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Introduzione al Modello Standard

- Una teoria di gauge per le interazioni deboli
- Masse e flavour-mixing
- Oltre il tree level: il potenziale efficace
- Anomalia delle correnti assiali

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Our starting point: The Fermi theory:

$$\mathcal{L} = -\frac{G_\beta}{\sqrt{2}}\overline{p}\gamma^\alpha(1-a\gamma_5)n\overline{e}\gamma_\alpha(1-\gamma_5)\nu_e$$
$$-\frac{G_\mu}{\sqrt{2}}\overline{\nu}_\mu\gamma^\alpha(1-\gamma_5)\mu\overline{e}\gamma_\alpha(1-\gamma_5)\nu_e,$$

$G_\mu \simeq 1.16639\times 10^{-5}\ \mathrm{GeV}^{-2}$; $G_\beta \simeq G_\mu$; $a \simeq 1.239\pm 0.09$.

Non unitary, non renormalizable.

BUT: gives us the structure of the currents involved:

$$J_\mu = \overline{\nu}_e \frac{1}{2}\gamma_\mu(1-\gamma_5)e.$$

We want to to rewrite J_μ in the form of a Noether current:

$$\overline{\psi}_i\gamma_\mu T^A_{ij}\psi_j,$$

Define

$$L = \frac{1}{2}(1-\gamma_5)\left(\begin{array}{c} \nu_e \\ e \end{array}\right) \equiv \left(\begin{array}{c} \nu_{eL} \\ e_L \end{array}\right),$$

Then

$$J_\mu = \overline{L}\gamma_\mu \tau^+ L,$$

with

$$\tau^+ = \frac{1}{2}(\tau_1 + i\tau_2) = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right],$$

$$J_\mu^\dagger = \overline{L}\gamma_\mu \tau^- L$$

$$J_3^\mu = \overline{L}\gamma^\mu[\tau^+, \tau^-]L = \overline{L}\gamma_\mu \tau_3 L$$

will also be present. No other current must be introduced, since $[\tau_3, \tau^\pm] = 2\tau^\pm$.

The currents correspond to an $SU(2)$ symmetry.

Gauge theory based on $SU(2)_L$: introduce vector fields via covariant derivative:

$$D^\mu = \partial^\mu - igW_i^\mu T_i,$$

where

$$T_i = \frac{\tau_i}{2}$$

for left-handed fields, and $T_i = 0$ for right-handed ones (no R field in the Fermi Lagrangian).

We end up with

$$\mathcal{L} = i\overline{L}\hat{D}L + i\overline{\nu}_{eR}\hat{D}\nu_{eR} + i\overline{e}_R\hat{D}e_R$$

which contains the usual kinetic term

$$\mathcal{L}^{kin} = i\overline{L}\hat{\partial}L + i\overline{\nu}_{eR}\hat{\partial}\nu_{eR} + i\overline{e}_R\hat{\partial}e_R$$

plus interaction terms

$$\mathcal{L}^W = \mathcal{L}_c^W + \mathcal{L}_n^W$$

where

$$\mathcal{L}_c^W = gW_1^\mu \overline{L}\gamma_\mu \frac{\tau_1}{2} L + gW_2^\mu \overline{L}\gamma_\mu \frac{\tau_2}{2} L$$

$$\mathcal{L}_n^W = gW_3^\mu \overline{L}\gamma_\mu \frac{\tau_3}{2} L = \frac{g}{2} W_3^\mu \left(\overline{\nu}_{eL}\gamma_\mu \nu_{eL} - \overline{e}_L \gamma_\mu e_L \right).$$

In terms of fields and currents with definite electric charge quantum numbers,

$$\mathcal{L}_c^W = \frac{g}{\sqrt{2}} \overline{L}\gamma^\mu \tau^+ L W_\mu^+ + \frac{g}{\sqrt{2}} \overline{L}\gamma^\mu \tau^- L W_\mu^-$$

where we have defined

$$W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \mp i W_\mu^2)$$

The neutral current $J_3^\mu = \overline{L}\gamma^\mu \tau_3 L$ cannot be identified with the electromagnetic current

$$\Downarrow$$

The gauge vector boson W_3^μ cannot be interpreted as the photon.

Extend the gauge group to include the EM current:

$$SU(2) \rightarrow SU(2) \otimes U(1).$$

NB: $U(1) \neq U(1)_{EM}$!

$$\psi \rightarrow \psi' = \exp\left[ig'\alpha\frac{Y(\psi)}{2}\right]\psi$$

$$D^\mu = \partial^\mu - igW_i^\mu T_i - ig'\frac{Y}{2}B^\mu$$

Only $\mathcal{L}_n^W$ is modified:

$$\begin{aligned}\mathcal{L}_n^W &= \frac{g}{2}W_3^\mu\left(\overline{\nu}_{eL}\gamma_\mu\nu_{eL} - \overline{e}_L\gamma_\mu e_L\right) \\ &+\frac{g'}{2}B^\mu\Big[Y(L)\left(\overline{\nu}_{eL}\gamma_\mu\nu_{eL} + \overline{e}_L\gamma_\mu e_L\right) \\ &+Y(\nu_{eR})\overline{\nu}_{eR}\gamma_\mu\nu_{eR} + Y(e_R)\overline{e}_R\gamma_\mu e_R\Big]\end{aligned}$$

Assign the quantum numbers Y so that the EM interaction term appear in the lagrangian. To do this, first rotate W_3 and B:

$$\begin{aligned} A^\mu &= \quad B^\mu \cos\theta_\mathrm{w} + W_3^\mu \sin\theta_\mathrm{w} \\ Z^\mu &= -B^\mu \sin\theta_\mathrm{w} + W_3^\mu \cos\theta_\mathrm{w}. \end{aligned}$$

Then identify one of the two (e.g. A_μ) with the photon field. You find

$$\begin{aligned} g \sin\theta_\mathrm{w} &= e \\ g' \cos\theta_\mathrm{w} &= e, \end{aligned}$$

where e is the positron charge, and

$$Y(L) = -1, \quad Y(\nu_{e\,R}) = 0, \quad Y(e_R) = -2.$$

In general

$$Y = 2(Q - T_3).$$

No coupling for right-handed neutrinos

Form a column vector Ψ with all the fermionic fields (left and right-handed components counted separately). Then

$$\mathcal{L}_n^W = e\left[\overline{\Psi}\gamma_\mu Q \Psi A^\mu + \overline{\Psi}\gamma_\mu Q_Z \Psi Z^\mu\right],$$

e the positron charge, Q the diagonal matrix of electromagnetic charges,

$$Q_Z = \frac{1}{\cos\theta_{\rm W}\sin\theta_{\rm W}}\left(T_3 - Q\sin^2\theta_{\rm W}\right).$$

The extension to include more lepton doublets is straightforward.

