutions of continuous integrable models in two dimensions. Next we discuss in more detail solvable models of statistical physics. The central role of the Yang Baxter equation is emphasized. The connection to Braids and Knots and certain algebras is mentioned.

The continuum limit of lattice models may yield a conformal invariant field theory. The Virasoro algebra is realized in a special manner. The central extension is related to the conformal anomaly. Unitary highest weight representations restrict both the coefficient in front of the anomaly and the conformal weights. The latter are directly related to the critical exponents. The operator product expansion as well as the classification of field operators is mentioned finally.

* *Lectures given at the*
2^{nd} *National Seminar of Theoretical Physics*
Parma, 1-12 September 1992.

1 Introduction

Various subjects became more and more interrelated recently:

1.1 Continuous Integrable Models

More than 150 years ago J. Scott-Russell observed solitons interacting with each other. 1898 Korteweg and de Vries wrote down an equation for $v(t,x)$:

$$v_t = 6vv_v - v_{xxx} \tag{1.1}$$

which was supposed to describe water waves in a channel. Fermi, Pasta and Ulam observed already by numerical experiments that certain modes of a dynamical system may dominate. But it was not before 1967 when Gardner, Green, Kruskal and Miura invented the inverse scattering method to *solve* the KdV equation. Especially the soliton solutions were obtained explicitly. Lax reformulated their scheme. Soon after these discoveries many hierarchies were obtained. Among them there are the Nonlinear-Schrödinger equation, the Sine-Gordon equation, the Toda lattices but more recently the Kadomtsev-Petviashvili type equations in three dimensions were also shown to be integrable. Especially during these more recent developments many more insights have been obtained. Special vertex operators "create" solitons. The Bose-Fermi correspondence allows to identify orbits of infinite-dimensional groups with nonlinear equations. An interesting connection to the Riemann-Hilbert problem resulted.

1.2 "Solvable" Models of Statistical Physics

The subject started when Lenz asked his student Ising to study the thermodynamics of a one-dimensional spin system. The teacher became disappointed since no nontrivial phase transition was found. Heisenberg wrote down a more general quantum spin model with Hamiltonian

$$H = -J\sum_{j=1}^{N} \vec{\sigma}_j\vec{\sigma}_{j+1}, \qquad \sigma_j^k = \mathbf{1}\otimes\ldots \underbrace{\sigma^k}_{j\text{-th place}} \otimes\mathbf{1}\ldots\otimes\mathbf{1}, \tag{1.2}$$

which is supposed to describe ferromagnetism for $J > 0$. Physically J results from the exchange integral, but even its sign is hard to evaluate. σ^k are Pauli matrices. If we allow only for spin up and down degrees of freedom the Ising model results. The great step forward was done by Onsager who calculated the free energy of the two-dimensional Ising model. If we allow for an anisotropic interaction in spin space of (1.2) we connect to vertex models. A special case of the six-vertex model yields a two-dimensional description of ice.

The more general eight-vertex models were studied by Baxter and others. Among the many other models we mention especially the Potts models, since the algebra obtained from operators entering the transfer matrix, the Temperley-Lieb algebra, is related to the Hecke algebra and finally to braids and knots.

1.3 Yang-Baxter Relation

Already in 1938 Bethe found an efficient method to obtain eigenfunctions of (1.2). The Bethe states are factorized in the same way as the S-matrices of certain two-dimensional models. From a quantized form of the inverse scattering method an algebraic scheme results. The algebras of operators entering this algebraic Bethe ansatz are closely connected to quantum groups.

It was realized that a large number of integrable models are obtained from solving consistency conditions, which are nowadays connected to the names of Yang and Baxter. The commutativity of the transfer matrices to different spectral values yields immediately an infinite number of conserved quantities and indicates the integrability of the model.

1.4 Braids and Knots

The algebraic steps by which we obtain the Yang-Baxter relations are closely connected to the braiding relations. It is therefore not surprising that invariant knot polynomials can be obtained from integrable lattice models. The constant entering the Temperley-Lieb algebra is related to the Jones index. The partition function of the Potts models connects to chromatic polynomials. The Beraha numbers play a special role. Jones polynomials can be obtained as correlation functions within a topological field theory. Jones obtained the special values for his index by studying embeddings of type II von Neumann algebras. Realizations of such algebras are given by conformal field theory.

1.5 Conformal Field Theory $d = 2$

The construction of correlation functions in the continuum limit consists of tedious steps and has been done explicitly only for the Ising model in two dimensions. Following standard wisdom we should follow renormalization group trajectories while taking the limit where the lattice constant goes to zero. The temperature should approach the critical one, and the correlation length should diverge. If we obtain a conformal invariant field theory, then the Fourier coefficients of the energy momentum tensor yield a representation of the Virasoro algebra. More precisely, a ray representation is obtained due to the occurrence of a central extension term. The coefficient in front of this conformal anomaly as well as the eigenvalue of one of the generators determine properties of the representation. The latter is called a conformal weight and enters the behaviour of correlation functions and determines critical exponents. For unitary representations both constants are severely restricted. This allows to classify possible two-dimensional conformal covariant models. Depending on the behaviour under conformal transformations we introduce the notion of primary and secondary fields. A further useful tool to study these models consists in using the operator product expansion. Conformal blocks form a closed algebra. Correlation functions are obtained as solutions of differential equations.

2 Integrable Continuum Models – Inverse Scattering Method – Solitons

2.1 A General Scheme for Solving (Linear) Problems

We are all familiar with integrable systems having a finite number of degrees of freedom. Most books and courses in mechanics deal with such systems; e.g. the oscillator or the Coulomb problem. In these examples we can find enough globally defined conserved quantities. More precisely consider a $2n$-dimensional phase space with local coordinates (q_i, p_i) and canonical two form $\omega = \sum_i dq_i \wedge dp_i$. Poisson brackets are given by $\{f, g\} = \omega(X_f, X_g)$ where X_f denotes the vector field generated by f and the equations of motion read $\dot{q} = \{q, H\}$, $\dot{p} = \{p, H\}$, with H being the Hamiltonian. A well-known theorem says:

Liouville-Arnold: If there exist n globally defined conserved quantities K_i in involution $\{K_i, K_j\} = 0$, then there exists a transformation to new variables φ_i, I_i (cyclic action-angle variables) such that the new time evolution is given by $\varphi_i(t) = \omega_i t + \varphi_i(0)$. I_i are constants and $\omega = \sum_i d\varphi_i \wedge dI_i$.

The scheme for solving the time evolution is therefore given by the diagram:

$$\begin{array}{ccccc} q_i(0), p_i(0) & \overset{\text{"direct" step}}{\longrightarrow} & \varphi_i(0), I_i(0) & & \\ ?\downarrow & & \downarrow & & \text{free time evolution} \\ q_i(t), p_i(t) & \overset{\text{"inverse" step}}{\longleftarrow} & \varphi_i(t), I_i(t) & & \end{array}$$

As a further example, where a similar scheme applies, we solve the linear diffusion equation $\Phi_t = \Phi_{xx}$ with infinite number of degrees of freedom. From the dispersion law $\omega = -ik^2$ we obtain the solution of the initial value problem through Fourier transformation (F.T.):

$$\Phi(t,x) = \int_{-\infty}^{\infty} dk e^{ikx} \tilde{\Phi}(t,x), \qquad \tilde{\Phi}(t,x) = e^{-k^2 t} \tilde{\Phi}(0,k). \tag{2.1}$$

A rapid spreading of wave packets occurs. The scheme for solving looks similar as before

$$\begin{array}{ccccc} \Phi(0,x) & \overset{\text{F.T., "direct" step}}{\longrightarrow} & \tilde{\Phi}(0,k) & & \\ ?\downarrow & & \downarrow & & \text{free time evolution} \\ \Phi(t,x) & \overset{(\text{F.T.})^{-1}\text{, "inverse" step}}{\longleftarrow} & \tilde{\Phi}(t,k) & & \end{array}$$

We note that there exists an infinite number of local constants of motion. First of all we may integrate the equation of motion and realize that the spatial integral of $\Phi(t,x)$

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx \Phi(t,x) = \int_{-\infty}^{\infty} dx \frac{\partial^2}{\partial x^2} \Phi(t,x) = 0 \tag{2.2}$$

is conserved. Moreover all the Fourier coefficients of $\Phi(t,x)$

$$e^{q^2 t} \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-iqx} \Phi(t,x) = \tilde{\Phi}(0,k) \tag{2.3}$$

provide us with constants of motion.

2.2 The Direct Step

The above mentioned scheme may be generalized to a large class of nonlinear evolution equations. As an example we shall describe the solution of the KdV equation (1.1). As for the direct step we consider the spectral problem for the one-dimensional Schrödinger operator

$$(-\frac{d^2}{dx^2} + v(x))\psi(x) = E\psi(x), \tag{2.4}$$

for real potential v, such that $(1+x^2)|v| \in L^1(\mathbf{R})$. We introduce Jost solutions f_1, f_2 with spatial behaviour at infinity

$$\lim_{x\to\infty} f_1(k,x)e^{-ikx} = 1, \qquad \lim_{x\to-\infty} e^{ikx} f_2(k,x) = 1. \tag{2.5}$$

Since $f_1(-k,x)$ solves equ. (2.4) too, there exists a relation between these three functions

$$f_2(k,x) = a(k)f_1(-k,x) + b(k)f_1(k,x). \tag{2.6}$$

The physical solution $\psi(k,x)$ should describe scattering from one side and is connected to Jost solutions by

$$\psi(k,x) = T(k)f_2(k,x) = R(k)f_1(k,x) + f_1(-k,x), \tag{2.7}$$

where we introduced reflection and transmission coefficients of the one-dimensional scattering problem. We compare (2.6) to (2.7) and relate $a = R/T$ and $b = 1/T$ to R and T.

