On-Shell Methods in Field Theory

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Lecture I

Tools for Computing Amplitudes

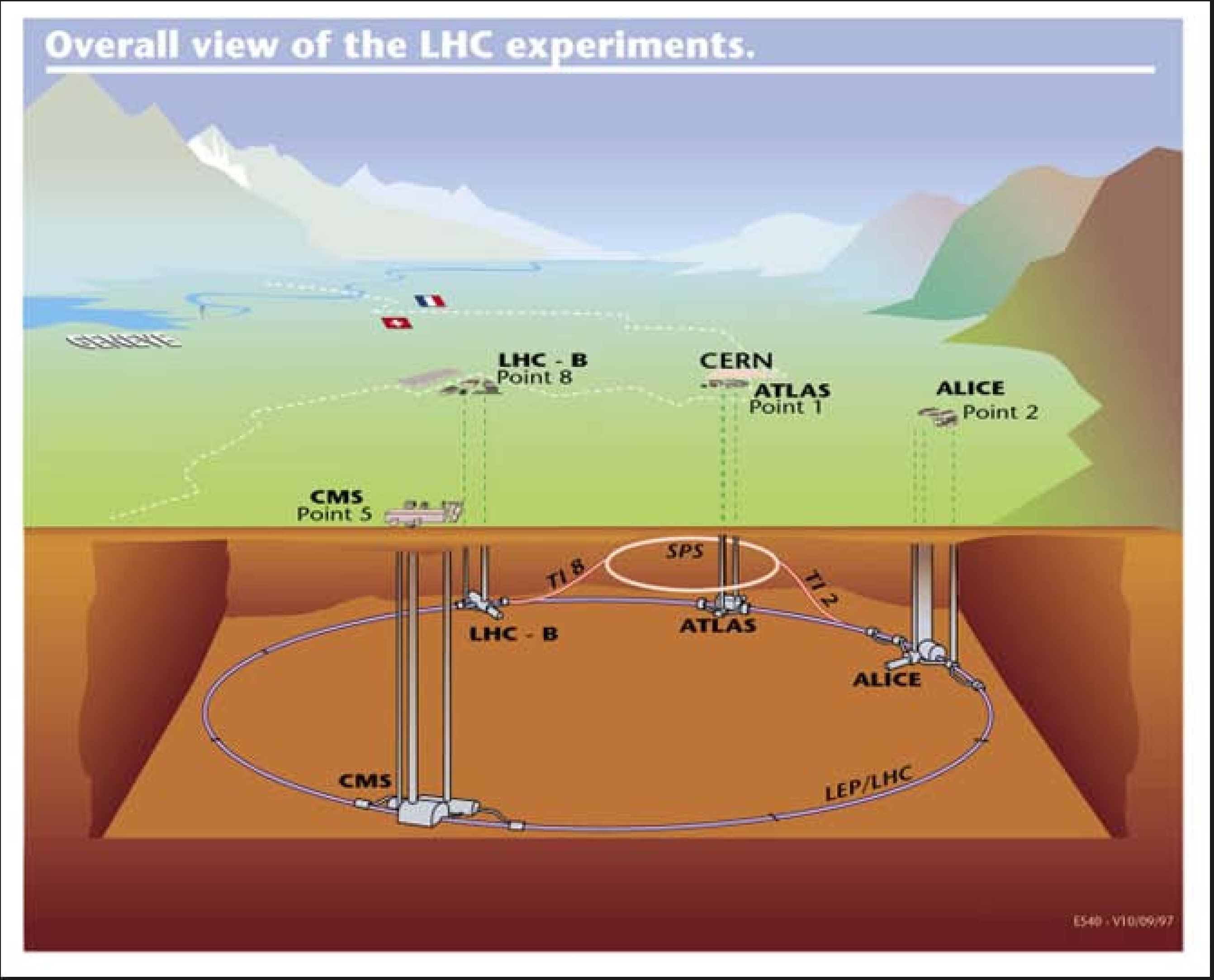
- Focus on gauge theories
  …but they are useful for gravity too