Including hadrons

Start from the hadronic current of β and strange particle decays:

$$J^\mu_{had} = \cos\theta_c \overline{u}\gamma^\mu \frac{1}{2}(1-\gamma_5)d + \sin\theta_c \overline{u}\gamma^\mu \frac{1}{2}(1-\gamma_5)s,$$

where θ_c is the Cabibbo angle ($\theta_c \sim 13^\circ$). The obvious way is the

WRONG WAY:

Define

$$Q = \frac{1}{2}(1-\gamma_5)\begin{bmatrix} u \\ d \\ s \end{bmatrix} \equiv \begin{bmatrix} u_L \\ d_L \\ s_L \end{bmatrix}$$

with

$$T^+ = \begin{bmatrix} 0 & \cos\theta_c & \sin\theta_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that

$$J^\mu_{had} = \overline{Q}\gamma^\mu T^+ Q.$$

Then, the current corresponding with

$$T_3 = [T^+, T^-] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos^2\theta_c & -\cos\theta_c\sin\theta_c \\ 0 & -\cos\theta_c\sin\theta_c & -\sin^2\theta_c \end{bmatrix}$$

contains flavor-changing nutral currents (FCNC), such as $\overline{d}_L\gamma^\mu s_L$, with couplings of the same order of flavor conserving ones. They are instead strongly suppressed: you don't observe e.g.

$$K^0 \to \pi^0 e^+ e^-$$

at the expected rate. Which is then the

CORRECT WAY

to proceed? Introduce a fourth quark c (for *charm*) with charge $2/3$; assume $m_c \gg m_u, m_d$ and assume

$$\begin{aligned} J^\mu_{had} &= \cos\theta_c \overline{u}\gamma^\mu \frac{1}{2}(1-\gamma_5)d + \sin\theta_c \overline{u}\gamma^\mu \frac{1}{2}(1-\gamma_5)s \\ &- \sin\theta_c \overline{c}\gamma^\mu \frac{1}{2}(1-\gamma_5)d + \cos\theta_c \overline{c}\gamma^\mu \frac{1}{2}(1-\gamma_5)s. \end{aligned}$$

Now

$$Q = \begin{bmatrix} u_L \\ c_L \\ d_L \\ s_L \end{bmatrix}$$

and

$$T^+ = \begin{bmatrix} 0 & 0 & \cos\theta_c & \sin\theta_c \\ 0 & 0 & -\sin\theta_c & \cos\theta_c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

No flavour-changing neutral current is now present. In fact,

$$[T^+, T^-] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

This is the Glashow-Iliopoulos-Maiani (GIM) mechanism of FCNC suppression.

The current J^{μ}_{had} is usually written as

$$J^{\mu}_{had} = (\overline{u}_L \overline{d}'_L)\gamma^{\mu}\tau^{+}\left(\begin{array}{c} u_L \\ d'_L \end{array}\right) + (\overline{c}_L \overline{s}'_L)\gamma^{\mu}\tau^{+}\left(\begin{array}{c} c_L \\ s'_L \end{array}\right),$$

where

$$\left(\begin{array}{c} d'_L \\ s'_L \end{array}\right) = V \left(\begin{array}{c} d_L \\ s_L \end{array}\right), \quad V = \left[\begin{array}{cc} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{array}\right].$$

The pairs (u, d), (c, s) are called quark families. The structure outlined above can be extended to n quark families; then, V becomes an $n \times n$ unitary matrix. (more on this when we'll discuss CP violation).

The charged-current interaction term is now given by

$$\mathcal{L}_c^W = \frac{g}{\sqrt{2}} \sum_{f=1}^{n} \left[\overline{L}_f \gamma^\mu \tau^+ L_f + \overline{Q}_f \gamma^\mu \tau^+ Q_f \right] W_\mu^+ + h.c.,$$

where

$$L_f = \begin{pmatrix} \nu_{e\,L} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu\,L} \\ \mu_L \end{pmatrix}, \ldots$$

$$Q_f = \begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \begin{pmatrix} c_L \\ s'_L \end{pmatrix}, \ldots,$$

* * *

To conclude, the pure Yang-Mills term:

$$\mathcal{L}_{YM} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W_i^{\mu\nu},$$

where

$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$$

$$W_i^{\mu\nu} = \partial^\mu W_i^\nu - \partial^\nu W_i^\mu + g \epsilon_{ijk} W_j^\mu W_k^\nu .$$

Masses

The W boson must be very heavy (with respect to light fermions). Consider the amplitude for β decay:

$$\frac{G_\beta}{\sqrt{2}}\overline{u}\gamma^\mu(1-\gamma_5)d\overline{e}\gamma_\mu(1-\gamma_5)\nu_e.$$

In the standard model the same process is induced by the exchange of a W boson:

$$\left(\frac{g}{\sqrt{2}}\overline{u}_L\gamma^\mu d_L\right)\frac{1}{q^2-m_{\rm W}^2}\left(\frac{g}{\sqrt{2}}\overline{e}_L\gamma_\mu\nu_{eL}\right),$$

To match the Fermi amplitude in the $q\rightarrow 0$ limit, it must be

$$\frac{G_\beta}{\sqrt{2}}=\left(\frac{g}{2\sqrt{2}}\right)^2\frac{1}{m_{\rm W}^2}.$$

Recalling that $g=e/\sin\theta_{\rm W}$,

$$m_{\rm W}\geq 37.3 \ \ {\rm GeV}.$$

However, gauge boson mass terms are not gauge-invariant.

Inserting a mass term for the W boson by hand leads to a non-renormalizable theory:

$$\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2} m_A^2 A^\mu A_\mu,$$

Work out the propagator $\Delta^{\mu\nu}$ for A^μ in momentum space:

$$\Delta^{\mu\nu} = \frac{i}{k^2 - m_A^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_A^2}\right).$$

$\Delta^{\mu\nu}$ has not the correct behaviour for large values of the momentum k: for $k \to \infty$ $\Delta \sim k^0$ rather than vanishing as k^{-2}, thus violating power-counting and making the theory unrenormalizable.

We need something else.

We consider scalar electrodynamics:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\phi)^\dagger D_\mu\phi - V(\phi),$$

where $D^\mu = \partial^\mu + ieA^\mu$, and

$$V(\phi) = m^2 \mid \phi \mid^2 + \lambda \mid \phi \mid^4 .$$

We look for constant field configurations that minimize the energy of the system.

$m^2 \geq 0$: minimum for $\phi = 0$.

$m^2 < 0$: the potential has an infinite number of degenerate minima:

$$\mid \phi \mid^2 = -\frac{m^2}{2\lambda} \equiv \frac{1}{2}v^2.$$

connected by gauge transformations.

When the system chooses one of the minimum configurations, spontaneous breaking of the gauge symmetry takes place.

Not really a symmetry breaking: the Lagrangian is still gauge invariant, currents are still conserved.

Shift ϕ to one of the degenerate minima:

$$\phi(x) = \frac{1}{\sqrt{2}} \left[v + H(x) + iG(x) \right]$$

One of the two fields H and G could in principle be removed from the lagrangian by an appropriate gauge transformation. For example, one could eliminate G by choosing a gauge transformation that brings ϕ to be real. For the moment, we keep both H and G in the lagrangian; we will come back to this point later.

$$V(\phi) = (m^2 v + \lambda v^3) H + \frac{1}{2}(m^2 + 3\lambda v^2) H^2$$
$$+ \frac{1}{2}(m^2 + \lambda v^2) G^2 + \lambda v H (H^2 + G^2) + \frac{\lambda}{4}(H^2 + G^2)^2.$$

$\lambda v^2 = -m^2 \rightarrow$ terms proportional to H and G^2 vanish. The coefficient of the H^2 term is now $(-2m^2)/2$, and has therefore the correct sign to be interpreted as a mass term (remember that m^2 is negative).

Covariant derivative term:

$$
\begin{aligned}
(D^\mu \phi)^\dagger D_\mu \phi \quad = \quad & \frac{1}{2}\partial^\mu H \partial_\mu H + \frac{1}{2}\partial^\mu G \partial_\mu G \\
& + \frac{1}{2}e^2 (H^2 + G^2 + 2vH) A^\mu A_\mu \\
& + e A_\mu (H \partial^\mu G - G \partial^\mu H) \\
& + ev A^\mu \partial_\mu G \\
& + \frac{1}{2} e^2 v^2 A^\mu A_\mu .
\end{aligned}
$$

A_μ has acquired a mass $m_A = ev$

Gauge-fixing:

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi}(\partial^\mu A_\mu - b\xi G)^2,$$

ξ arbitrary constant (the gauge parameter). Gauge-fixing condition: $\partial^\mu A_\mu - b\xi G = 0$. Choose $b = ev = m_A$; then the $A^\mu \partial_\mu G$ term is cancelled.