From the Volterra integral equation obeyed by f_1 we deduce analyticity properties of that function and the fact that $|f_1(k,x) - \exp(ikx)| \in L^2((-\infty,\infty), dk)$. A theorem, similar to the Paley-Wiener one, due to Boas allows to deduce support properties of the Fourier tranform in k. This yields a representation for f_1 of the type

$$f_1(k,x) = e^{ikx} + \int_x^\infty dy K(x,y)e^{iky}. \tag{2.8}$$

Note that $K(x,y)$ is independent of the spectral parameter k.

2.3 The Inverse Step

The inverse problem for (2.4) consists in recovering the potential from scattering data. The two-dimensional drum problem became famous: M. Kac asked the question as to whether one can hear the shape of a drum. We consider the operator $(-\Delta)|_\Omega$ with Dirichlet boundary conditions on Ω. Does the knowledge of all frequencies determine the shape? We may expand the trace of the heat kernel operator

$$\mathrm{Tr}\ \exp(\beta(-\Delta)|_\Omega) \overset{\beta\searrow 0}{\cong} \frac{c_0|\Omega|}{\beta} + \frac{c_1 L}{\beta^{1/2}} + c_2(1-n) + \ldots \tag{2.9}$$

and observe that the area of the drum $|\Omega|$, the length of the boundary L and the number of holes n are determined together with the frequencies. Since $|\Omega| \leq L^2/4\pi$ is a standard isoperimetric inequality which becomes an equality iff the drum is a circle. We deduce uniqueness of the inverse problem for the circular case. The general problem is very complicated. A counterexample exists in 16 dimensions.

Coming back to the Schrödinger potential problem we first connect the kernel $K(x,y)$ to the potential. We apply the Schrödinger operator to $f_1 - e^{ikx}$ and Fourier transform in k:

$$(-\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+v(x))K(x,y)=-v(x)\delta(x-y). \tag{2.10}$$

Introduce light cone coordinates $\xi = y+x$, $\eta = y-x$, integrate (2.10) from $-\varepsilon$ to ε, observe that $K(x,y)$ vanishes for $x>y$ and deduce that

$$-2\frac{d}{dx}K(x,x)=v(x). \tag{2.11}$$

The connection of $K(x,y)$ to the scattering data $\{R(k),\varepsilon_\ell,c_\ell\}$, where ε_ℓ denote the energy eigenvalues and c_ℓ bound state wave function normalization constants, is more tricky. We rewrite (2.7) so that all quantities have a Fourier transform

$$(T(k)-1)f_2(k,x)=R(k)(f_1(k,x)-e^{ikx})+(f_1(-k,x)-e^{-ikx})-(f_2(k,x)-e^{-ikx})+R(k)e^{ikx}. \tag{2.12}$$

We obtain contributions to the F.T. of the l.h.s. from bound states

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}(T(k)-1)e^{iky}f_2(k,x)=-\sum_{\ell=1}^{N}c_\ell^2e^{-\kappa_\ell y}f_1(i\kappa_\ell,x), \tag{2.13}$$

where we used the proportionality of f_1 and f_2 at $k=i\kappa_\ell$. If we define a kernel G through the scattering data by

$$G(x,y)=\sum_{\ell=1}^{N}c_\ell^2e^{-\kappa_\ell(x+y)}+\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(x+y)}R(k), \tag{2.14}$$

the F.T. of (2.12) is rewritten and becomes the **G**elfand-**L**evitan-**M**archenko equation

$$K(x,y)+G(x,y)+\int_x^\infty dzK(x,z)G(z,y)=0,\qquad x<y. \tag{2.15}$$

We note the meaning of c_ℓ: If $f_1(i\kappa_\ell,x)\overset{x\to\infty}{\simeq}e^{-\kappa_\ell x}$ then $c_\ell^{-2}=\int dx f_1^2(i\kappa_\ell,x)$.

2.4 Solutions of the GLM Equation for $R\equiv 0$.

All one-dimensional totally reflectionless mirrors can be obtained from (2.15) easily. They form all soliton solutions of the KdV equation. For $R(k)\equiv 0$ (2.15) becomes a Fredholm equation with a separable kernel. From the ansatz

$$K(x,y)=-\sum_\ell c_\ell\psi_\ell(x)e^{-\kappa_\ell y}\Longrightarrow(1+C(x))\psi_\ell=e^{-\kappa_\ell x}c_\ell,$$

$$C_{\ell m}(x)=\frac{c_\ell c_m}{\kappa_\ell+\kappa_m}\exp(-(\kappa_\ell+\kappa_m)x) \tag{2.16}$$

we obtain

$$K(x,x) = \operatorname{Tr}(1+C)^{-1}\frac{d}{dx}C = \frac{d}{dx}\ln\det(1+C) \tag{2.17}$$

and all reflectionless potentials are given by

$$V(x) = -2\frac{d^2}{dx^2}\ln\det(1+C(x)). \tag{2.18}$$

Among them are the standard examples $-n(n+1)/\cosh^2 x$ with $n \in \mathbf{N}$.

2.5 Solving the KdV Equation

The scheme mentioned in (2.1) applies also here: We take $v(t,x)$ as a potential in a Schrödinger problem and transform to scattering data:

$$\begin{array}{ccccc} v(0,x) & \stackrel{\text{Schrödinger equ., "direct" step}}{\longrightarrow} & \{R_0(k), \varepsilon_\ell(0), c_\ell(0)\} & & \\ ?\downarrow & & \downarrow & & \text{"free" time evolution} \\ v(t,x) & \stackrel{\text{GLM equ., "inverse" step}}{\longleftarrow} & \{R_t(k), \varepsilon_\ell(t), c_\ell(t)\} & & \end{array}$$

In order to "solve" the initial value problem, we have to determine the time volution of the scattering data of the auxiliary problem assuming that $v(t,x)$ evolves according to the KdV equation. As a first example we evaluate the change of ε_ℓ:

$$\delta\varepsilon_\ell(t) = \int_{-\infty}^{\infty} dx\varphi_\ell^2(6vv_x - v_{xxx}) = 0. \tag{2.19}$$

By partial integrations we obtain the invariance of the spectrum of the Schrödinger operator under the KdV flow: $\varepsilon_\ell(t) = \varepsilon_\ell(0)$. This indicates the stability of solitons. In a similar way we obtain the time evolution of the other data

$$R_t(k) = R_0(k)e^{-8ik^3t}, \qquad T_t(k) = T_0(k), \qquad c_\ell(t) = c_\ell(0)e^{4\kappa_\ell^3 t}, \tag{2.20}$$

where $\kappa_\ell^2 = -\varepsilon_\ell$. (2.20) can be more easily obtained from the Lax pair which is behind the integrability. Note that we have found an infinite number of conserved quantities. Scattering data are the analog of action-angle variables. Similarly to the drum problem we may expand $\operatorname{Tr}\exp(-\beta H) \stackrel{\beta\searrow 0}{\simeq} \sum_n \beta^n I_n(v)/\beta^{1/2}$ and obtain a hierarchy of invariants.

2.6 Lax Pairs

We are familiar with invariant spectra if we think of unitary transformations of operators in Hilbert space. Assume that there exists a pair (L,B) such that $\dot{L} = [B,L]$ is equivalent to a nonlinear evolution equation. The pure point spectrum of $L(t)\psi(t) = \lambda(t)\psi(t)$, with $L^\dagger = L$, will then remain invariant. We differentiate, use the Heisenberg type equation and obtain

$$(L-\lambda)(\dot{\psi} - B\psi) = \dot{\lambda}\psi. \tag{2.21}$$

Take the scalar product of (2.21) with ψ and conclude that $\dot\lambda = 0$. As for the continuous spectrum we assume that there exists a unitary operator U such that

$$\frac{\partial U}{\partial t} = BU, \qquad B^\dagger = -B, \qquad \dot L = [B, L]. \tag{2.22}$$

Differentiating $\tilde L(t) = U^\dagger(t) L(t) U(t)$ and doing simple algebra proves that $\tilde L$ is time independent and equals $L(0)$. $L(t)$ is therefore obtained from $L(0)$ by a unitary transformation.

As a first example we evaluate the commutator between $L = -\frac{d^2}{dx^2} + v$ and $B = -4\frac{\partial^3}{\partial x^3} + 3v\frac{\partial}{\partial x} + 3\frac{\partial}{\partial x}v$ and obtain the KdV equation. From the operator B we get the k^3 dispersion law which determines the time evolution of $R_t(k)$ and $c_\ell(t)$.

This scheme works for hierarchies connected to the Toda lattice, the Sine-Gordon equation and the nonlinear Schrödinger equation

$$i\dot\psi = -\psi_{xx} + 2|\psi|^2\psi. \tag{2.23}$$

The appropriate Lax operator for (2.23) becomes a Dirac operator

$$L = \begin{pmatrix} \frac{1}{i}\frac{d}{dx} & \psi \\ \psi^* & -\frac{1}{i}\frac{d}{dx} \end{pmatrix}, \qquad L\chi = \lambda\chi, \tag{2.24}$$

where λ denotes the spectral parameter. Jost solutions with asymptotic behaviour

$$F(k,x) \stackrel{x\to\infty}{\longrightarrow} \begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{pmatrix}, \qquad G(k,x) \stackrel{x\to-\infty}{\longrightarrow} \begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{pmatrix} \tag{2.25}$$

are connected through the transition matrix $T(\lambda)$

$$F(\lambda, x) = T(\lambda) G(\lambda, x), \qquad T(\lambda) = \begin{pmatrix} a_\lambda & b_\lambda \\ b_\lambda^* & a_\lambda^* \end{pmatrix}. \tag{2.26}$$

2.7 Remarks

a) In the latter case a possible Poisson bracket is defined by

$$\{f(\psi,\psi^*), g(\psi,\psi^*)\} = i\int dx \left(\frac{\partial f}{\partial\psi}\frac{\partial g}{\partial\psi^*} - \frac{\partial f}{\partial\psi^*}\frac{\partial g}{\partial\psi}\right). \tag{2.27}$$