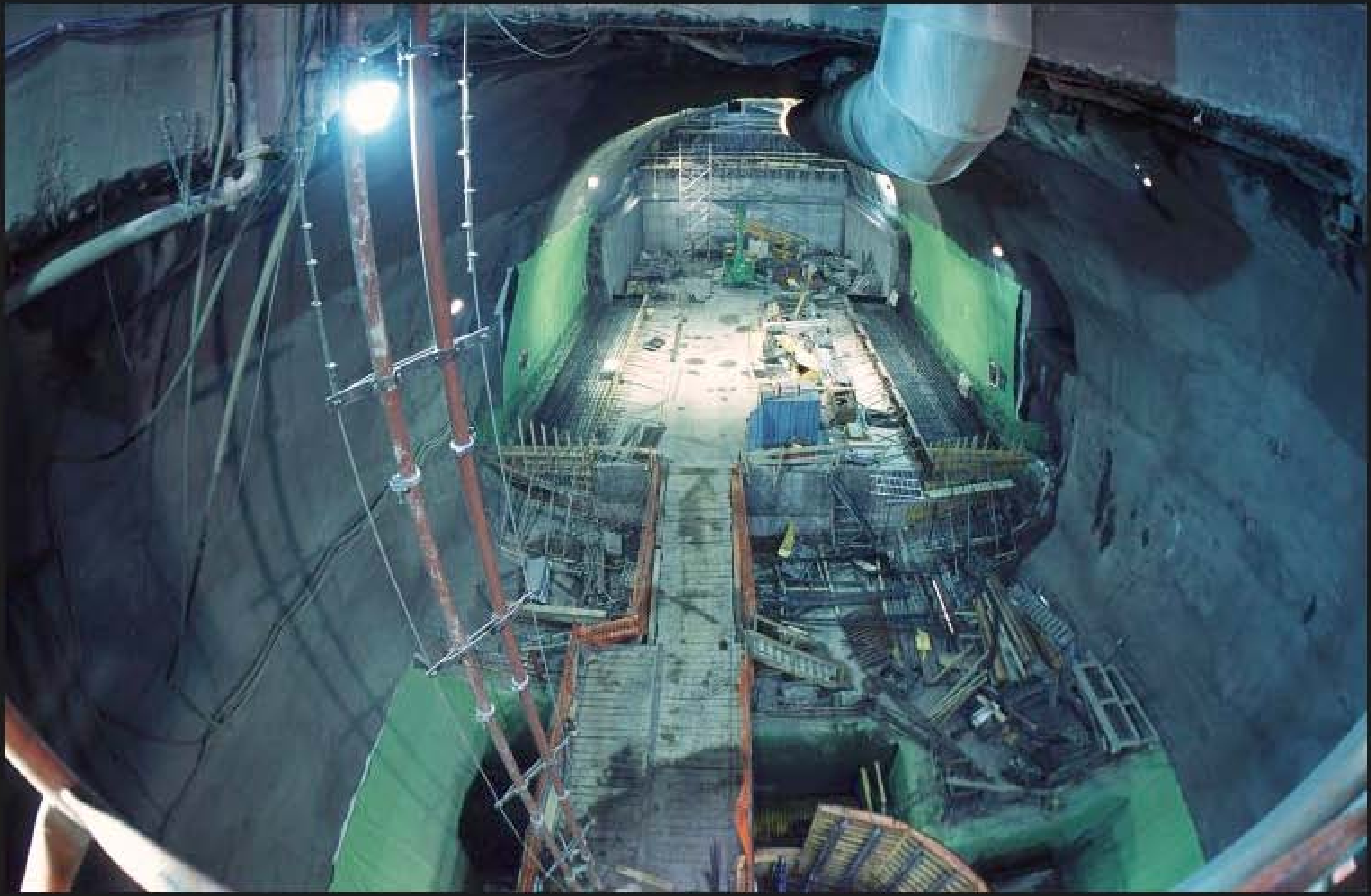
- Motivations and connections
  - Particle physics
  - $\mathcal{N}=4$ supersymmetric gauge theories and AdS/CFT
  - Witten's twistor string