A term

$$-\frac{1}{2}\xi b^2 G^2 = -\frac{1}{2}\xi m_A^2 G^2$$

arises, which gives a squared mass ξm_A^2 to the unphysical field G.

Let us now compute the propagator. We have

$$-\frac{1}{2}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu) + \frac{1}{2} m_A^2 A^\mu A_\mu - \frac{1}{2\xi}(\partial^\mu A_\mu)^2,$$

which gives

$$\Delta_\xi^{\mu\nu} = \frac{i}{k^2 - m_A^2}\left[-g^{\mu\nu} + \frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi m_A^2}\right].$$

The propagator has now the correct behaviour at large momenta.

New singularity at $k^2 = \xi m_A^2$. Its contribution to phyisical quantities exactly cancelled by the contribution of G exchange.

Two common choices: the Feynman gauge, $\xi = 1$, which gives

$$\Delta_F^{\mu\nu} = -\frac{i g^{\mu\nu}}{k^2 - m_A^2}$$

and the Landau gauge, $\xi = 0$, for which

$$\Delta_L^{\mu\nu} = \frac{i}{k^2 - m_A^2}\left[-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right].$$

In the standard model:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}; \qquad Y(\phi) = 1.$$

$$V(\phi) = m^2 \mid \phi \mid^2 + \lambda \mid \phi \mid^4,$$

minimum at

$$\mid \phi \mid^2 = -\frac{m^2}{2\lambda} \equiv \frac{1}{2} v^2.$$

We can reparameterize ϕ as

$$\phi = \frac{1}{\sqrt{2}} e^{i\tau^i \theta^i(x)/v} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix},$$

convenient in the unitary gauge: θ_i can be rotated away by an $SU(2)$ gauge transformation. The scalar potential takes the form

$$V = \frac{1}{2}(2\lambda v^2) H^2 + \lambda v H^3 + \frac{1}{4}\lambda H^4$$

and $m_{\rm H}^2 = 2\lambda v^2$.

Covariant derivative term:

$$D^\mu \phi = \left(\partial^\mu - i\frac{g}{2}\tau^i W^i_\mu - i\frac{g'}{2}B_\mu\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix} - \frac{i}{2}(v+H)\begin{pmatrix} g(W_1^\mu - iW_2^\mu) \\ -gW_3^\mu + g' B^\mu \end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix} - \frac{i}{2}\left(1 + \frac{H}{v}\right)\begin{pmatrix} gvW^{\mu +} \\ -\sqrt{(g^2 + g'^2)/2}vZ^\mu \end{pmatrix}$$

$$|(D^\mu \phi)|^2 = \frac{1}{2}\partial^\mu H \partial_\mu H$$
$$+ \left[\frac{1}{4}g^2 v^2 W^{\mu +} W^-_\mu + \frac{1}{8}(g^2 + g'^2)v^2 Z^\mu Z^-_\mu\right]\left(1 + \frac{H}{v}\right)^2.$$

Vector boson masses:

$$m^2_{\rm W} = \frac{1}{4}g^2 v^2; \quad m^2_{\rm Z} = \frac{1}{4}(g^2 + g'^2)v^2; \quad m^2_\gamma = 0$$

The value of v can be obtained:

$$v = \sqrt{\frac{1}{G_\mu \sqrt{2}}} \simeq 246.22 \text{ GeV}.$$

Fermion masses:

$$-m\overline{\psi}\psi = -m(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L),$$

not invariant under a chiral transformation.

Hadrons:

$$Q'_f = \begin{pmatrix} u'_{f\,L} \\ d'_{f\,L} \end{pmatrix} \quad U'_f = u'_{f\,R} \quad D'_f = d'_{f\,R}.$$

A Yukawa interaction term can be added to the lagrangian:

$$\mathcal{L}_Y^{hadr} = -(\overline{Q'}\phi h'_D D' + \overline{D'}\phi^\dagger h'^\dagger_D Q') - (\overline{Q'}\phi_c h'_U U' + \overline{U'}\phi_c^\dagger h'^\dagger_U Q'),$$

where h'_U and h'_D are generic $n \times n$ constant matrices in the generation space,

$$\phi_c = \begin{pmatrix} \phi^{0\,*} \\ -\phi^- \end{pmatrix}.$$

Define new quark fields u and d by

$$u'_L = V_{\rm L}^{\rm U} u_L, \quad u'_R = V_{\rm R}^{\rm U} u_R$$
$$d'_L = V_{\rm L}^{\rm D} d_L, \quad d'_R = V_{\rm R}^{\rm D} d_R,$$

where $V_{\rm L,R}^{\rm U,D}$ are unitary matrices, chosen so that

$$h_U \equiv V_{\rm L}^{\rm U\,\dagger} h'_U V_{\rm R}^{\rm U}$$

and

$$h_D \equiv V_{\rm L}^{\rm D\,\dagger} h'_D V_{\rm R}^{\rm D}$$

are diagonal with real, non-negative entries (it is always possible).

In the unitary gauge

$$\mathcal{L}_Y^{hadr} = -\frac{1}{\sqrt{2}}(v+H)\sum_{f=1}^{N}(h_D^f \overline{d}_f d_f + h_U^f \overline{u}_f u_f),$$

where $h_{U,D}^f$ are the diagonal entries of the matrices $h_{U,D}$.

We can now identify the quark masses by

$$m_U^f = \frac{vh_U^f}{\sqrt{2}}, \quad m_D^f = \frac{vh_D^f}{\sqrt{2}}.$$

The charged hadronic weak current takes the form

$$J^\mu_{hadr} = \overline{Q'}\gamma^\mu \tau^+ Q' = \sum_{f,f'} \overline{u}_L^f \gamma^\mu V_{ff'} d_L^{f'},$$

where

$$V = V_{\rm L}^{\rm U\,\dagger} V_{\rm L}^{\rm D}$$

is the Cabibbo-Kobayashi-Maskawa (CKM) matrix.

How many independent real parameters are needed to specify the CKM matrix?

Generic $N \times N$ unitary matrix $\rightarrow N^2$ independent real parameters. Split them into "angles" and "phases":

$$N^2 = N_{\text{angles}} + \hat{N}_{\text{phases}}$$

Clearly,

$$N_{\text{angles}} = \begin{pmatrix} N \\ 2 \end{pmatrix} = \frac{1}{2}N(N-1).$$

What about phases? We have

$$\hat{N}_{\text{phases}} = N^2 - N_{\text{angles}} = \frac{1}{2}N(N+1).$$

However, some $(2N-1)$ of them can be eliminated by redefining the left-handed quark fields. So

$$N_{\text{phases}} = \hat{N}_{\text{phases}} - (2N-1) = \frac{1}{2}(N-1)(N-2).$$

Leptons: same procedure, but no Yukawa coupling involving the conjugate scalar field ϕ_c (no R neutrinos):

$$\mathcal{L}_Y^{lept} = -(\overline{L'}\phi h'_E E' + \overline{E'}\phi^\dagger {h'}_E^\dagger L')\,,$$

diagonalized by

$$h_E = {V_{\rm L}^{\rm E}}^\dagger h'_E V_{\rm R}^{\rm E}\,.$$

In this case we may redefine the left-handed neutrino fields using *the same* matrix $V_{\rm L}^{\rm E}$ that rotates charged leptons:

$$\nu'_L = V_{\rm L}^{\rm E} \nu_L$$
$$e'_L = V_{\rm L}^{\rm E} e_L , \quad e'_R = V_{\rm R}^{\rm E} e_R \,.$$