A direct calculation of the Poisson brackets between a_λ and b_λ is tedious [1]:

$$\begin{aligned} &\{a_\lambda, a_\mu\} = \{a_\lambda, a_\mu^*\} = \{b_\lambda, b_\mu\} = 0 \\ &\{a_\lambda, b_\mu\} = \frac{1}{\lambda - \mu + i\varepsilon} a_\lambda b_\mu, \qquad \{a_\lambda, b_\mu^*\} = -\frac{1}{\lambda - \mu + i\varepsilon} a_\lambda b_\mu^* \\ &\{b_\lambda, b_\mu^*\} = 2\pi i |a_\lambda|^2 \delta(\lambda - \mu). \end{aligned} \tag{2.28}$$

This indicates that the new variables play almost the role of action-angle variables. The calculation of (2.28) simplifies if we introduce the monodromy matrix $M(x, y|\lambda)$ as the solution of

$$\frac{\partial}{\partial x} M(x, y|\lambda) = \ell(x, \lambda) M(x, y|\lambda), \qquad \ell(x, \lambda) = i \begin{pmatrix} -\lambda & \psi \\ -\psi^* & \lambda \end{pmatrix} \tag{2.29}$$

with $M(x, x|\lambda) = 1$. The calculation of the Poisson brackets of elements of $\ell(x, \lambda)$ with elements of $\lambda(y, \mu)$ is simple. They imply the relation

$$\{M(x, y|\lambda) \overset{\otimes}{,} M(x, y|\mu)\} = [r(\lambda - \mu), M(x, y|\lambda) \otimes M(x, y|\mu)],$$

$$r(\lambda) = -\frac{1}{\lambda} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{2.30}$$

On the l.h.s. the bracket between elements of the tensor product is meant. Carefully taking the limits which define $T(\lambda)$ finally yields (2.28). (2.30) follows from the relation which yields the Yang-Baxter algebra. $r(\lambda)$ solves the classical Yang-Baxter relation (see next chapter).

b) More recently integrable models in $2+1$-dimensions have been found (Kadomtsev-Petviashvili hierarchies). For this generalization the Bose-Fermion correspondence plays a crucial role. It helps to relate subspaces as well as operators of the fermionic Fock space to subspaces and operators in the bosonic one.

If one studies the orbit of the vacuum vector under the group GL_∞ on the fermionic side it turns out that the appropriate subspaces on the bosonic side are characterized by nonlinear integrable models. Solitons turn out to be created by a particular vertex operator $\Gamma(u, v)$ depending on parameters u, v. The n-soliton solutions are built up by applying the operator

$$(1 + a_1 \Gamma(u_1, v_1)) \ldots (1 + a_n \Gamma(u_n, v_n))\Omega \tag{2.31}$$

to the vacuum.

3 Integrable Lattice Systems

3.1 Introduction

Many phase transitions occur in nature. The ferromagnetic one may serve as an example. Fe, Co, Ni and certain alloys show a spontaneous magnetization M_s below the Curie temperature T_c. Let $M(T, h)$ be the magnetization as a function of temperature T and magnetic field h. $M_s(T) = \lim_{h \searrow 0} M(T, h)$. M_s serves as an order parameter distinguishing between a disordered phase for $T \geq T_c$ and ordered phases for $T < T_c$. It vanishes for $T \geq T_c$.

In statistical mechanics we may describe properties within the canonical ensemble. We introduce a density matrix $\rho(q, p) = e^{-\beta H(q,p)}/Z$ for inverse temperature $\beta = 1/T$ and the partition function $Z = \text{ Tr } e^{-\beta H} = e^{-\beta F}$. F denotes the free energy of the system. Expectation values of observables are defined by $\langle A \rangle = \text{ Tr } A\rho/Z$. Differentiating Z with respect to β yields $\langle H \rangle = U = F + TS$, $S = -\partial F/\partial T$ and standard thermodynamic relations result. Classically Tr means integration over phase space. Quantum mechanically ρ becomes a positive trace class operator in Hilbert space.

The simplest model to describe ferromagnetism is the Ising model. Lenz asked his student Ising in 1925 to study a system supposed to describe phase transitions. Start from a hypercubical lattice and assign to each lattice point i a spin variable $s_i \in \{1, -1\}$ with possible values ± 1. The set $\{s_i\}_{i=1}^N$ form a spin field configuration. A model is defined through the definition of the energy of the configuration. The "exchange" interaction term suggests to take

$$H_N(\{s_i\}) = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i, \tag{3.1}$$

where $\langle ij \rangle$ denotes nearest neighbours and h the magnetic field. The dipol-dipol interaction would give a term of the form $s_i s_j$ too, but it is much too weak. The partition function for N spins is given by summing up Boltzmann factors over all configurations

$$Z_N = \sum_{\{s_i\}} e^{-\beta H_N(\{s_i\})} = e^{-\beta F_N}, \qquad \frac{F_N}{N} = -\frac{1}{\beta} \frac{\ln Z_N}{N} \stackrel{N \to \infty}{\longrightarrow} f(\beta, h) \tag{3.2}$$

and the mean free energy $f(\beta, h)$ is obtained in the thermodynamic limit. Phase transitions show up through points of nonanalyticity of f.

As the simplest example we may solve the one-dimensional Ising model with the help of the transfer matrix T. We rewrite Z_N as

$$Z_N = \sum_{s_1, \ldots, s_N} T_{s_1, s_2} \ldots T_{s_N s_1}, \quad T = V^{1/2} W V^{1/2}, \quad V_{s_1' s_1} = \delta_{s_1' s_1} e^{h s_1/2}, \quad W_{s_1 s_2} = e^{J s_1 s_2}. \tag{3.3}$$

If we denote the eigenvalues of T by Λ_i, $\Lambda_1 \geq \Lambda_2$, we obtain

$$f(\beta, h) = \lim_{N \to \infty} \frac{F_N}{N} = -\frac{1}{\beta} \ln \Lambda_1 \tag{3.4}$$

that the largest eigenvalue of the transfer matrix determines the thermodynamic properties. For $d = 1$ no nontrivial phase transition occurs ($T_c = 0$). Even a simple energy-entropy argument shows this. Configurations where all spins are aligned become unstable even for very low temperature. Since the free energy $F = U - TS$ should be minimal $\delta F = \delta U - T\delta S$ should be positive. But changing a part of the chain of N spins leads to δU independent of N while δS becomes proportional to $\ln N$. In higher dimensions we might switch inlands and thus δU depends on the length of the contour. Estimation of the probability of occurrence of such Peierls contours allows to conclude that a nontrivial phase transition occurs for $d \geq 2$. In case there is an internal symmetry (like in (1.2)), Bloch walls show up and the phase transition occurs only in dimensions $d \geq 3$. If we allow the classical spins to vary continuously on a circle we obtain the plane rotator for which vortices configurations occur. We remark that properties of these systems are determined by the occurrence of topological configurations like kinks, solitons, vortices (monopoles and instantons).

Polyacetylen is an example of a system for which solitons occur. It forms long $(\mathrm{CH})_x$-chains whose trans-form is stable. Denote by u_n deviations of the n-th (CH) molecule from its equilibrium position. Besides lattice vibrations there is hopping of electrons along the chain. A simple Yukawa type interaction yields the Su-Schrieffer-Heeger Hamiltonian

$$H_N = \sum_{n=1}^{N} (c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1})(t + \alpha(u_{n+1} - u_n)) + \frac{\omega^2}{2} \sum_{n=1}^{N} (u_{n+1} - u_n)^2. \tag{3.5}$$

If one takes $u_n = (-)^n u$ with constant u and evaluates the fermionic ground state energy $E_N^F(u)$ one obtains a double well potential indicating a phase transition and spontaneous symmetry breaking. This Peierls instability leads to a gap and soliton sectors occur. Kink solutions interpolate between the degenerate minima of $E_N^F(u)$. Charged solitons may be responsible for the enhancement in conductivity under doping.

3.2 Ising and Potts Models

For the one-dimensional Ising model we introduced matrices $V = e^{h\sigma^3}$ describing the interaction at a point and $W = e^J + \sigma^1 e^{-J}$ describing the interaction between points. For $d = 2$ and a $M \times N$ lattice we write

$$Z_{M,N} = \sum_{\{s_{m,n}\}} \exp[\frac{K_h}{2} \sum_{m,n} s_{m,n} s_{m,n+1} + \frac{K_v}{2} \sum_{m,n} s_{m,n} s_{m+1,n}] = \sum_{S_1', S_1 \dots S_M} V_{S_1' S_1} W_{S_1 S_2'} V_{S_2' S_2} \dots \tag{3.6}$$

where $S_m = \{s_{m_1}, \dots s_{m_N}\}$. V is again diagonal, why W is not. They can be represented by the $2^N \times 2^N$ matrices $\sigma_n^j = \mathbf{1} \otimes \dots \otimes \sigma^j \otimes \mathbf{1} \dots \otimes \mathbf{1}$ where σ^j is put to the n-th factor. This yields

$$V = \exp[\frac{K_h}{2} \sum_n \sigma_n^3 \sigma_{n+1}^3], \qquad W = \prod_n (e^{K_v/2} + e^{-K_v/2} \sigma_n^1). \tag{3.7}$$