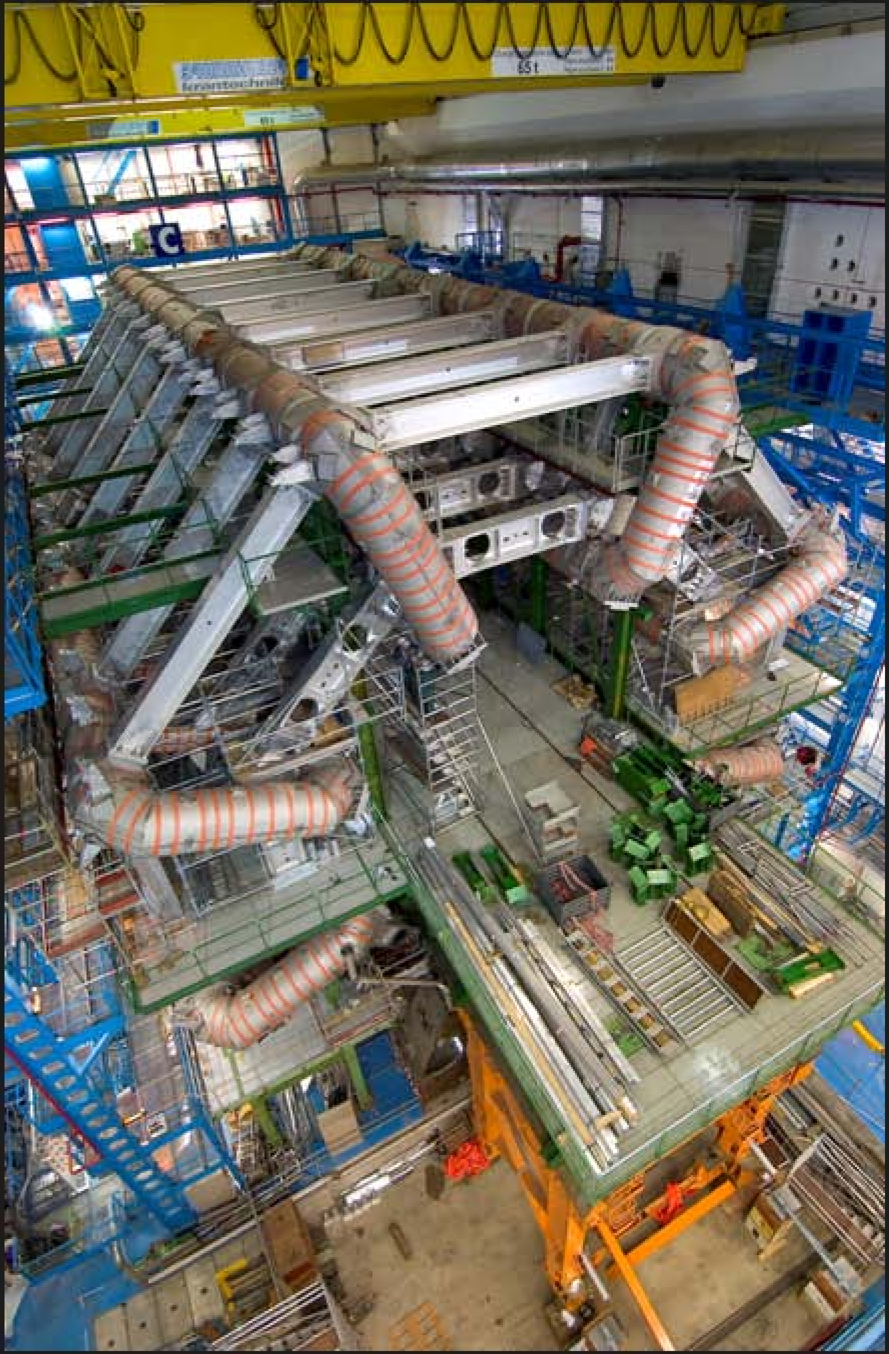
Particle Physics

- Why do we
- Why do we
- What quan

Now 450 to 600 days away…


Overall view of the LHC experiments.
LHC - B
Point 8
CERN
ATLAS
Point 1
ALICE
Point 2
CMS
Point 5
SPS
TI 8
TI 2
LHC - B
ATLAS
ALICE
CMS
LEP/LHC
E540 - V10/09/97