This puts the Yukawa interaction in diagonal form,

$$\mathcal{L}_Y^{lept} = -\sum_{f=1}^{N} h_E^f (\overline{L}_f \phi e_R^f + \overline{e}_R^f \phi^\dagger L_f) \,,$$

but leaves the interaction term unchanged:

$$J^\mu_{lept} = \overline{L'} \gamma^\mu \tau^+ L' = \overline{L} \gamma^\mu \tau^+ L = \sum_f \overline{\nu}_L^f \gamma^\mu e_L^f .$$

No lepton flavor mixing! (without ν_R).

$$\mathcal{L}_Y^{lept} = -\sum_{f=1}^{N} \frac{h_E^f}{\sqrt{2}} (v + H) \overline{e}_f e_f ,$$

$$m_E^f = \frac{v h_E^f}{\sqrt{2}} .$$

Beyond the tree level

$$Z[J] = < 0|Te^{i\int d^4x\, J(x)\phi(x)}|0 > = < 0|0 >_J,$$

Functional derivatives of $Z[J]$ with respect to J at $J=0$ $\leftrightarrow$ Green's functions of the theory. Define the functional for connected Green's functions

$$W[J] = -i \log Z[J]$$

$$\phi_c(x) = \frac{\delta W[J]}{\delta J(x)} = \frac{< 0|\phi(x)|0 >_J}{< 0|0 >_J}$$

and the effective action $\Gamma[\phi_c]$ as

$$\Gamma[\phi_c] = W[J] - \int d^4x J(x)\phi_c(x).$$

The effective action has an expansion in powers of the classical field,

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n \phi_c(x_1) \ldots \phi_c(x_n) \Gamma_n(x_1, \ldots x_n),$$

whose coefficients $\Gamma_n(x_1, \ldots x_n)$ are the connected, one-particle irreducible Green's functions of the theory.

Spontaneous symmetry breaking if $\phi_c \neq 0$ even when the source $J = 0$. On the other hand, for $J = 0$

$$\frac{\delta \Gamma[\phi_c]}{\delta \phi_c} = 0.$$

We conclude that spontaneous symmetry breaking takes place when the classical field that minimizes the effective action is different from zero.

Fourier transform:

$$\begin{aligned} \Gamma_n(x_1, \ldots x_n) &= \int \frac{dp_1}{(2\pi)^4} \cdots \frac{dp_n}{(2\pi)^4} e^{i(p_1 x_1 + \ldots p_n x_n)} \\ & (2\pi)^4 \delta(p_1 + \ldots + p_n) \tilde{\Gamma}_n(p_1, \ldots p_n), \end{aligned}$$

and expand $\tilde{\Gamma}_n$ in powers of momenta around $p_i = 0$,

$$\tilde{\Gamma}_n(p_1, \ldots, p_n) = \tilde{\Gamma}_n(0) + \ldots \; .$$

The effective action becomes

$$\begin{aligned}\Gamma[\phi_c] &= \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n \phi_c(x_1) \ldots \phi_c(x_n) \\ & \int \frac{dp_1}{(2\pi)^4} \cdots \frac{dp_n}{(2\pi)^4} e^{i(p_1 x_1 + \ldots p_n x_n)} \\ & \int d^4 x e^{-ix(p_1 + \ldots + p_n)} \left[\tilde{\Gamma}_n(0) + \ldots \right] \\ &= \int dx \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Gamma}_n(0) \phi_c^n(x) + \ldots\end{aligned}$$

The first term in this expansion is usually written as

$$-\int dx V(\phi_c),$$

where

$$V(\phi_c) = -\sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Gamma}_n(0) \phi_c^n$$

$V(\phi_c)$ is called the effective potential (no field derivatives). The neglected terms, originating from higher powers of momenta in the expansion of $\tilde{\Gamma}_n$, contain instead two or more derivatives of ϕ_c.

Minimum condition:

$$\frac{\delta}{\delta\phi_c}\int dx V(\phi_c) = \frac{dV(\phi_c)}{d\phi_c} = 0$$

if we require translational invariance of the vacuum state.

Direct computation of V:

$$V_0(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4.$$

One-loop Green's functions at $p = 0$:

$$\tilde{\Gamma}_{2n}(0) = -i\, S_n \left(-4!\, \frac{i\lambda}{4} \right)^n \int \frac{d^4k}{(2\pi)^4} \left[\frac{i}{k^2 - m^2 + i\eta} \right]^n,$$

where

$$S_n = \frac{(2n)!}{2^n 2n}.$$

Therefore

$$V_1(\phi_c) = \frac{i}{2} \sum_{n=1}^{\infty} \left(3\lambda \phi_c^2 \right)^n \frac{1}{n} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\eta)^n}.$$

$n = 1, 2 \rightarrow$ UV-divergent integrals.

Finite part, $n > 2$:

$$V_1^{\text{finite}} = \frac{i}{2} \frac{i}{(4\pi)^2} \sum_{n=3}^{\infty} \left(3\lambda \phi_c^2 \right)^n \frac{(-1)^n}{n} \frac{\Gamma(n-2)}{\Gamma(n)}\, m^{4-2n},$$

or, defining $z = 3\lambda\phi_c^2/m^2$,

$$V_1^{\text{finite}} = -\frac{m^4}{32\pi^2} \sum_{n=3}^{\infty} \frac{(-1)^n z^n}{n(n-1)(n-2)}$$

The serie can be summed, using

$$\frac{1}{n(n-1)(n-2)} = \frac{1}{2n} - \frac{1}{n-1} + \frac{1}{2(n-2)}$$

and the log expansion. We get

$$\begin{aligned} V_1^{\mathrm{finite}} &= \frac{m^4}{64\pi^2}\left[(1+z)^2\log(1+z) - z - \frac{3}{2}z^2\right] \\ &= \frac{1}{64\pi^2}\Big[\left(m^2+3\lambda\phi_c^2\right)^2\log\frac{m^2+3\lambda\phi_c^2}{m^2} \\ &\qquad\qquad - 3\lambda\phi_c^2 m^2 - \frac{3}{2}(3\lambda\phi_c^2)^2\Big]. \end{aligned}$$

Divergent parts:

$$\begin{aligned} V_1^{\rm div} &= \frac{i}{2}\left[\left(3\lambda\phi_c^2\right)\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\eta}\right. \\ &+ \left.\frac{1}{2}\left(3\lambda\phi_c^2\right)^2\int \frac{d^4k}{(2\pi)^4}\frac{1}{\left(k^2-m^2+i\eta\right)^2}\right]. \end{aligned}$$

After regularization,

$$V_1^{\rm div} = A(\lambda, m, \text{cutoff})\phi_c^2 + B(\lambda, m, \text{cutoff})\phi_c^4,$$

Some renormalization prescription must be assigned. For example, require

$$\tilde{\Gamma}_2(0) = -m^2; \quad \tilde{\Gamma}_4(0) = -6\lambda.$$

Already true at tree level potential; moreover, the finite part of the one-loop corrections starts with ϕ_c^6. Then

$$V_1^{\rm ct} = -A\phi_c^2 - B\phi_c^4,$$

so that, in this case,

$$V_1 = V_1^{\rm finite}.$$

Another (among infinite) possibility: minimal subtraction (MS): subtractions of poles in $\epsilon = (d-4)/2$.