Diagonalization of $T = V^{1/2} W V^{1/2}$ is possible, but cumbersome. We may add a constant term to each $s_i \cdot s_j$ term of (3.6) and use the identity $(1 + s_i s_j)/2 = \delta_{s_i s_j}$ to rewrite

$$Z_{M,N} = const. \sum_{\{s_i\}} \exp[K_v \sum_{\langle i,j \rangle^v} \delta_{s_i s_j} + K_h \sum_{\langle i,j \rangle^h} \delta_{s_i s_j}] \tag{3.8}$$

where $\langle i, j \rangle^{h,v}$ means nearest neighbour pairs in the horizontal and vertical direction. If we allow $s_j \in \{1, \dots, Q\}$ to take values one up to Q we obtain the Potts models; $Q = 2$ gives Ising. V and W may be expressed in terms of $Q \times Q$ matrices $(E_{ij})_{\alpha\beta} = \delta_{\alpha i} \delta_{\beta j}$ which fulfill the algebra $E_{ij} E_{k\ell} = \delta_{jk} E_{i\ell}$. We define matrices e_i acting in $\bigotimes_{j=0}^{N-1} \mathbf{C}^Q$ with $1 \le i \le 2N - 1$ [2]

$$\begin{aligned} e_{2j} &= Q^{1/2} \mathbf{1} \otimes \dots \otimes \sum_{k=1}^{Q} \underbrace{E_{kk}}_{j\text{-th place}} \otimes E_{kk} \otimes \mathbf{1} \dots \\ e_{2j-1} &= Q^{-1/2} \mathbf{1} \otimes \dots \otimes \sum_{k,\ell=1}^{Q} \underbrace{E_{k\ell}}_{j\text{-th place}} \otimes \mathbf{1} \dots, \end{aligned} \tag{3.9}$$

and obtain

$$V = \prod_{j=1}^{N-1} (\mathbf{1} + \xi_h e_{2j}), \qquad W = \prod_{j=1}^{N} Q^{1/2} (\xi_v \mathbf{1} + e_{2j-1}) \tag{3.10}$$

with $\xi_{h,v} = Q^{-1/2} (e^{K_{h,v}} - 1)$, as a simple calculation shows.

The matrices e_j fulfill the Temperley-Lieb algebra

$$e_j^2 = \sqrt{Q}\, e_j, \qquad e_i e_j = e_j e_i \text{ for } |i-j| \geq 2, \qquad e_i e_{i\pm 1} e_i = e_i, \tag{3.11}$$

and the partition function is completely determined by these algebraic relations. We shall come back to (3.11) later on.

3.3 Vertex Model

There exists a class of "ice"-like models. Ice forms a lattice of H_2O molecules. A simplified $d = 2$ model is obtained by placing O-atoms at the lattice points and H-atoms in between. Assume two possible positions for the H-atoms and indicate them by an arrow: or . Surround each O-atom by exactly two H-atoms. This allows six possible vertices to which we assign weights $\omega_j = \exp(-\beta\varepsilon_j)$:

$\omega_1 = \omega_2 = a$ $\qquad$ $\omega_3 = \omega_4 = b$ $\qquad$ $\omega_5 = \omega_6 = c$

We assumed a symmetry under reflections and obtained three independent constants. The partition function is given by $Z = \sum_{\text{Conf.}} \prod_{n,m} \omega_j(n,m)$.

The model becomes critical for $|a^2 + b^2 - c^2| \leq 2|ab|$. Correlation functions decay then algebraically. If we add the vertices

and

we obtain the eight-vertex model.

For the 6-vertex model we define the matrix

$$L^{\alpha_1\alpha_2}_{\beta_1\beta_2'} = \begin{array}{c|c} & \beta_1' \\ \hline \alpha_1 & \qquad \alpha_2 \\ & \beta_1 \end{array} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \tag{3.11}$$

acting in $h \otimes \mathbf{C}^2$, where the auxiliary space $h = \mathbf{C}^2$ too. The monodromy matrix describes

$$M_N = \begin{array}{c|c|c} & \beta_1' & \beta_N' \\ \hline \alpha_1 & \alpha_2 \ldots\ldots & \qquad \alpha_1' \\ & \beta_1 & \beta_N \end{array} = L_N \ldots L_1 \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.12}$$

and acts in $h \otimes (\mathbf{C}^2)^N$. Finally the transfer matrix is defined by $\mathrm{tr}_h\, M_N = T_N$ and $Z = \mathrm{Tr}\, T_N^M$.

As a suitable parametrization we choose $a = \rho \sin(\gamma - \lambda)$, $b = \rho \sin\lambda$, $c = \rho\sin\gamma$, put $\rho = 1$ and denote λ as the spectral parameter.

3.4 Connection to Quantum Spin Models

Proposition: We introduce the Hamiltonian of the XXZ model

$$H_{XXZ} = \frac{1}{2}\sum_{j=1}^{N}(\sigma_j^1\sigma_{j+1}^1 + \sigma_j^2\sigma_{j+1}^2 + \Delta(\sigma_j^3\sigma_{j+1}^3 + 1)) \tag{3.13}$$

and claim that

$$H_{XXZ} = \sin\gamma \frac{d}{d\lambda}\ln T_N(\lambda)\Big|_{\lambda=0} \qquad \text{with } \Delta = -\cos\gamma. \tag{3.14}$$

Proof: Note that $L_n(0) = \sin\gamma P_{0n}$, where P_{0n} denotes the permutation operator between vector space h and the n-th $\mathbf{C}^2$. $\sin\gamma^{-N} T_N(0) = \mathrm{tr}_h P_{0N}\ldots P_{01} = \mathrm{tr}_h P_{12}P_{23}\ldots P_{N-1,N}P_{N0} = U$, where U denotes the shift operator. Let $(\partial L_n/\partial\lambda)|_{\lambda=0} \equiv \bar{L}_{0n}$. We calculate the logarithmic derivative of the transfer matrix at $\lambda = 0$ and obtain

$$\sin\gamma\frac{d}{d\lambda}\ln T(\lambda)\Big|_{\lambda=0} = U^{-1}\sum_{n=1}^{N}\mathrm{tr}_h(P_{01}\ldots P_{0,n-1}\bar{L}_{0n}P_{0,n+1}\ldots P_{0N}) = \sum_{n=1}^{N}P_{n-1,n}\bar{L}_{n-1,n} =$$

$$= \sum_{n=1}^{N}\left(\begin{array}{cc|cc} -\cos\gamma & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\cos\gamma \end{array}\right)_{n-1,n}. \tag{3.15}$$

The indices indicate the vector space in which the matrices act.

Remark: L_n corresponds to a Lax operator; let $L_n\phi_n = \phi_{n+1}$. There exists an operator M_n with $\dot{\phi}_n = M_n\phi_n$ such that the integrability condition between the two equations $\dot{L}_n = M_{n+1}L_n - L_nM_n$ is equivalent to the equation of motion for spins $\dot{\vec{\sigma}}_n = i[H_{XXZ}, \vec{\sigma}_n]$ in the XXZ model.

3.5 Integrability of the Lattice Model

We take γ fixed and consider $L_n(\lambda)$ as a function of λ. We take the tensor product of two subsidiary spaces and consider $L_n(\lambda)\otimes L_n(\mu)$ acting in $h\otimes h\otimes(\mathbf{C}^2)^N$. It is remarkable that there exists an operator $R \in \mathrm{End}(h\otimes h)$ such that the Yang-Baxter relation [2]

$$R(\lambda-\mu)L_n(\lambda)\otimes L_n(\mu) = L_n(\mu)\otimes L_n(\lambda)R(\lambda-\mu) \tag{3.16}$$

holds. A possible solution for the quantum R matrix is given by $R(\lambda) = PL(\lambda)$ where P denotes again the permutation operator. If we put indices one and two for the first two vector spaces h and index three for remaining space we may rewrite (3.16) as

$$R_{12}(\lambda-\mu)P_{13}R_{13}(\lambda)P_{23}R_{23}(\mu) = P_{13}R_{13}(\mu)P_{23}R_{23}(\lambda)R_{12}(\lambda-\mu)$$

or

$$R_{23}(\lambda-\mu)R_{12}(\lambda)R_{23}(\mu) = R_{12}(\mu)R_{23}(\lambda)R_{12}(\lambda-\mu). \tag{3.17}$$

There are various other formulations. The S-matrix factorization conditions are identical to (3.17). The quantum space need not be identical to the subsidiary space. (3.16) may hold nevertheless.

Since L_n matrices to different indices commute it follows from (3.16) that

$$R(\lambda-\mu)M(\lambda)\otimes M(\mu)=M(\mu)\otimes M(\lambda)R(\lambda-\mu) \tag{3.18}$$

and by taking the trace in $\mathbf{C}^4$ we conclude that

$$[T(\lambda),T(\mu)]=0. \tag{3.19}$$

We have therefore obtained an infinite number of conserved quantities, one signal of integrability. From the Yang-Baxter relation (3.18) we deduce "commutation" relations among the elements A, B, C, D of the monodromy matrix. They are used for the algebraic Bethe ansatz.

(3.17) can be considered as a sufficient condition such that associativity of the tensor product holds. We consider operators L^1, L^2 and L^3 acting in $h\otimes h\otimes h\otimes(\mathbf{C}^2)^N$. We transform from $L^1L^2L^3$ to $L^2L^1L^3$ with the help of R_{12}, apply next R_{23} and finally R_{12} again. We obtain $L^3L^2L^1$. The same result can be obtained by first transforming with R_{23}, then using R_{12} and finally R_{23} again. Consistency is obtained from (3.17).