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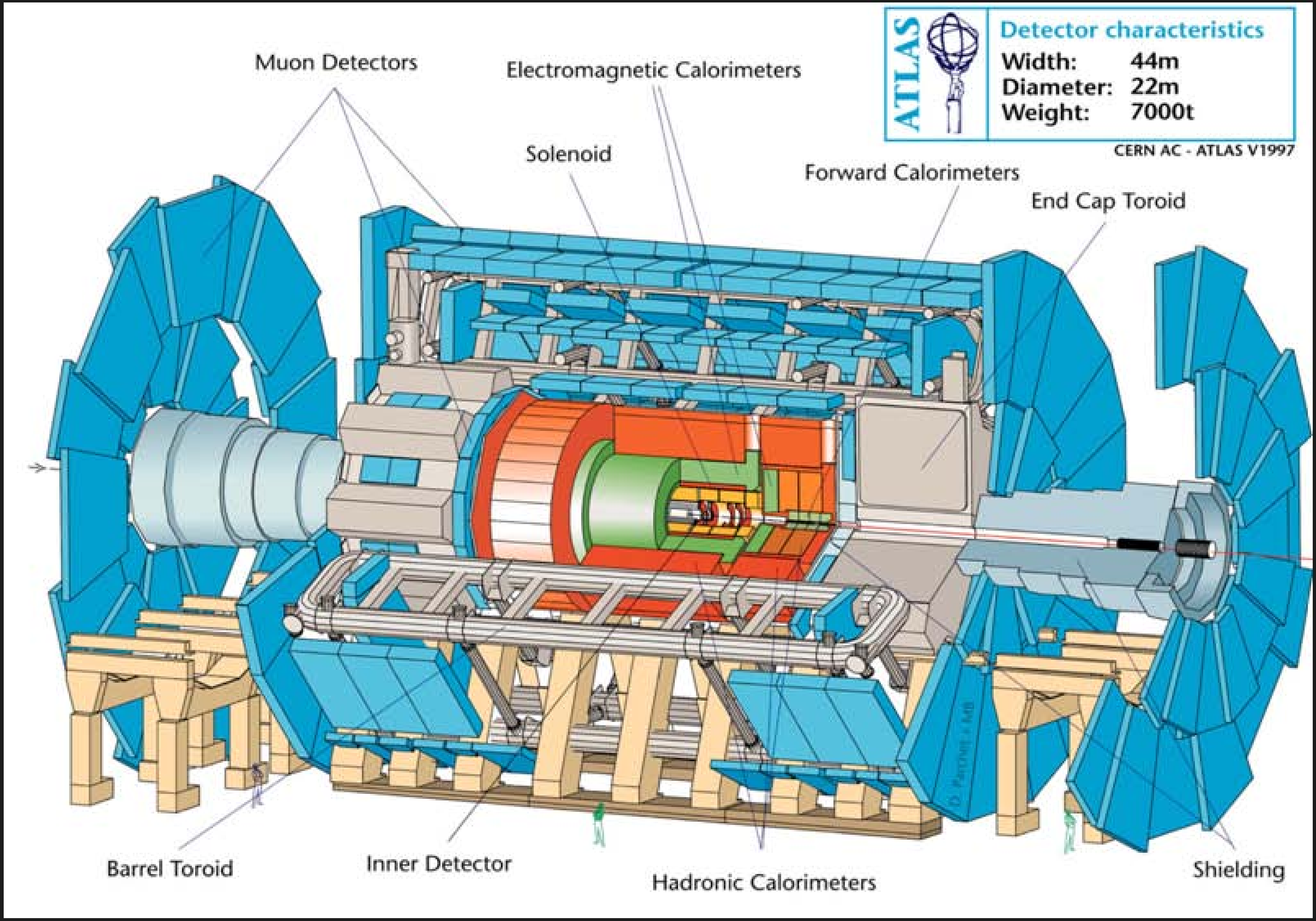