Modified version ($\overline{MS}$): subtracting terms proportional to

$$\frac{1}{\epsilon} - \gamma + \log(4\pi)$$

We find

$$V_1^{\rm div} = -\frac{1}{64\pi^2}\Big[6\lambda\phi_c^2 m^2 + 6\lambda\phi_c^2\left(m^2 + \frac{3}{2}\lambda\phi_c^2\right)\left(\frac{1}{\epsilon} - \gamma + \log(4\pi) + \log\frac{\mu^2}{m^2}\right)\Big]$$

and finally

$$V_1^{\overline{MS}} = \frac{1}{64\pi^2}\left(m^2 + 3\lambda\phi_c^2\right)^2\left[\log\frac{m^2 + 3\lambda\phi_c^2}{\mu^2} - \frac{3}{2}\right],$$

A more efficient way to compute the effective potential: define a new theory by

$$\phi \to \phi + \omega .$$

The corresponding effective potential is

$$V'(\phi_c) = -\sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Gamma}_n(0) \, (\phi_c + \omega)^n = -\sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Gamma}'_n(\omega, 0) \, \phi_c^n ,$$

where the Green's functions $\tilde{\Gamma}'_n$ can be computed in terms of $\tilde{\Gamma}_n$. We have in particular

$$\tilde{\Gamma}'_1(\omega, 0) = \sum_{n=1}^{\infty} \frac{1}{n!} \tilde{\Gamma}_n(0) \, n \omega^{n-1}$$

and therefore

$$\int_0^{\phi_c} d\omega \, \tilde{\Gamma}'_1(\omega, 0) = \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Gamma}_n(0) \, \phi_c^n = -V(\phi_c).$$

A tree-level test:

$$V'_0(\phi) = \frac{1}{2} m^2 (\phi + \omega)^2 + \frac{1}{4} \lambda (\phi + \omega)^4 .$$

Tree-level tadpole:

$$-m^2 \omega - \lambda \omega^3 ,$$

One-loop correction:

$$\begin{aligned}
\tilde{\Gamma}'_1(\omega, 0) &= -3\lambda\omega \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2 - 3\lambda\omega^2}. \\
&= -3\lambda\omega \, \frac{(4\pi)^\epsilon}{(4\pi)^2} \, \Gamma(-1+\epsilon)(m^2 + 3\lambda\omega^2)^{1-\epsilon} \\
&= \frac{3\lambda\omega}{(4\pi)^2}(m^2 + 3\lambda\omega^2) \\
& \quad \left[\frac{1}{\epsilon} - \gamma + \log(4\pi) - \log \frac{m^2 + 3\lambda\omega^2}{\mu^2} + 1\right] \\
& \quad + \mathcal{O}(\epsilon),
\end{aligned}$$

which gives

$$\begin{aligned}
& V_1(\phi_c) \\
= \; & \frac{1}{(4\pi)^2} \int_0^{\phi_c} d\omega \, 3\lambda\omega (m^2 + 3\lambda\omega^2) \left(\log \frac{m^2 + 3\lambda\omega^2}{\mu^2} - 1\right) \\
= \; & \frac{1}{64\pi^2}(m^2 + 3\lambda\phi_c^2)^2 \left[\log \frac{m^2 + 3\lambda\phi_c^2}{\mu^2} - \frac{3}{2}\right].
\end{aligned}$$

The same procedure, applied to the standard model, gives

$$
\begin{aligned}
V(\phi) \quad = \quad & \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda(\phi^2)^2 \\
& + \frac{1}{64\pi^2}\left[H^2\left(\log\frac{H}{\mu^2} - \frac{3}{2}\right) + 3G^2\left(\log\frac{G}{\mu^2} - \frac{3}{2}\right)\right. \\
& \qquad + 6W^2\left(\log\frac{W}{\mu^2} - \frac{5}{6}\right) + 3Z^2\left(\log\frac{Z}{\mu^2} - \frac{5}{6}\right) \\
& \qquad \left. -12T^2\left(\log\frac{T}{\mu^2} - \frac{3}{2}\right)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
H &= m^2 + 3\lambda\phi^2 \\
G &= m^2 + \lambda\phi^2 \\
W &= \frac{1}{4}g^2\phi^2 \\
Z &= \frac{1}{4}(g^2 + {g'}^2)\phi^2 \\
T &= \frac{1}{2}h_t^2\phi^2.
\end{aligned}
$$

This is the standard model effective potential at one loop in the Landau gauge.

Renormalization scale dependence:

$$\frac{dV(\phi)}{dt} = 0, \quad t = \log \mu^2$$

since

$$\left(\frac{\partial}{\partial t} + \beta_\lambda \frac{\partial}{\partial \lambda} + m^2 \gamma_m \frac{\partial}{\partial m^2} + n\gamma \right) \tilde{\Gamma}_n = 0,$$

where

$$\frac{d\lambda}{dt} = \beta_\lambda \quad \frac{dm^2}{dt} = \gamma_m m^2 \quad \frac{d\phi^2}{dt} = 2\gamma\phi^2.$$

On the other hand, dV/dt can be computed explicitly:

$$\begin{aligned} \frac{dV(\phi)}{dt} &= \frac{\phi^4}{4}\Bigg\{ \beta_\lambda + 4\lambda\gamma \\ &\quad - \frac{1}{16\pi^2}\left[12\lambda^2 + \frac{3}{8}g^4 + \frac{3}{16}(g^2 + g'^2)^2 - 3h_t^4 \right] \Bigg\} \\ &\quad + \frac{1}{2}m^2\phi^2 \left[\gamma_m + 2\gamma - \frac{12\lambda}{32\pi^2} \right] \end{aligned}$$

We have therefore

$$\beta_\lambda + 4\lambda\gamma = \frac{1}{16\pi^2}\left[12\lambda^2 + \frac{3}{8}g^4 + \frac{3}{16}(g^2 + {g'}^2)^2 - 3h_t^4\right]$$

$$\gamma_m + 2\gamma = \frac{12\lambda}{32\pi^2}.$$

* * *

Behaviour of $V(\phi)$ for large ϕ_i. We require $V(\phi) \to +\infty$ for large ϕ^2.

Assume $\phi^2 \sim \Lambda^2$, $\Lambda \gg G_{\rm F}^{-1/2}$. We have

$$\begin{aligned} V(\phi) \simeq & \frac{1}{4}\phi^4\left\{\lambda + \frac{1}{16\pi^2}\left[12\lambda^2 + \frac{3}{8}g^4 + \frac{3}{16}(g^2 + {g'}^2)^2 \right.\right. \\ & \left.\left. -3h_t^4\right]\log\frac{\Lambda^2}{\mu^2}\right\} \\ & +\frac{1}{2}m^2\phi^2\left[1 + \frac{12\lambda}{32\pi^2}\log\frac{\Lambda^2}{\mu^2}\right], \end{aligned}$$

or

$$
\begin{aligned}
V(\phi) \quad \simeq & \quad \frac{1}{4}\phi^4 \left[\lambda + (\beta_\lambda + 4\lambda\gamma)\log\frac{\Lambda^2}{\mu^2}\right] \\
& + \frac{1}{2}m^2\phi^2 \left[1 + (\gamma_m + 2\gamma)\log\frac{\Lambda^2}{\mu^2}\right].
\end{aligned}
$$

Now observe that

$$
\begin{aligned}
\lambda(\Lambda) &\simeq \lambda + \beta_\lambda \log\frac{\Lambda^2}{\mu^2} \\
m^2(\Lambda) &\simeq m^2\left(1 + \gamma_m \log\frac{\Lambda^2}{\mu^2}\right) \\
\phi^2(\Lambda) &\simeq \phi^2\left(1 + 2\gamma\log\frac{\Lambda^2}{\mu^2}\right),
\end{aligned}
$$

with $\lambda = \lambda(\mu)$, $m^2 = m^2(\mu)$, $\phi^2 = \phi^2(\mu)$. So

$$
V_{RG}(\phi) = \frac{1}{2}m^2(\Lambda)\phi^2(\Lambda) + \frac{1}{4}\lambda(\Lambda)\phi^4(\Lambda).
$$

We see that the stability condition for the potential is simply the positivity of the running coupling constant $\lambda(\Lambda)$ at large scales.