3.6 Bethe States

Bethe obtained already in 1931 exact eigenstates for the one-dimensional Heisenberg chain (1.2) assuming periodic boundary conditions. The principal idea is simple. We start from a reference vector, e.g. $|\Omega\rangle=|\uparrow\ldots\uparrow\rangle$ where all spins are up. The total spin $\vec{S}=\sum_{n=1}^N\vec{\sigma}_n/2$ is obviously conserved. $|\Omega\rangle$ is a first eigenstate of H. A new one can be obtained by taking superpositions of the form $\sum_{n=1}^N e^{ik_n}\sigma_n^-|\Omega\rangle\equiv|k\rangle$. These eigenstates form spin waves (or magnons) to wave vectors k and are quasiparticle excitations. We next may try to flip two spins. Since $(\sigma_n^-)^2=0$ two magnons repel each other. This leads to a phase shift $\Theta_{k_1k_2}$ and an interaction between spin waves. The Bethe ansatz

$$|k_1,k_2\rangle=\sum_{n_1<n_2}\psi_{n_1,n_2}^{k_1,k_2}\sigma_{n_1}^-\sigma_{n_2}^-|\Omega\rangle \tag{3.20}$$

$$\psi_{n_1,n_2}^{k_1,k_2}=\exp[\frac{i}{2}\Theta_{k_1k_2}]\exp[i(k_1n_1+k_2n_2)]+\exp[-\frac{i}{2}\Theta_{k_1,k_2}]\exp[i(k_2n_1+k_1n_2)]$$

leads to an eigenstate of H iff $\Theta_{k_1k_2}$ solves

$$2\cot\frac{\Theta_{k_1k_2}}{2}=\cot\frac{k_1}{2}-\cot\frac{k_2}{2}. \tag{3.21}$$

Note that the effective repulsion is built in since $\exp(i\Theta_{kk})=-1$ and $\psi_{n_1,n_2}^{kk}=0$. For M interacting magnons the ansatz

$$|k_1,\ldots,k_M\rangle=\sum_{n_1<n_2<\ldots<n_M}\psi_{\underline{n}}^{\underline{k}}\sigma_{n_1}^-\ldots\sigma_{n_M}^-|\Omega\rangle \tag{3.22}$$

$$\psi_{\underline{n}}^{\underline{k}} = \sum_{\text{Perm}\,\pi} \exp[i\sum_{\alpha} k_{\pi_\alpha} n_\alpha + \frac{i}{2}\sum_{\alpha<\beta}\Theta_{k_{\pi_\alpha}k_{\pi_\beta}}]$$

leads to eigensolutions iff Θ and k solve the Bethe equations

$$2\cot\frac{\Theta_{k_\alpha k_\beta}}{2} = \cot\frac{k_\alpha}{2} - \cot\frac{k_\beta}{2}$$

and

$$Nk_\alpha + \sum_{\beta\neq\alpha}\Theta_{k_\alpha k_\beta} = 2\pi\lambda_\alpha, \qquad \lambda_\alpha = 0,\dots,N-1. \tag{3.23}$$

Lieb and Liniger solved the one-dimensional Bose gas using the Bethe ansatz. Let $\psi(x)$ denote a Bose field with $[\psi(x),\psi^\dagger(y)] = \delta(x-y)$. The Hamiltonian operator $H = \int_0^L dx(\psi_x^\dagger\psi_x + \lambda\psi^\dagger\psi^\dagger\psi\psi)$ leads on the N-particle sector to the many body interaction

$$(-\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j} + 2\lambda\sum_{i<j}\delta(x_i - x_j))\psi_N = E\psi_N. \tag{3.24}$$

There exist again Bethe states of the form

$$\psi_N = \sum_{\text{Perm}\,\pi}\exp[i\sum_{j=1}^{N}k_{\pi_j}x_j + \frac{i}{2}\sum_{i<j}\Theta_{k_{\pi_i}k_{\pi_j}}], \qquad x_1 < x_2 < \dots x_n, \tag{3.25}$$

$$\exp[i\Theta_{k_ik_j}] = \frac{k_i - k_j + i\lambda}{k_i - k_j - i\lambda}$$

where periodic boundary conditions require that

$$k_\ell L = 2\pi I_\ell - 2\sum_{j=1}^{N}\arctan(\frac{k_\ell - k_j}{\lambda}). \tag{3.26}$$

In the thermodynamic limit $L\to\infty$, $N\to\infty$, with $\rho = N/L$ fixed, one introduces a distribution function for the roots of equ. (3.26). This leads to an integral equation.

3.7 Algebraic Bethe Ansatz

The Yang-Baxter relations allow to establish an algebraic procedure such that the Bethe states are built up. Among the 16 relations which determine the algebra of operators A, B, C, D we quote

$$\begin{aligned} A_\lambda B_\mu &= \frac{a(\mu-\lambda)}{b(\mu-\lambda)}B_\mu A_\lambda - \frac{c(\mu-\lambda)}{b(\mu-\lambda)}B_\lambda A_\mu \\ D_\lambda B_\mu &= \frac{a(\lambda-\mu)}{b(\lambda-\mu)}B_\mu D_\lambda - \frac{c(\lambda-\mu)}{b(\lambda-\mu)}B_\lambda D_\mu. \end{aligned} \tag{3.27}$$

We intend to solve the family of eigenvalue problems $T(\lambda)\psi = (A_\lambda + D_\lambda)\psi = E\psi$, since the free energy is determined by the largest eigenvalue. We observe that the reference vector $|\Omega\rangle_N = \otimes_{j=1}^N|\uparrow\rangle_j$ acts like a "pseudo"vacuum since

$$L_n|\uparrow\rangle_n = \left(\begin{array}{c|c} a_\lambda|\uparrow\rangle_n & c_\lambda|\downarrow\rangle_n \\ \hline 0 & b_\lambda|\uparrow\rangle_n \end{array}\right), \tag{3.28}$$

$$A_\lambda|\Omega\rangle_N = a_\lambda^N|\Omega\rangle_N, \qquad D_\lambda|\Omega\rangle_N = b_\lambda^N|\Omega\rangle_N, \qquad C_\lambda|\Omega\rangle_N = 0.$$

$|\Omega\rangle_N$ is eigenvector of $T(\lambda)$ to eigenvalue $a_\lambda^N + b_\lambda^N$. We may try to take B_λ as a creation operator of quasiparticles. If we consider only the first terms in equs. (3.27) the state $\psi_N = B_{\lambda_1}\ldots B_{\lambda_N}|\Omega\rangle_N$ would be an eigenstate of $T(\lambda)$ to eigenvalue

$$a_\lambda^N \prod_{j=1}^N \frac{a(\lambda_j-\lambda)}{b(\lambda_j-\lambda)} + b_\lambda^N \prod_{j=1}^N \frac{a(\lambda-\lambda_j)}{b(\lambda-\lambda_j)}.$$

But this holds true if we require vanishing of all other terms obtained from the last terms of (3.27). These Bethe equations become

$$\left(\frac{a(\lambda_\ell)}{b(\lambda_\ell)}\right)^N = \prod_{j\neq\ell}^n \frac{a(\lambda_\ell-\lambda_j)b(\lambda_j-\lambda_\ell)}{b(\lambda_\ell-\lambda_j)a(\lambda_j-\lambda_\ell)}, \tag{3.29}$$

if $c(\lambda)b(-\lambda) = -c(-\lambda)b(\lambda)$, as it is true in most examples. For the 6-vertex model (3.29) goes into

$$e^{ip_\ell N} \equiv \left(\frac{\sin\lambda_\ell}{\sin(\gamma-\lambda_\ell)}\right)^N = \prod_{j\neq\ell}^N \frac{\sin(\lambda_j-\lambda_\ell-\gamma)}{\sin(\lambda_j-\lambda_\ell+\gamma)} \equiv \prod_{j\neq\ell}^n e^{i\phi(\lambda_j-\lambda_\ell)} \tag{3.30}$$

or identically

$$Np_\ell + \sum_{j\neq\ell}^N \phi(\lambda_j-\lambda_\ell) = 2\pi I_\ell, \qquad I_\ell \in \mathbf{Z} \tag{3.31}$$

an equation similar to (3.26). We note that $b(\lambda_\ell-\lambda)$ vanishes at $\lambda=\lambda_\ell$, but the transfer matrix should be regular. Vanishing of the residuum at these poles explains (3.29) too.

3.8 Knots, Links and Braids

A knot is a closed line embedded into $\mathbf{R}^3$ without crossings. A link is a set of knoted knots. A complete classification of links is complicated. In order to define braids we choose n points in $\mathbf{R}^2$ and consider mappings γ_i from $[0,1] \to \mathbf{R}^2 \times [0,1] \subset \mathbf{R}^3$, $i = 1,\ldots,n$ with the properties that $\dot\gamma_i^3(t) > 0$ for $t \in [0,1]$, $\gamma_i(t) \neq \gamma_j(s)$ for $i \neq j$ and all $s,t \in [0,1]$ and $\gamma_i(0) = x_i$, $\gamma_i(1) = x_{\pi(i)}$. We identify all mappings with the mentioned properties which can be obtained by continuous deformations. The resulting object forms a braid of n strings $\in B_n$.

Algebraically B_n consists of words generated from $\{e, b_i, b_i^{-1}|i=1,\ldots,n\}$. b_i exchanges the i-th and the $(i+1)$-st string, so that one lies above the other. b_i^{-1} puts them in the opposite way. There are three types of Reidemeister moves: $e = b_ib_i^{-1} = b_i^{-1}b_i$, $b_ib_j = b_jb_i$ for $|i-j| \geq 2$ and $b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}$. The last one is the essential braiding relation. It is obviously related to the algebraic procedure which led to the Yang-Baxter relations [3].

If we identify upper and lower endings of a braid we obtain a link. Many braids lead to the same link.

Markov Theorem: Braids which corresponds to the same link can be obtained from each other by successive applications of Markov moves of type I and II. If $A, B \in B_N$: $\mathcal{L}(AB) = \mathcal{L}(BA)$ and if $A \in B_n$, $b_n \in B_{n+1}$: $\mathcal{L}(Ab_n) = \mathcal{L}(A) = \mathcal{L}(Ab_n^{-1})$, $\mathcal{L}(\cdot)$ denotes an invariant function on links. The Burau representation of B_n is given by $n \times n$ matrices

$$b_j = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 1-t & t & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

whose (j,j) element is $1-t$. If we require that the eigenvalues of b_n are -1 and $t \in \mathbf{C}$, we restrict the algebra by imposing the condition $(g_i+1)(g_i-t) = 0$. Together with $g_i g_j = g_j g_i$, $|i-j| \geq 2$ and $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ the Hecke algebra is defined.