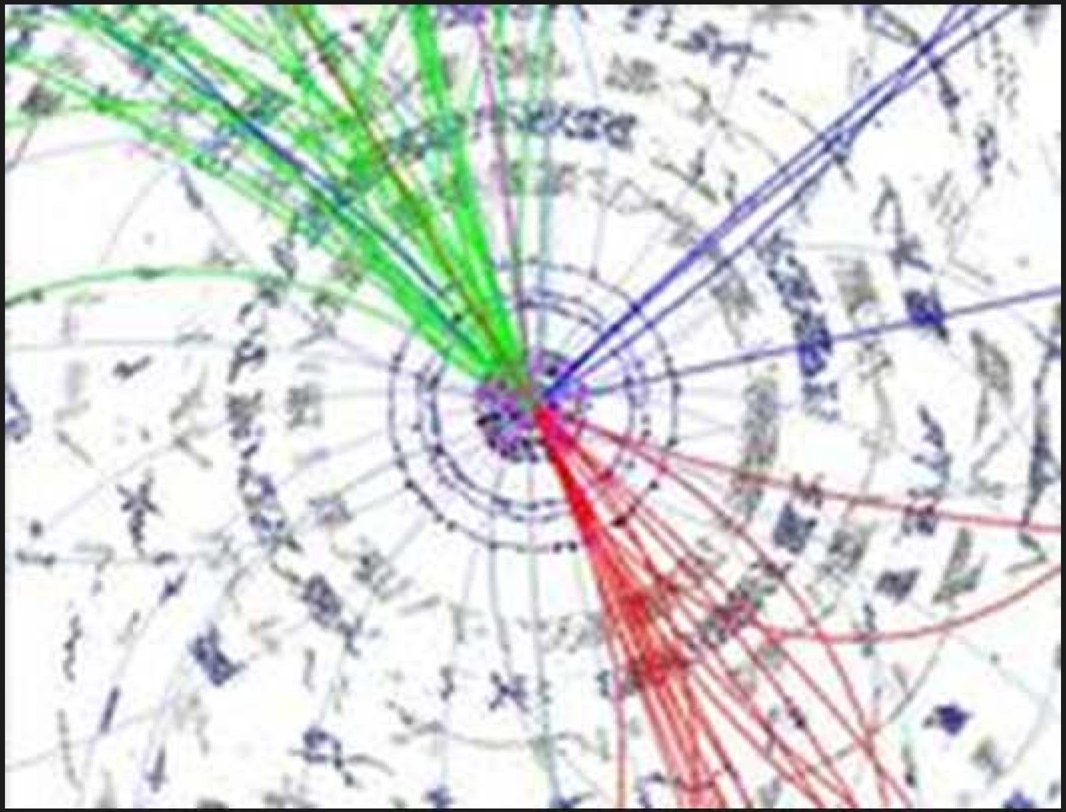
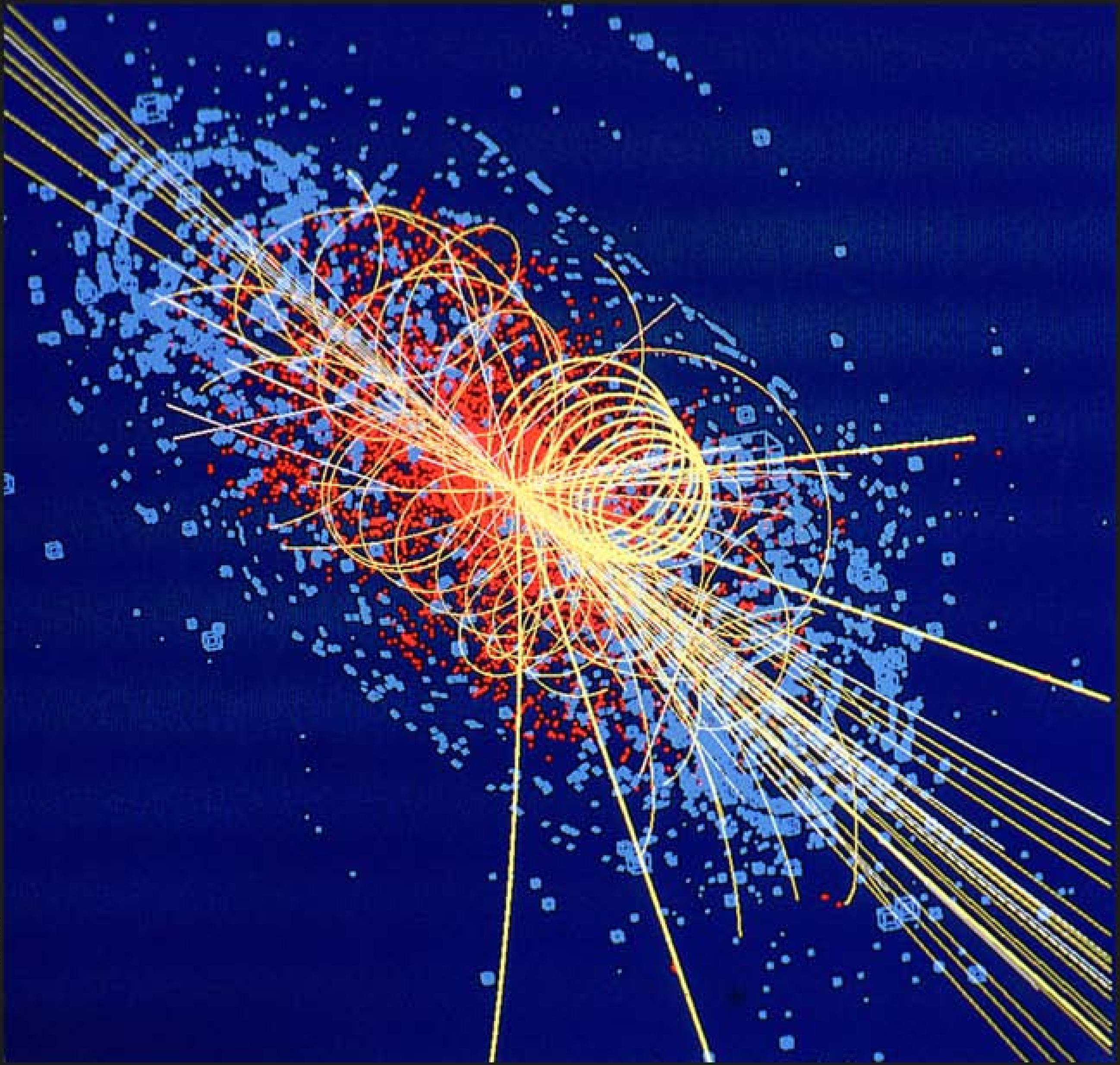
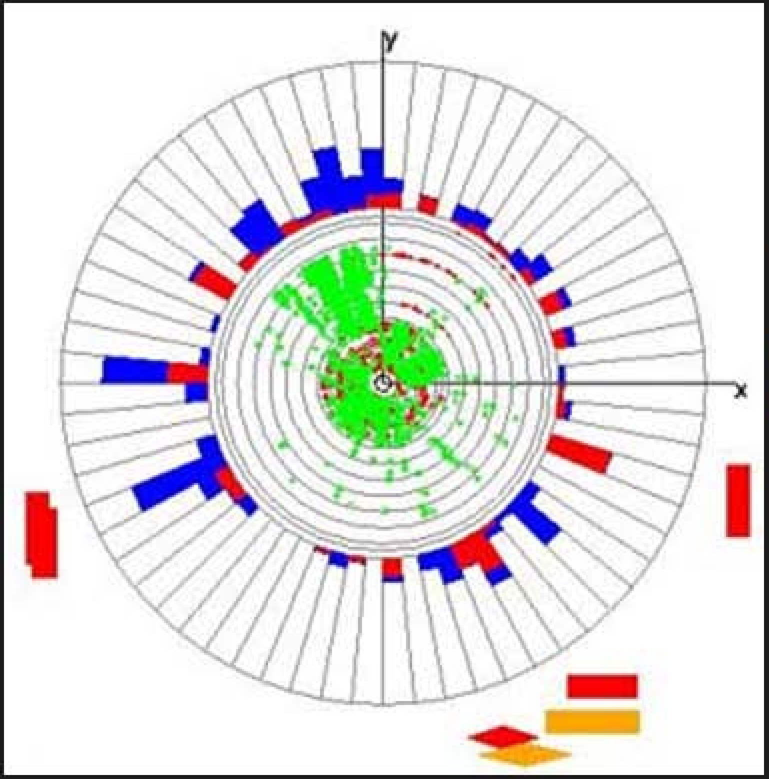