The stability condition can be translated into a lower limit for the Higgs boson mass.

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2}\Big[12\lambda^2 + \frac{3}{8}g^4 + \frac{3}{16}(g^2 + {g'}^2)^2$$
$$-3h_t^4 - 3\lambda g^2 - \frac{3}{2}\lambda(g^2 + {g'}^2) + 6\lambda h_t^2\Big].$$

This equation must be solved together with the one-loop renormalization group equations for gauge and Yukawa coupling constants, which in the standard model are given by

$$\frac{dg}{dt} = \frac{1}{32\pi^2}\left(-\frac{19}{6}g^3\right)$$
$$\frac{dg'}{dt} = \frac{1}{32\pi^2}\frac{41}{6}{g'}^3$$
$$\frac{dg_S}{dt} = \frac{1}{32\pi^2}(-7g_S^3)$$
$$\frac{dh_t}{dt} = \frac{1}{32\pi^2}\left[\frac{9}{2}h_t^3 - \left(8g_S^2 + \frac{9}{4}g^2 + \frac{17}{12}{g'}^2\right)h_t\right]$$

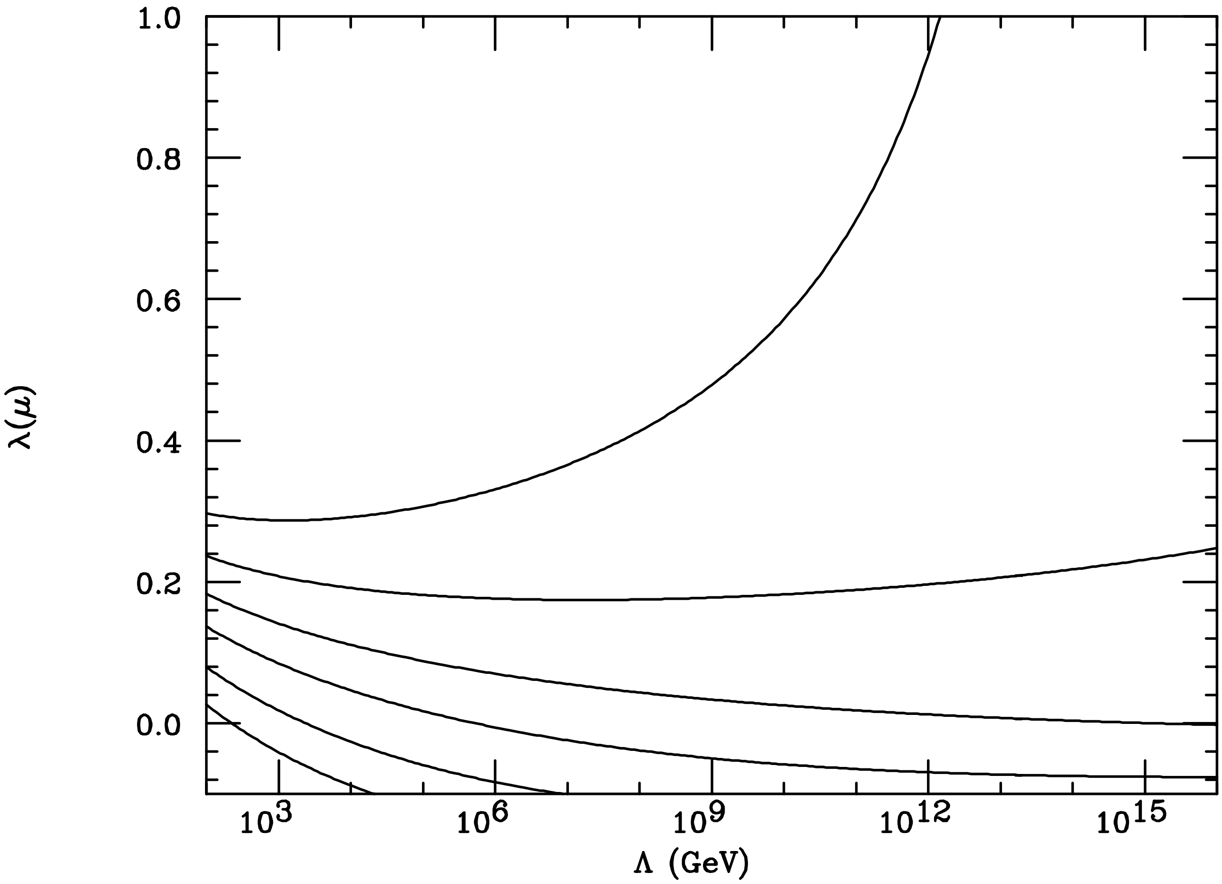


Figure 1:

The running coupling constant $\lambda(\mu)$ for different values $\lambda(m_{\rm z})$ corresponding to $m_H = 60, 100, 130, 150, 190$ GeV.