If we transform to e_i putting $g_i = (1+t)e_i - 1$, we obtain

$$(1+t)^2 e_i e_{i+1} e_i - t e_i = (1+t)^2 e_{i+1} e_i e_{i+1} - t e_{i+1}.$$

If we put both sides to zero the Temperley-Lieb algebra results.

We note finally that we may obtain link-invariants from solutions of the Yang-Baxter equation.

4 Conformal Field Theory

4.1 Introduction

Through the behaviour near the critical point we define critical exponents: $M_s \simeq |\tau|^\beta$, $\tau \equiv T - T_c$, specific heat $c_h \simeq |\tau|^{-\alpha}$ and susceptibility $\chi_T \simeq \tau^{-\gamma}$. The two point function of the Ising model decays away from T_c like an exponential (Ornstein-Zernike behaviour)

$$|\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle|_{|i-j| \to \infty} \simeq c \frac{\exp[-\frac{|i-j|}{\xi}]}{|i-j|^{d-2}}, \qquad T \neq T_c \tag{4.1}$$

while at $T = T_c$ we obtain a second order phase transition and the correlation length diverges (long range order): $\xi(T) \simeq |\tau|^{-\nu}$, in addition

$$|\langle s_i s_j \rangle^c|_{|i-j| \to \infty} \simeq \frac{const}{|i-j|^{d-2+\eta}}. \tag{4.2}$$

Renormalization group ideas have been used to get reliable answers for nontrivial exponents. Mean field methods underestimate fluctuations and $\alpha = 0$, $\beta = 1/2$, $\nu = 1/2$, $\eta = 0$ result. For the two-dimensional Ising model we get $\alpha = 0$, $\beta = 1/8$, $\gamma = 7/4$, $\nu = 1$, $\eta = 1/4$ [4].

Already in '65 Widom suggested a scaling behaviour close to the critical point for the free energy $\lambda f(\tau, h) \simeq f(\lambda^a\tau, \lambda^b h)$. a and b determine all critical exponents. Block spin methods led to a justification of this ansatz, which we illustrate in one dimension. We may integrate out each second spin and observe that

$$\sum_{s_2} \exp[J(s_1s_2-1)+\frac{h}{2}(s_1+2s_2+s_3)+J(s_2s_3-1)+c] = \exp[\bar{J}(s_1s_3-1)+\frac{\bar{h}}{2}(s_1+s_3)+\bar{c}]. \quad (4.3)$$

The unique fix point of the mapping $(J, h, c) \to (\bar{J}, \bar{h}, \bar{c})$ determines the critical point $\mu \equiv e^{-J} = e^{-\bar{J}} \equiv \bar{\mu} = 0$, $h = \bar{h} = 0$. The flow around this point $e^{-\bar{J}} = \bar{\mu} \simeq \sqrt{2}\mu$, $\bar{h} \simeq 2h$ determines the behaviour of thermodynamic functions $\ln \Lambda_1 \simeq \mu^2\sqrt{1 + h^2/\mu^2}$ which agrees with the renormalization group behaviour $f(\mu, h) \simeq \mu^2 f(1, h/\mu^2)$.

In the continuum limit $(a \to 0)$ the transfer matrix determines the Hamiltonian $\mathcal{H}$

$$T = \begin{pmatrix} e^h & e^{-2J} \\ e^{-2J} & e^{-h} \end{pmatrix} \simeq 1 + h\sigma^3 + \mu^2\sigma^1 = 1 + a\mathcal{H} \quad (4.4)$$

where $\mathcal{H} = \lambda\sigma^3 + \sigma^1$, $\mu^2 = a$, $h = \lambda\mu^2$. We note that the limit $a \to 0$ is connected to the limit $T \to T_c$. Keeping $\xi(a) \cdot a = const$ requires that $\xi(a = 0) = \infty$. According to renormalization group ideas we have to follow renormalization group trajectories for $a \to 0$. This has been achieved explicitly only for the $d = 2$ Ising model. A massless Majorana field theory is obtained in the limit. This model is not only scale invariant but invariant under the conformal group. η becomes determined by the anomalous scale dimension of a field $s(x)$. A fruitful hypothesis for $d = 2$ turns out to be the assumption that the continuum model is conformal covariant. A classification of possible critical exponents results. Operators fall into representations of the Virasoro algebra.

4.2 Conformal Invariance

We start with $\mathbf{R}^d$ and metric $g = g_{\mu\nu}dx^\mu \otimes dx^\nu$, $g_{\mu\nu} = \eta_{\mu\nu}$, $p + q = d$ and signature (p, q). A mapping from $\mathbf{R}^d \cup \{\infty\}$ to $\mathbf{R}^d \cup \{\infty\}$ such that $g_{\mu\nu} \to f(x)g_{\mu\nu}$ is called a conformal transformation.

Examples of conformal transformations are translations $x^\mu \to x^\mu + a^\mu$, $a^\mu \in \mathbf{R}^d$ and pseudorotations $x^\mu \to \Lambda^\mu_\nu x^\nu$, $\Lambda \in O(p, q)$. In both cases $f = 1$. In addition dilatations $x^\mu \to \lambda x^\mu$, $\lambda \in \mathbf{R}^+$ lead to $f(x) = \lambda^{-2}$. Finally there are special conformal transformations

$$x^\mu \xrightarrow{\text{refl.}} \frac{x^\mu}{x^2} \xrightarrow{\text{transl.}} \frac{x^\mu + b^\mu}{(x+b)^2} \xrightarrow{\text{refl.}} \frac{x^m u + b^\mu x^2}{(1 + 2(bx) + b^2x^2)}, \qquad b^\mu \in \mathbf{R}^d \quad (4.5)$$

which yield $f(x) = (1 + 2(bx) + b^2x^2)^2$. The $(d + 1)(d + 2)/2$ generators of these transformations form the Lie-algebra of $so(p + 1, q + 1)$. These are the only transformations which can be globally defined on $\mathbf{R}^d \cup \{\infty\}$. Calculation of the Jacobi determinant yields $|\partial x'/\partial x| = (1 + 2(bx) + b^2x^2)^{-d}$ for the special conformal translations. For N points we obtain $(N - 3)$ invariant quantities which are the anharmonic quotients, for example $|x_1 - x_3||x_2 - x_4|/|x_1 - x_2||x_3 - x_4|$.

The hypothesis of covariance under global conformal transformations asserts that there exists quasiprimary fields $\{A_\ell\}$ which transform under conformal transformations $x \to x'$ as

$$A_\ell(x) \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta_\ell/d} A_\ell(x'). \tag{4.6}$$

All other fields are linear combinations of quisiprimary fields A_ℓ and their derivations. Correlation functions transform covariantly, the vacuum is invariant. Δ_ℓ is called the anomalous dimension. Correlation functions are therefore restricted. For example, two- and three-point functions are given by

$$\langle A_1(x_1)A_2(x_2)\rangle = \frac{c_{12}}{x_{12}^{\Delta_1+\Delta_2}}, \qquad x_{12} = |x_1 - x_2|, \qquad c_{12} = 0 \text{ if } \Delta_1 \neq \Delta_2,$$

$$\langle A_1(x_1)A_2(x_2)A_3(x_3)\rangle = \frac{c_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{12}^{\Delta_1+\Delta_3-\Delta_2}}. \tag{4.7}$$

The four-point function depends on the anharmonic quotient.

4.3 Local Conformal Transformations $d = 2$

As we know from hydordynamics there exist conformal transformations in $\mathbf{R}^2$ for which certain singularities occur. Requiring that $g_{\mu\nu} \to h(x,y)g_{\mu\nu}$ implies that the mapping $(x,y) \to (x',y')$ fulfills the Cauchy-Riemann differential equations. In terms of $z = x+iy$, $\bar{z} = x - iy$, $z' = f(z)$ and $\bar{z}' = \bar{f}(\bar{z})$ have to be analytic resp. antianalytic: $\partial_{\bar{z}} z' = 0$, $\partial_z \bar{z}' = 0$. This group becomes infinite-dimensional. We remark that $(z,\bar{z})$ may be considered as a point in $\mathbf{C}^2$.

It is an easy exercise that the mapping $z \to z'$ is globally defined iff $f(z) = (az+b)/(cz+d)$. No essential singularities or branch point singularities may occur. $ad - bc$ has to be non-zero and is put to one. 6 real parameters occur. These transformations can be mapped by a homomorphism to $s\ell(2,\mathbf{C})$. The composition $(\tilde{f} \circ f)(z)$ corresponds to matrix multiplication. The six parameters correspond to translations:

$$z \to z + b \leftrightarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad b \in \mathbf{C},$$

to dilatations

$$z \to \lambda z \leftrightarrow \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}, \qquad \lambda \in \mathbf{R}^+,$$

to rotations

$$z \to e^{i\theta} z \leftrightarrow \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \qquad \theta \in \mathbf{R},$$

and to special conformal transformations

$$z \to z/(cz+1) \leftrightarrow \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \qquad c \in \mathbf{C}.$$

The infinitesimal transformations $z \to z' = z - \varepsilon_n z^{n+1}$ can be represented on functions by $F(z) \to F(z') \cong F(z) - \varepsilon_n z^{n+1} F'(z)$. The algebra of the generators $\ell_n = -z^{n+1}\partial_z$

becomes the Virasoro algebra $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$. This is the algebra of the diffeomorphism group of S^1 too. $\bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$ fulfills an isomorphic algebra. Everything becomes doubled.

The subalgebra $\{\ell_{-1}, \ell_0, \ell_1\}$ forms an $s\ell(2, \mathbf{C})$ and generates global conformal transformations. Assume that $|h, \bar{h}\rangle$ is an eigenvector of ℓ_0 and $\bar{\ell}_0$ to eigenvalues h and $\bar{h}$. h, $\bar{h}$ are called conformal weights. Since $i(\ell_0 - \bar{\ell}_0)$ generates rotations $|h - \bar{h}|$ may be called spin. Since $\ell_0 + \bar{\ell}_0$ generates dilatations $h + \bar{h}$ becomes the anomalous dimension.