ATLAS
Detector characteristics
Width: 44m
Diameter: 22m
Weight: 7000t
CERN AC - ATLAS V1997
Muon Detectors
Electromagnetic Calorimeters
Solenoid
Forward Calorimeters
End Cap Toroid
Barrel Toroid
Inner Detector
Hadronic Calorimeters
Shielding

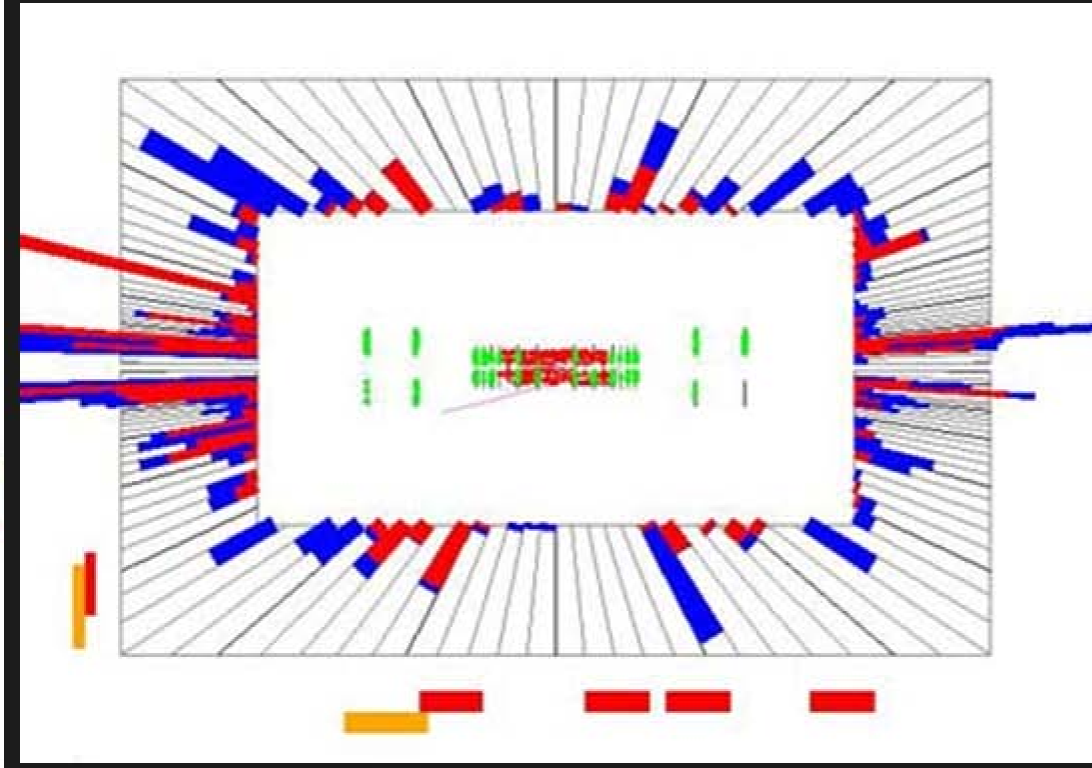


CDF event



CMS


y
x



$SU(3) \times SU(2) \times U(1)$ Standard Model

- Known physics, and background to new physics
- Hunting for new physics beyond the Standard Model
- Discovery of new physics
- Compare measurements to predictions — need to calculate signals
- Expect to confront backgrounds
- Backgrounds are large

Event rates

Event production rates at $L=10^{33}\ cm^{-2}\ s^{-1}$ and statistics to tape

Process	Events/s	Evts on tape, 10 fb^{-1}
$W \to e\nu$	15	10^8
$Z \to ee$	1	10^7
$t\bar{t}$	1	10^6
gluinos, m=1 TeV	0.001	10^3
Higgs, m=130 GeV	0.02	10^4
Minimum bias	10^8	10^7
$b\bar{b} \to \mu X$	10^3	10^7
QCD jets p_T>150 GeV/c	10^2	10^7

assuming 1% of trigger bandwidth

⇒ statistical error negligible after few days!

⇒ dominated by systematic errors (detector understanding, luminosity, theory)

Hunting for New Physics

- Yesterday's new physics is tomorrow's background
- To measure new physics, need to understand backgrounds in detail
- Heavy particles decaying into SM or invisible states
  - Often high-multiplicity events
  - Low multiplicity signals overwhelmed by SM:
    Higgs $\longrightarrow b\bar{b} \longrightarrow$ 2 jets

- Predicting backgrounds requires precision calculations of known Standard Model physics

- Complexity is due to QCD

- Perturbative QCD:

  Gluons & quarks → gluons & quarks

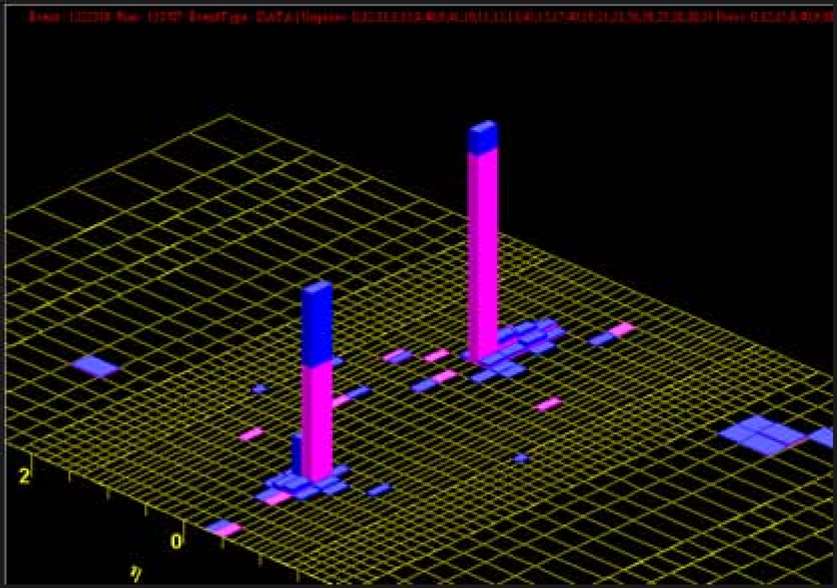
- Real world:

  Hadrons → hadrons with hard physics described by pQCD

- Hadrons → jets *narrow nearly collimated streams of hadrons*

Jets

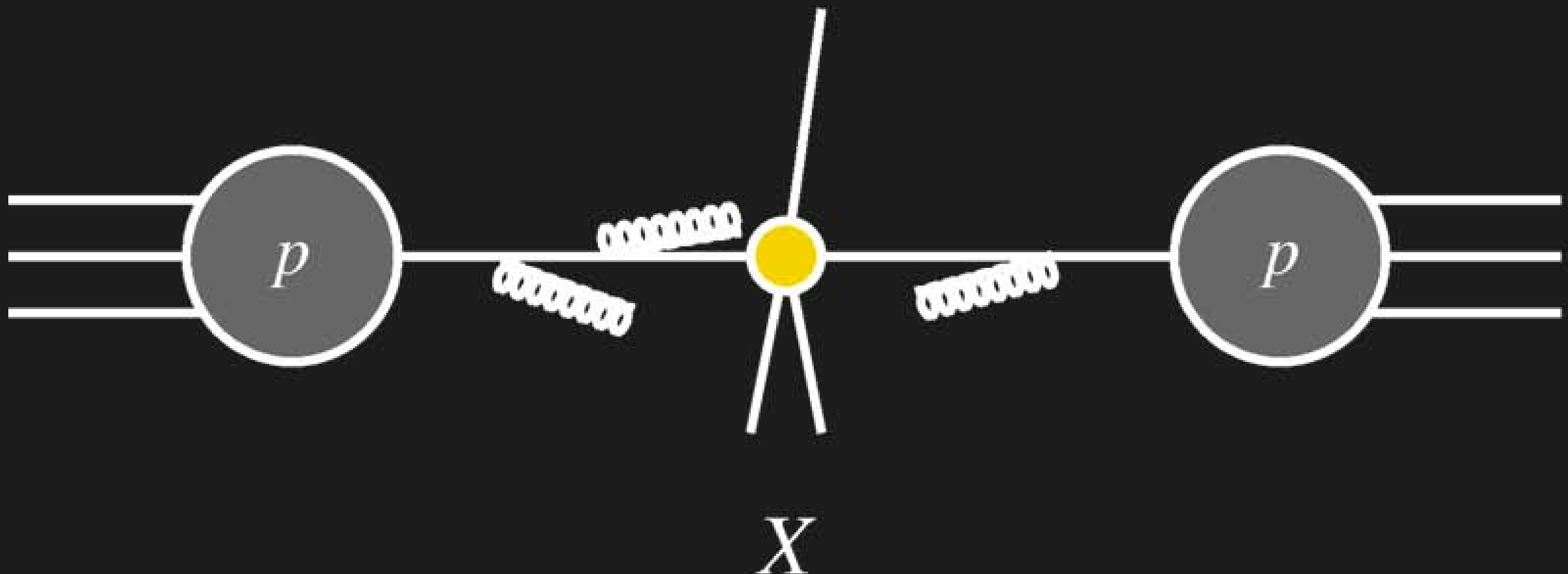
- Defined by an experimental resolution parameter
  - invariant mass in e^+e^-
  - cone algorithm in hadron colliders: cone size in $R = \sqrt{(\Delta\eta)^2 + (\Delta\phi)^2}$ and minimum E_T
  - k_T algorithm: essentially by a relative transverse momentum



CDF (Lefevre 2004)
1374 GeV

In theory, theory and practice are the same. In practice, they are different — Yogi Berra

QCD-Improved Parton Model



$$\int dx_a dx_b \, d\text{Phase} \; f_a f_b \, \sigma_{ab} \delta(v - \text{Observable})$$

The Challenge

- Everything at a hadron collider (signals, backgrounds, luminosity measurement) involves QCD
- Strong coupling is not small: $\alpha_s(M_Z) \approx 0.12$ and running is important
  - $\Rightarrow$ events have high multiplicity of hard clusters (jets)
  - $\Rightarrow$ each jet has a high multiplicity of hadrons
  - $\Rightarrow$ higher-order perturbative corrections are important
- Processes can involve multiple scales: $p_T(W)$ & M_W
  - $\Rightarrow$ need resummation of logarithms
- Confinement introduces further issues of mapping partons to hadrons, but for suitably-averaged quantities (infrared-safe) avoiding small E scales, this is not a problem (power corrections)

Approaches

- General parton-level fixed-order calculations
  - Numerical jet programs: general observables
  - Systematic to higher order/high multiplicity in perturbation theory
  - Parton-level, approximate jet algorithm; match detector events only statistically
- Parton showers
  - General observables
  - Leading- or next-to-leading logs only, approximate for higher order/high multiplicity
  - Can hadronize & look at detector response event-by-event
- Semi-analytic calculations/resummations
  - Specific observable, for high-value targets
  - Checks on general fixed-order calculations