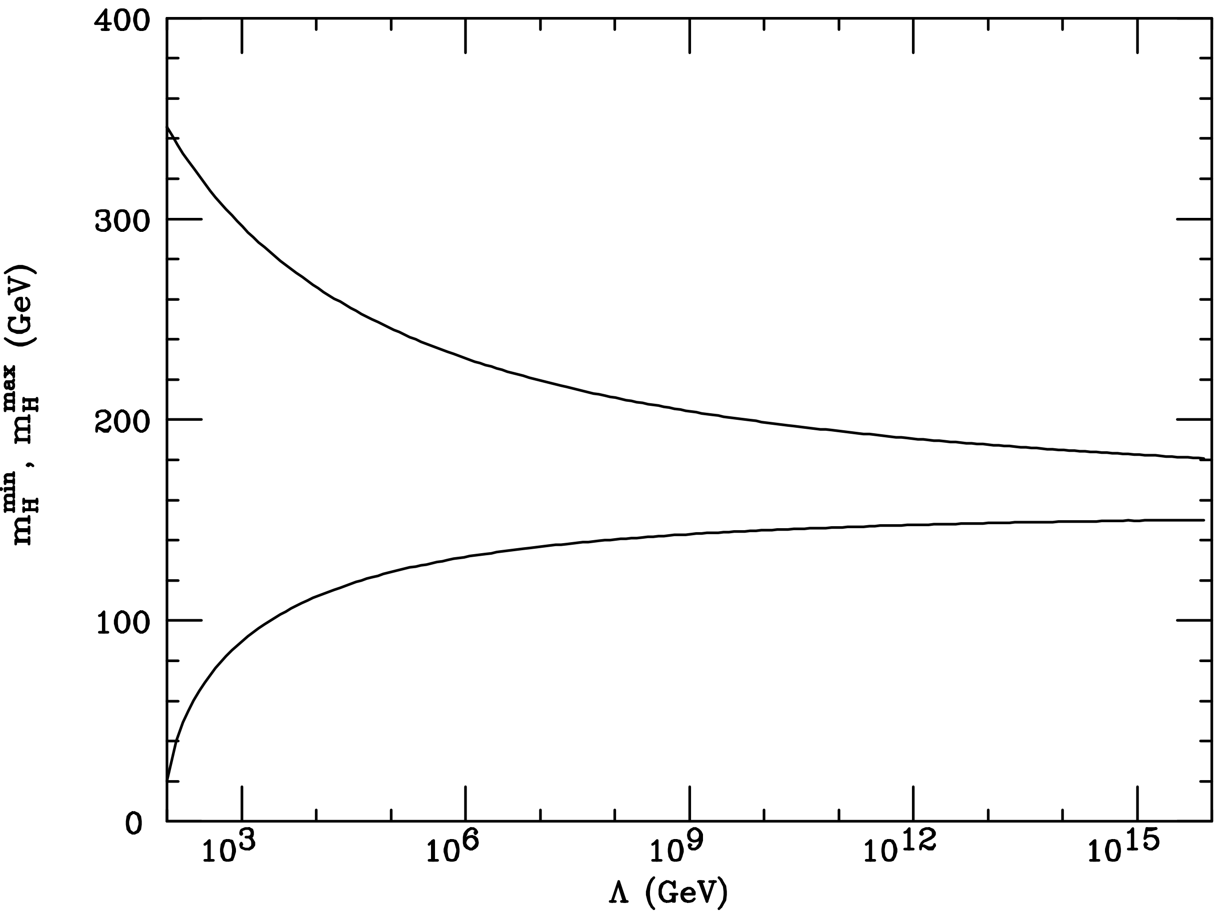


Figure 2:

Theoretical upper and lower bounds on the Higgs mass.

Anomalies

QED with one massive fermion, ψ. Consider the operators

$$J_V^\mu = \overline{\psi}\gamma^\mu\psi$$
$$J_A^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi$$
$$J_P = \overline{\psi}\gamma_5\psi.$$

It is easy to show, using the equations of motion, that

$$\partial_\mu J_V^\mu = 0; \qquad \partial_\mu J_A^\mu = 2imJ_P.$$

Now consider the Green functions

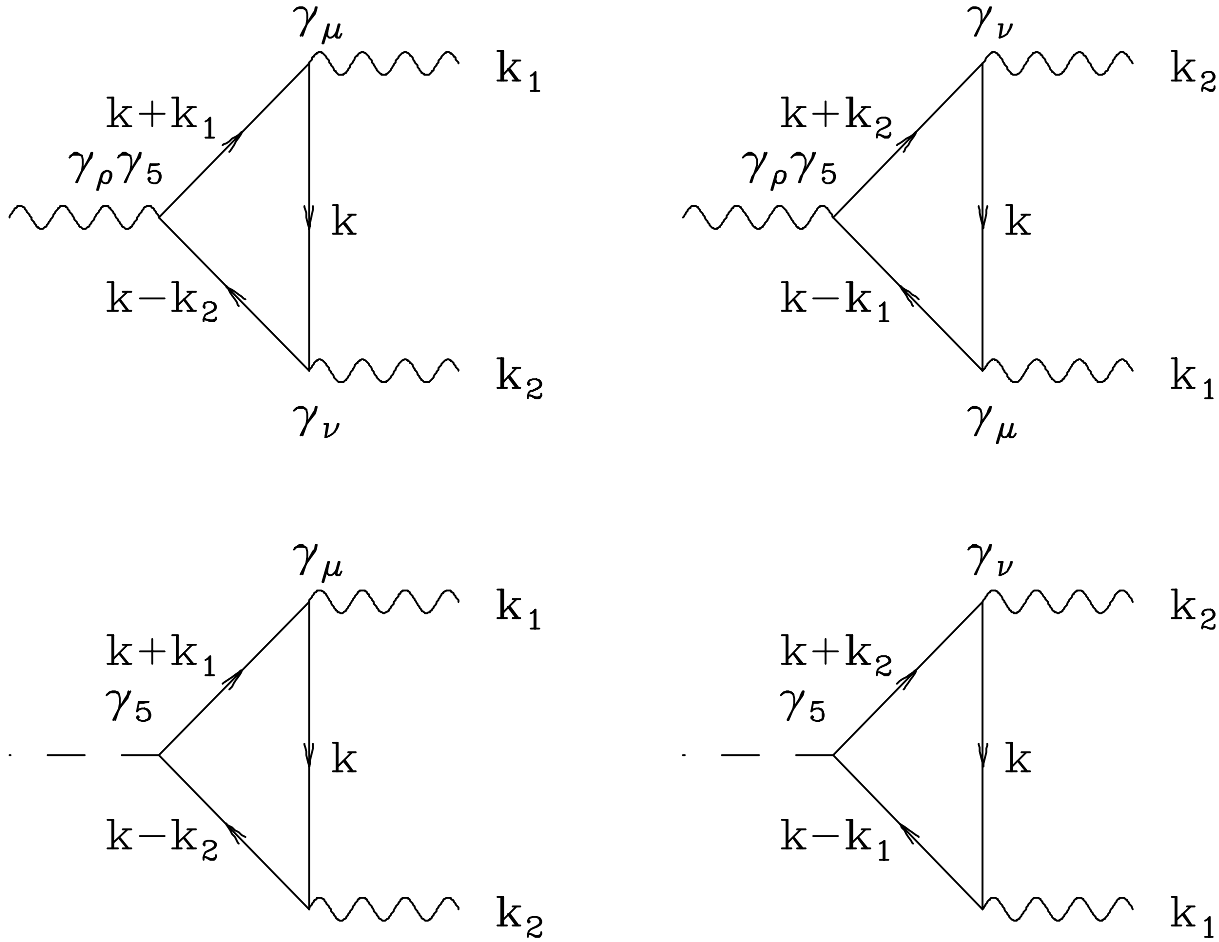
$$\begin{aligned} T^{\mu\nu\rho}(k_1,k_2) &= i\int d^4x_1 d^4x_2 e^{ik_1x_1+ik_2x_2} \\ &\quad < 0|T[J_V^\mu(x_1)J_V^\nu(x_2)J_A^\rho(0)]|0 > \\ T^{\mu\nu}(k_1,k_2) &= i\int d^4x_1 d^4x_2 e^{ik_1x_1+ik_2x_2} \\ &\quad < 0|T[J_V^\mu(x_1)J_V^\nu(x_2)J_P(0)]|0 > . \end{aligned}$$

They formally satisfy the Slavnov-Taylor identities

$$k_1^\mu T_{\mu\nu\rho} = k_2^\nu T_{\mu\nu\rho} = 0$$
$$q^\rho T_{\mu\nu\rho} = 2m T_{\mu\nu},$$

where $q = k_1 + k_2$.

Are they satisfied in perturbation theory? The answer is not obviously yes, because of regularization procedures. At one loop



We have

$$T^{\mu\nu\rho}(k_1,k_2) = T_1^{\mu\nu\rho}(k_1,k_2) + T_2^{\mu\nu\rho}(k_1,k_2)$$
$$T^{\mu\nu}(k_1,k_2) = T_1^{\mu\nu}(k_1,k_2) + T_2^{\mu\nu}(k_1,k_2),$$

where

$$T_1^{\mu\nu\rho} = -i\int \frac{d^4k}{(2\pi)^4} Tr\left[\frac{i}{\hat{k}+\hat{k}_1-m}\gamma^\rho\gamma_5\frac{i}{\hat{k}-\hat{k}_2-m}\gamma^\nu\frac{i}{\hat{k}-m}\gamma^\mu\right]$$
$$T_1^{\mu\nu} = -i\int \frac{d^4k}{(2\pi)^4} Tr\left[\frac{i}{\hat{k}+\hat{k}_1-m}\gamma_5\frac{i}{\hat{k}-\hat{k}_2-m}\gamma^\nu\frac{i}{\hat{k}-m}\gamma^\mu\right]$$

and

$$T_2^{\mu\nu\rho}(k_1,k_2) = T_1^{\nu\mu\rho}(k_2,k_1)$$
$$T_2^{\mu\nu}(k_1,k_2) = T_1^{\nu\mu}(k_2,k_1).$$

The overall minus sign is due to the presence of a fermion loop.