4.4 Three Implications

Energy-Momentum Tensor: We start from the action functional $S = \int d^dx\mathcal{L}(\phi, \partial\phi)$ and vary ϕ by $\delta\phi = \varepsilon^\mu\partial_\mu\phi$. s is changed by $\delta S = \int d^dx\partial^\nu\varepsilon^\mu T_{\mu\nu}$, where $T_{\mu\nu}$ denotes the energy-momentum tensor $T_{\mu\nu} = -g_{\mu\nu}\mathcal{L} + \partial^\mu\phi(\partial\mathcal{L}/\partial\partial_\nu\phi)$. In two dimensions the traceless and symmetric tensor T becomes

$$T = \begin{pmatrix} H & -P \\ -P & H \end{pmatrix}.$$

Since according to Noether it is conserved $\partial^\mu T_{\mu\nu} = 0$, we obtain $(\partial_0 + \partial_x)(H + P) = 0$ and $(\partial_0 - \partial_x)(H - P) = 0$. In terms of $T_\pm = \frac{1}{2}(H \mp P)$ we get $\partial_+ T_- = \partial_- T_+ = 0$ where $\partial_\pm$ denote derivatives with respect to light cone coordinates. In the Euclidean formulation $t = -i\tau$, $x_+ = \bar{z} = x - i\tau$ etc. and the two components of the energy-momentum tensor $(T_{11} - iT_{12})/2 = T(z)$, $(T_{11} + iT_{12})/2 = \bar{T}(\bar{z})$ depend only on z resp. $\bar{z}$.

Ward Identity: The hypothesis that correlation functions transform covariantly under global conformal transformations $\delta_{\varepsilon(z)}\langle\phi_1(z_1, \bar{z}_1)\ldots\phi_N(z_N, \bar{z}_N)\rangle = 0$ yields

$$\begin{aligned} &\textstyle\sum_k \partial_{z_k}\langle\phi_1\ldots\phi_N\rangle = 0 \\ &\textstyle\sum_k (h_k + z_k\partial_{z_k})\langle\phi_1\ldots\phi_N\rangle = 0 \\ &\textstyle\sum_k (2z_k h_k + z_k^2\partial_{z_k})\langle\phi_1\ldots\phi_N\rangle = 0 \end{aligned} \tag{4.8}$$

where $\phi_j \equiv \phi_j(z_j, \bar{z}_j)$. From conservation of the energy momentum tensor we deduce that

$$\frac{\partial}{\partial z_0^i}\langle T_{ij}(z_0)\phi_1\ldots\phi_N\rangle = 0 \qquad \text{if } z_i \neq z_j,\ i = 0, \ldots, N. \tag{4.9}$$

The next question concerns the behaviour of correlation functions under local transformations. The Ward identity asserts that the change is determined by T_{ij} generating (4.8):

$$\delta_{\varepsilon(x)}\langle\mathcal{F}(A)\rangle = \int d^2x\partial_i\varepsilon_j(x)\langle T_{ij}(x)\mathcal{F}(A)\rangle \tag{4.10}$$

where $\mathcal{F}(A)$ denotes a product of operators $A_\ell(z_\ell, \bar{z}_\ell)$. We have to be careful since such correlation functions are singular at coinciding points. Doing a partial integration modifies (4.10) to

$$\delta_\varepsilon\langle\mathcal{F}(A)\rangle = \sum_\alpha \oint_{C_\alpha} d\ell_i\varepsilon_j(x)\langle T_{ij}(x)\mathcal{F}(A)\rangle - \int d^2x\varepsilon_j(x)\partial_i\langle T_{ij}(x)\mathcal{F}(A)\rangle. \tag{4.11}$$

Here C_α denotes a circle surrounding the singular point. Next we introduce $T(z)$ and $\bar{T}(\bar{z})$ and obtain the final form of the Ward identity

$$\delta_{\varepsilon(x)}\langle\mathcal{F}(A)\rangle = \sum_\alpha \oint_{C_\alpha} \frac{dz}{2\pi i}\varepsilon(z)\langle T(z)\mathcal{F}(A)\rangle + \sum_\beta \oint_{\bar{C}_\beta} \frac{d\bar{z}}{2\pi i}\bar{\varepsilon}(\bar{z})\langle \bar{T}(\bar{z})\mathcal{F}(A)\rangle. \tag{4.12}$$

Schwinger Term: We may ask the question how $T(z)$ transforms under local conformal transformations $z \to z' = z + \varepsilon(z)$. Classically we would expect that

$$T(z)dz^2 \to T'(z)dz^2 \equiv T(z')dz^2 \cong (T(z) + \varepsilon(z)\frac{dT}{dz} + 2\frac{d\varepsilon}{dz}T)dz^2 + O(\varepsilon^2).$$

Quantum mechanically, due to normal ordering, an additional c-number term shows up:

$$T(z)dz^2 \to T'(z)dz^2 \cong T(z')dz'^2 + \frac{c}{12}\{z', z\}dz^2. \tag{4.13}$$

The c-number term is called a cocycle. It is restricted due to the Jacobi identity. Requiring that it should be a "local" expression determines its form to be the Schwarz derivative except for a constant c:

$$\{w, z\} = \frac{w_{zzz}}{w_z} - \frac{3}{2}\frac{w_{zz}^2}{w_z^2}. \tag{4.14}$$

Infinitesimally $\{z', z\} \cong \varepsilon'''(z)$ which vanishes for $\varepsilon(z) = -\varepsilon_0 - \varepsilon_1 z - \varepsilon_2 z^2$. The transformation law finally becomes

$$T(z) \to T(z) + \varepsilon(z)T'(z) + 2\varepsilon'(z)T(z) + \frac{c}{12}\varepsilon'''(z). \tag{4.15}$$

4.5 The Virasoro Algebra

Through the expansion $T(z) = \sum_{n=-\infty}^{\infty} L_n/z^{n+2}$ we may introduce operators L_n and similarly $\bar{T}(\bar{z})$ yields operators $\bar{L}_n$. We put T instead of $\mathcal{F}(A)$ in equ. (4.12) and use the transformation law (4.15). This yields the celebrated Virasoro algebra with central extension term

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\delta_{n,-m}n(n^2-1). \tag{4.16}$$

The Schwinger term vanishes for the $sL(2, \mathbf{C})$ algebra spanned by $\{L_{-1}, L_0, L_1\}$. Global transformations remain unbroken.

In order to proceed we have to quote a suitable conjugation. $z = 0$ and $z = \infty$ correspond to $t = -\infty$ and $t = \infty$ within the in-out formalism (radial quantization). Motivated by reflection positivity we require that

$$A(z, \bar{z})^+ = A(\frac{1}{\bar{z}}, \frac{1}{z})\frac{1}{\bar{z}^{2h}}\frac{1}{z^{2\bar{h}}}. \tag{4.17}$$

This conjugation yields for $T(z)$ that $L_{-m} = L_m^+$. We note that a nonvanishing value for c is very essential. If we require for the vacuum state that $L_{-1}|0\rangle = L_0|0\rangle = L_1|0\rangle = 0$ and $T(z)$ is regular so that $L_m|0\rangle = 0$ for $m \geq -1$, we obtain in addition that $\langle 0|L_m = 0$ for $m \leq 1$. Calculation of the correlation

$$\langle 0|T(z_1)T(z_2)|0\rangle = \frac{c/2}{(z_1 - z_2)^4} \tag{4.18}$$

shows the relevance of $c \neq 0$.

Example: The scaling limit of the $d = 2$ Ising model gives a Majorana field with $\mathcal{L} = \bar{\psi} i\gamma_i \partial_i \psi$ and equations of motion $\partial_{\bar{z}}\psi_1 = 0$, $\partial_z \psi_2 = 0$. The conformal weights of $\psi \equiv \psi_1$ are $(1/2, 0)$ and $T(z) = \frac{1}{2} i\psi \partial_z \psi_i$. The mode expansion

$$\psi(z) = \sum_{k\in\mathbf{Z}} a_k z^{-k} = \frac{1}{\sqrt{z}} \sum_{q=k-1/2} b_q z^{-q}, \qquad \begin{array}{ll} a_k|0\rangle = 0 & k > 0 \\ a_k^\dagger|0\rangle = 0 & k < 0 \end{array} \tag{4.19}$$

with $b_q^\dagger = b_{-q}$ fixes a representation of the Virasoro algebra and yields after explicit calculations $c = 1/2$.

It is remarkable that the Virasoro algebra admits unitary highest weight representations. All of them are classified. Let L_0 be diagonalized $L_0|h\rangle = h|h\rangle$. The representation will be characterized by (c, h). We require not only that $L_1|h\rangle = 0$ but also $L_2|h\rangle = 0$ and therefore $L_j|h\rangle = 0$ for all $j > 0$.

A Verma modul is built up from the representation space which is spanned by $L_{-k_1} \dots L_{-k_n}|h\rangle \equiv |k_1 \dots k_n, h\rangle$ for $1 \le k_1 \le k_2 \le \dots \le k_n$. We note that

$$L_0|k_1 \dots k_n, h\rangle = L_{-k_1} L_0|k_2, \dots\rangle + k_1|k_1 \dots\rangle = (\sum_{i=1}^n k_i + h)|k_1 \dots k_n, h\rangle. \tag{4.20}$$

This suggests to introduce the level of a vector $\nu(|\ \rangle) = \sum_{i=1}^n k_i$. All terms of (4.16) leave the level invariant. Let $P(N)$ be the number of basis vectors to level N. $P(N)$ equals the number of possible partitions of the integer N into positive integers. The generating function is given by

$$\prod_{N=1}^{\infty} \frac{1}{1 - x^N} = \sum_{M=0}^{\infty} x^M P(M). \tag{4.21}$$

We quote the first few examples: $N = 0$, $|h\rangle$; $N = 1$, $L_{-1}|h\rangle$; $N = 2$, $L_{-1}L_{-1}|h\rangle$, $L_{-2}|h\rangle$; $N = 3$, $L_{-1}L_{-1}L_{-1}|h\rangle$, $L_{-1}L_{-2}|h\rangle$, $L_{-3}|h\rangle$.