Conservation of the vector current:

$$\hat{k}_1 = (\hat{k} + \hat{k}_1 - m) - (\hat{k} - m) \;\; \text{in} \;\; T_1^{\mu\nu\rho}$$
$$\hat{k}_1 = (\hat{k} - m) - (\hat{k} - \hat{k}_1 - m) \;\; \text{in} \;\; T_2^{\mu\nu\rho}$$

$$\left[k_1^{\mu} T_{\mu\nu\rho}\right]_M$$
$$= -i \int \frac{d^4k}{(2\pi)^4} Tr \left[\frac{i}{\hat{k} + \hat{k}_1 - m} \gamma_\rho \gamma_5 \frac{i}{\hat{k} - \hat{k}_2 - m} \gamma_\nu \frac{i}{\hat{k} - m} \hat{k}_1 \right.$$
$$\left. + \frac{i}{\hat{k} + \hat{k}_2 - m} \gamma_\rho \gamma_5 \frac{i}{\hat{k} - \hat{k}_1 - m} \hat{k}_1 \frac{i}{\hat{k} - m} \gamma_\nu - (m \to M) \right]$$
$$= \int \frac{d^4k}{(2\pi)^4}$$
$$Tr \left[\gamma_\rho \gamma_5 \frac{i}{\hat{k} - \hat{k}_2 - m} \gamma_\nu \frac{i}{\hat{k} - m} - \frac{i}{\hat{k} + \hat{k}_1 - m} \gamma_\rho \gamma_5 \frac{i}{\hat{k} - \hat{k}_2 - m} \gamma_\nu \right.$$
$$+ \frac{i}{\hat{k} + \hat{k}_2 - m} \gamma_\rho \gamma_5 \frac{i}{\hat{k} - \hat{k}_1 - m} \gamma_\nu - \frac{i}{\hat{k} + \hat{k}_2 - m} \gamma_\rho \gamma_5 \frac{i}{\hat{k} - m} \gamma_\nu$$
$$\left. - (m \to M) \right]$$

Now, shifting $k \to k + k_2$ in the first term and shifting $k \to k - k_1 + k_2$ in the second one, they cancel against the fourth and second terms, respectively. We have therefore

$$[k_1^\mu T_{\mu\nu\rho}]_M = 0,$$

and also

$$[k_2^\nu T_{\mu\nu\rho}]_M = 0$$

by an analogous argument. The limit $M \to \infty$ can then be taken safely, thus obtaining the announced results.

We may use a similar procedure to check the identity for the axial current. Using

$$\hat{q}\gamma_5 = 2m\gamma_5 + (\hat{k} + \hat{k}_1 - m)\gamma_5 + \gamma_5(\hat{k} - \hat{k}_2 - m)$$

and

$$\hat{q}\gamma_5 = 2m\gamma_5 + (\hat{k} + \hat{k}_2 - m)\gamma_5 + \gamma_5(\hat{k} - \hat{k}_1 - m)$$

in $q_\rho T_1^{\mu\nu\rho}$ and $q_\rho T_2^{\mu\nu\rho}$ respectively (and making similar replacements in the terms with $m \rightarrow M$), we get

$$[q_\rho T^{\mu\nu\rho}]_M = [2mT^{\mu\nu}]_M + [R^{\mu\nu}]_M \,,$$

where

$$R^{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} Tr$$

$$\left[\frac{i}{\hat{k} + \hat{k}_1 - m}\gamma_5\gamma^\nu \frac{i}{\hat{k} - m}\gamma^\mu - \frac{i}{\hat{k} - \hat{k}_2 - m}\gamma_5\gamma^\nu \frac{i}{\hat{k} - m}\gamma^\mu \right.$$

$$\left. + \frac{i}{\hat{k} + \hat{k}_2 - m}\gamma_5\gamma^\mu \frac{i}{\hat{k} - m}\gamma^\nu - \frac{i}{\hat{k} - \hat{k}_1 - m}\gamma_5\gamma^\mu \frac{i}{\hat{k} - m}\gamma^\nu \right].$$

It is now easy to see that $[R^{\mu\nu}]_M = 0$. Therefore,

$$[q_\rho T^{\mu\nu\rho}]_M = [2mT^{\mu\nu}]_M \,.$$

Let us now compute $[2mT^{\mu\nu}]_M$ explicitly. We find ($k_1^2 = k_2^2 = 0$)

$$\begin{aligned} [2mT_{\mu\nu}]_M &= \frac{1}{\pi^2}\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma \\ &\quad \int_0^1 dx \int_0^{1-x} dy \left[\frac{m^2}{m^2 - q^2xy} - \frac{M^2}{M^2 - q^2xy}\right]. \end{aligned}$$

Notice that the RHS is finite when $M \to \infty$. The limit can now be taken safely, giving

$$q^\rho T_{\mu\nu\rho} = 2mT_{\mu\nu} - \frac{1}{2\pi^2}\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma .$$

or equivalently

$$\partial_\mu J_A^\mu = 2imJ_P + \frac{1}{(4\pi)^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$$

Non-abelian gauge theories: the condition for anomaly cancellation is

$$Tr(\{T^a, T^b\}T^c) = 0$$

In the standard model

$$Tr(\{\tau^a, \tau^b\}\tau^c) = 2\delta^{ab}Tr(\tau^c) = 0.$$

Since $\tau^a = 0$ for right-handed fermions, we have

$$Tr(\{\tau^a, \tau^b\}Y) = 2\delta^{ab}Tr(Y_L),$$

Since $Y = 1/3$ for the doublets of left-handed quarks, and $Y = -1$ for the doublets of left-handed leptons, we find

$$Tr(Y_L) = n_q \times 3 \times 2 \times \frac{1}{3} + n_l \times 2 \times (-1) = 2(n_q - n_l),$$

Trivially

$$Tr(Y^2\tau^c) = 0$$

Finally we must prove that $Tr(Y^3) = 0$. To show this, it is convenient to write the axial current as

$$\bar{\psi}\gamma^\mu\gamma_5\psi = \bar{\psi}\gamma^\mu\frac{1}{2}(1+\gamma_5)\psi - \bar{\psi}\gamma^\mu\frac{1}{2}(1-\gamma_5)\psi.$$

In this way, it is clear that left-handed fermions and right-handed fermions contribute to the axial anomaly with opposite signs. We have therefore

$$Tr(Y^3) = Tr(Y_L^3) - Tr(Y_R^3).$$

Using $Y = 2(Q - T_3)$ we find

$$Tr(Y_L^3) = 6n_q\left(\frac{1}{3}\right)^3 + 2n_l(-1)^3$$

$$Tr(Y_R^3) = 3n_q\left[\left(\frac{4}{3}\right)^3 + \left(-\frac{2}{3}\right)^3\right] + n_l(-2)^3,$$

and therefore

$$Tr(Y^3) = -6(n_q - n_l).$$

It is easy to prove that, because of the axial anomaly, the currents associated with the leptonic and barionic numbers,

$$L^\mu = \sum_{i=1}^{n_l} \left[\bar{e}_i \gamma^\mu e_i + \bar{\nu}_i \gamma^\mu \nu_i\right]$$

$$B^\mu = \frac{1}{3} \sum_{i=1}^{n_q} \left[\bar{u}_i \gamma^\mu u_i + \bar{d}_i \gamma^\mu d_i\right]$$

are anomalous. This results in a (numerically negligible) non-conservation of leptonic and barionic numbers L and B, due to instanton effects. The difference $B - L$ is however conserved.