In order to define the notion of unitarity we need a scalar product:

$$\langle h|L_{\ell_m} \dots L_{\ell_1} L_{-k_1} \dots L_{-k_m}|h\rangle$$

becomes zero for vectors with different levels. The representation becomes unitary iff all matrices M_ν formed from scalar products on level ν (with $P^2(\nu)$ elements) are positive. The representations need not be irreducible. We calculate, for example, $\|L_{-1}|h\rangle\|^2 = 2h \ge 0$ and $\|L_{-n}|h\rangle\|^2 = 2nh + \frac{c}{12} n(n^2 - 1) \ge 0$ from which we conclude that $h \ge 0$ and $c \ge 0$. Next we calculate

$$M_2 = \begin{pmatrix} \langle h|L_1 L_1 L_{-1} L_{-1}|h\rangle & \langle h|L_2 L_{-1} L_{-1}|h\rangle \\ \langle h|L_1 L_1 L_{-2}|h\rangle & \langle h|L_2 L_{-2}|h\rangle \end{pmatrix} = \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h + \frac{c}{2} \end{pmatrix} \tag{4.22}$$

and obtain $\det M_2 = 32(h - h_{11}(c))(h - h_{12}(c))(h - h_{21}(c))$. The zeros of $\det M_2$ are given by $h_{11} = 0$ and

$$h_{\substack{12\\21}}(c) = \frac{5 - c \pm \sqrt{(1-c)(25-c)}}{16}.$$

If det $M_2 < 0$ both eigenvalues cannot be positive. Next one studies $h_{p,q}(c)$ as a function of c: For $c \leq 1$ there exist two curves ending at $c = 1$ and $h = 1/4$. A part of the strip $0 \leq c \leq 1$ and $h \geq 0$ is excluded. For $1 < c < 25$ both zeros turn out to be complex. For $c > 25$ h_{12} and h_{21} are negative. One even is able to calculate

$$\det M_N(c,h) = \alpha_N \prod_{p,q\geq 1, pq\leq N} (h - h_{p,q}(c))^{P(N-pq)} \tag{4.23}$$

which are called Kac determinants. At each level parts of the above mentioned strip is excluded. Finally in the strip only certain points remain. A long argumentation reveals the

Theorem: There exist unitary heighest weight representations of the Virasoro algebra iff either $c \geq 1$, $h \geq 0$ or $c \in \{c_m\}$, $c_m = 1 - 6/(m(m+1))$, $m = 3, 4, \ldots$ and

$$h_{p,q}(m) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)} \quad \text{with } 1 \leq q \leq p \leq m-1. \tag{4.24}$$

As an example we quote the Ising case: $c = 1/2$ is obtained for $m = 3$. Three unitary representations to $h = 0, 1/16$ and $1/2$ are allowed. $(c,h) = (1/2, 0)$ corresponds to $T(z)$; $(1/2, 1/16)$ to $s(z,\bar{z})$ and $(1/2, 1/2)$ to the Majorana field $\psi(z)$. Note that

$$\langle s(z_1,\bar{z}_1)s(z_2,\bar{z}_2)\rangle = \frac{c_{12}}{|z_1 - z_2|^{h_1+h_2}|\bar{z}_1 - \bar{z}_2|^{\bar{h}_1+\bar{h}_2}} \tag{4.25}$$

and since $h_1 = h_2 = \bar{h}_1 = \bar{h}_2 = 1/16$ we obtain (!) the η-exponent to be $\eta = 1/4$.

4.6 Correlation Functions

We defined quasiprimary field operators by demanding the transformation law

$$A(z,\bar{z}) \to |f'(z)|^h |\bar{f}'(\bar{z})|^{\bar{h}} A(f(z), \bar{f}(\bar{z})) \tag{4.26}$$

or infinitesimally

$$\delta_\varepsilon A(z) = \varepsilon \partial_z A + h\varepsilon' A.$$

We note that the transformation law of the energy momentum tensor is different

$$\delta_\varepsilon T = \varepsilon \partial_z T + 2\varepsilon' T + \frac{c}{2}\varepsilon'''(z) \tag{4.27}$$

since the c-number term is added.

Following BPZ [6] we call ϕ a primary operator if

$$\delta_\varepsilon \phi = \varepsilon \partial_z \phi + h_\phi \varepsilon' \phi \tag{4.28}$$

for global *and* local conformal transformations. Otherwise the operator is called a secondary one. Note that primary implies quasiprimary. A secondary operator transforms as

$$\delta_\varepsilon A(z) = \sum_{k=0}^{N} B^{k-1}(z) \frac{d^k}{dz^k}\varepsilon(z), \qquad B^{-1} = \partial_z A, \qquad B^0 = hA. \tag{4.29}$$

Since the conformal weight of B^{k-1} is $h-k+1$, $k=0,\ldots,N$ and h_{k-1} has to be positive, $h_{k-1}>N$ and the sum in (4.29) has to be a finite one.

Given a primary field operator ϕ, there exists a family of secondary quasiprimary operators with conformal weights $(h+i,\bar{h}+j)$, $i,j\in\mathbf{N}$. They form a family $[\phi]$. The operator algebra is built up as $\mathcal{A}=\bigoplus_n[\phi_n]$. $[\phi]$ gives a representation of the Virasoro algebra. $\mathcal{A}$ can be studied with the help of the operator product expansion. As an example we note the expansion

$$T(\zeta)\phi(z)=\sum_{k=0}^{\infty}(\zeta-z)^{k-2}\phi^{-k}(z), \tag{4.30}$$

where ϕ^{-k} has conformal weight $(h+k)$. It follows that

$$L_{-k}\phi=\phi^{-k} \quad \text{with} \quad L_{-k}(z)=\oint\frac{d\zeta}{2\pi i}\frac{T(\zeta)}{(\zeta-z)^{k-1}}$$

and more generally elements of the family $[\phi]$ are given by

$$\phi^{(-k_1,\ldots,-k_N)}(z)=L_{-k_1}(z)\ldots L_{-k_N}(z)\phi(z), \qquad k_N\geq\ldots\geq k_1\geq 1. \tag{4.31}$$

With the help of a primary operator we can obtain the heighest weight vector

$$|H\rangle=\phi(0)|0\rangle, \qquad \phi^{(-k_1,\ldots,-k_N)}(0)|0\rangle=L_{-k_1}\ldots L_{-k_N}|h\rangle. \tag{4.32}$$

Due to the proposed transformation laws correlation functions are severely restricted. Let $\phi_j\equiv\phi_j(z_j)$ be primary fields. The correlation function $\langle T(z)\phi_1\ldots\phi_N\rangle$ can be expressed in terms of $\langle\phi_1\ldots\phi_N\rangle$: From (4.28) we get for $\varepsilon_n(z)=-\varepsilon z^{n+1}$, $n\in\mathbf{Z}$

$$\delta_n\phi\equiv-\frac{1}{\varepsilon}\delta_{\varepsilon_n}\phi=[L_n,\phi]=z^{n+1}\partial_z\phi+h(n+1)z^n\phi, \tag{4.33}$$

and calculate

$$[\sum_{m=-1}^{\infty}\frac{L_m}{z^{m+2}},\phi(z_1)]=\frac{1}{z-z_1}\partial_{z_1}\phi(z_1)-\frac{h}{(z-z_1)^2}\phi(z_1)\equiv\mathcal{L}_{zz_1}\phi(z_1). \tag{4.34}$$

We therefore deduce that

$$\langle T(z)\phi_1\ldots\phi_N\rangle=\sum_{j=1}^{N}\mathcal{L}_{zz_j}\langle\phi_1\ldots\phi_N\rangle. \tag{4.35}$$

For the special values of (c,j) (equ. 4.24) we obtain differential equations for correlation functions. As an example we treat $m=3$ for which three unitary representations exist. It follows that the determinants $\det M_N$ vanish for all $N\geq 2$. This means that there exists a vector which is orthogonal to all vectors on this level and which has norm zero (the representation is reducible). Take for example level two. There exists a vector $|\chi\rangle$ such that $L_n|\chi\rangle=0$ for all $n>0$. $|\chi\rangle=(L_{-2}+aL_{-1}^2)|h\rangle$. Require that $L_1|\chi\rangle=L_2|\chi\rangle=0$ gives two conditions and $a=-3/2(2h+1)$ and $h=[5-c\pm\sqrt{(1-c)(25-c)}]/16$ which are the values we obtained before. It follows that $\langle\chi|\chi\rangle=0$ with

$$|\chi\rangle=(L_{-2}-\frac{3L_{-1}L_{-1}}{2(2h+1)})|h\rangle. \tag{4.36}$$

It is implied that within the family $[\phi_h]$ there exists an operator $\chi(z)$ with $\chi(0)|0\rangle = |\chi\rangle$. $\chi(z)$ is called the null field and the family is called to be degenerate. It follows that the correlation functions of primary fields are given by solving differential equations. Start from

$$\langle 0|\phi_1 \ldots \phi_N (L_{-2} - \frac{3L_{-1}L_{-1}}{2(2h+1)})\phi_h|0\rangle = 0, \tag{4.37}$$

and commute L_{-k} to the left and use finally $\langle 0|L_{-k} = 0$. The differential equation

$$(D^N_{-2} - \frac{3}{2(2h+1)} D^N_{-1}{}^2)\langle 0|\phi_1 \ldots \phi_N \phi_h(0)|0\rangle = 0 \tag{4.38}$$

with

$$D^N_{-k} = \sum_{i=1}^N (-z_i^{k+1}\partial_{z_i} - (1-k)h_i z_i^{-k})$$

follows.

Acknowledgement: These notes summarize seminars in Vienna. I would like to thank the enthusiastic participants for their interest.

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