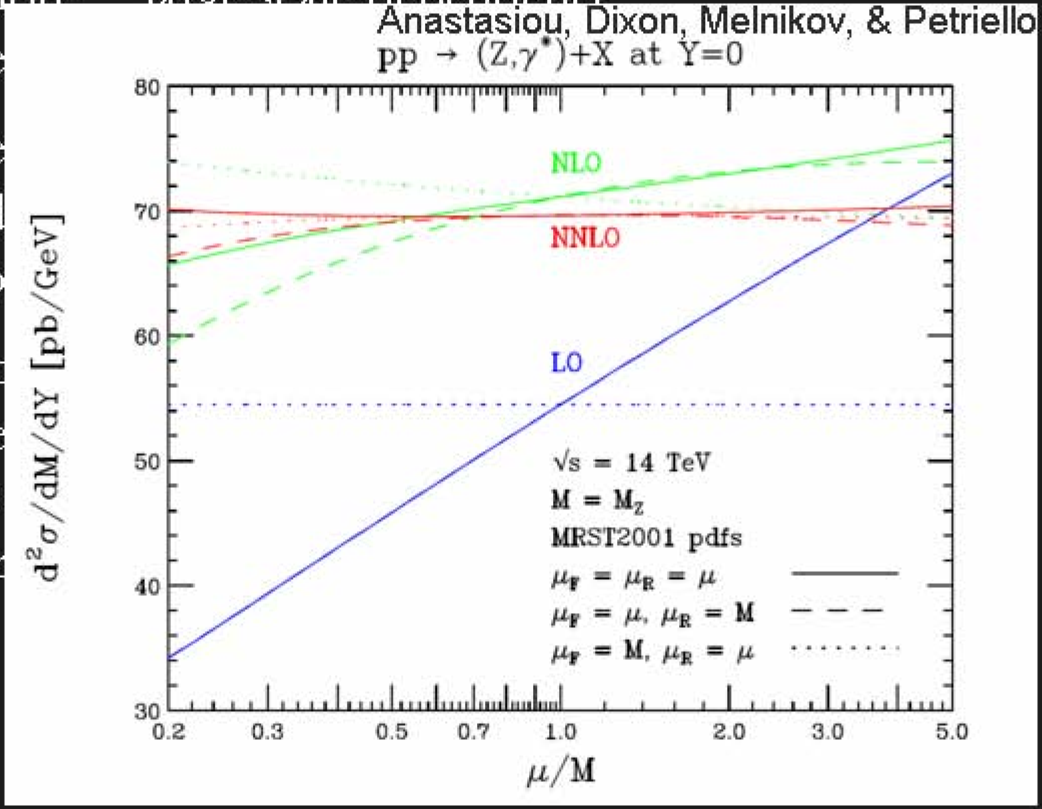
Precision Perturbative QCD

- Predictions of signals, signals+jets
- Predictions of backgrounds
- Measurement of luminosity
- Measurement of fundamental parameters (α_s, m_t)
- Measurement of electroweak parameters
- Extraction of parton distributions — ingredients in any theoretical prediction

Everything at a hadron collider involves QCD

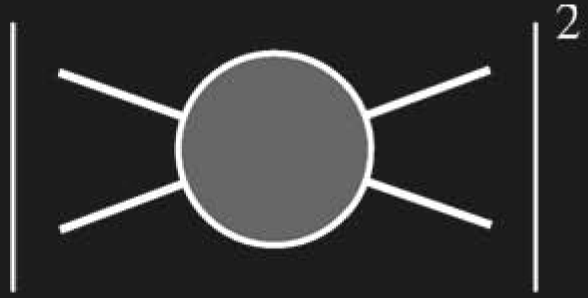
Leading-Order, Next-to-Leading Order

- LO: Basic shapes of distributions
  but: no quantitative predictio— large scale dependence
  missing sensitivity to je
- NLO: First quantitative pre
  improved scale depend
  basic approximation to
- NNLO: Precision predictic
  small scale dependence
  better correspondence
  understanding of theor

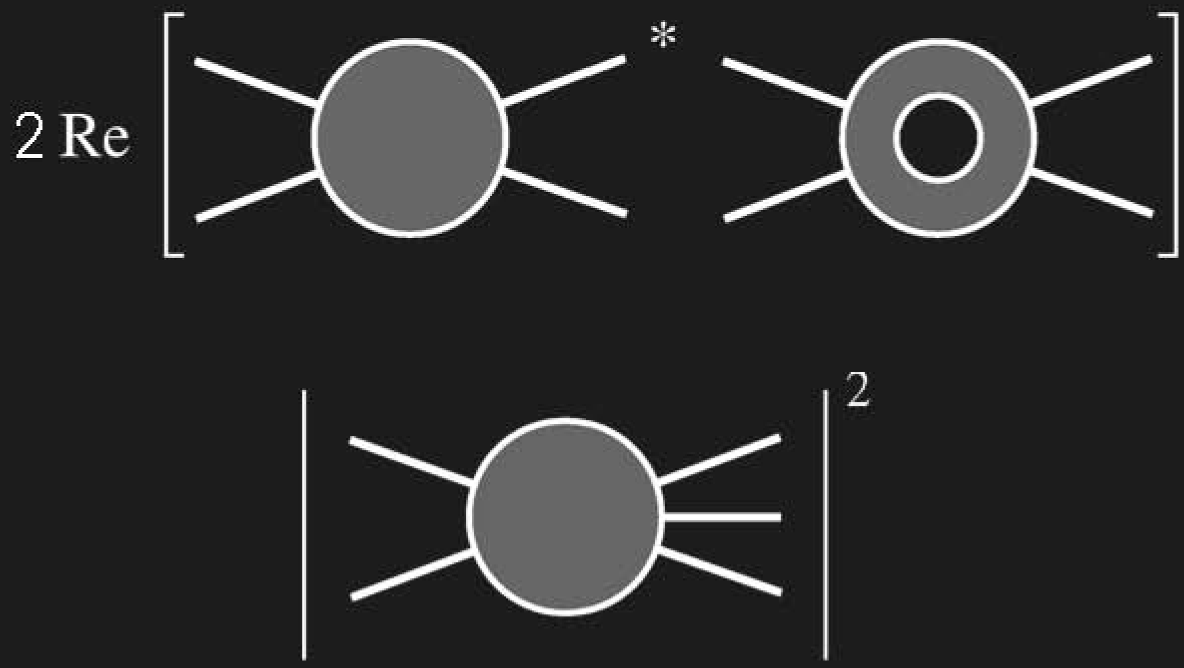


What Contributions Do We Need?

- Short-distance matrix elements to 2-jet production at leading order: tree level



- Short-distance matrix elements to 2-jet production at next-to-leading order: tree level + one loop + real emission



Real-Emission Singularities

$$e^+e^- \longrightarrow q\bar{q}g$$

Matrix element

$$|\mathcal{M}|^2 \propto \frac{(2-y_{qg})^2 + (2-y_{\bar{q}g})^2}{y_{qg}\,y_{\bar{q}g}}$$

Integrate

$$\int_{0\le y_{qg}+y_{\bar{q}g}\le 1} dy_{qg}\,dy_{\bar{q}g}\;|\mathcal{M}|^2$$

$$y_{qg} = 0 = y_{\bar{q}g} \qquad \text{soft}$$

$$\left.\begin{array}{l} y_{qg} = 0\,,\, y_{\bar{q}g} \neq 0 \\ y_{qg} \neq 0\,,\, y_{\bar{q}g} = 0 \end{array}\right. \qquad \text{collinear}$$

- Physical quantities are finite
- Depend on resolution parameter
- Finiteness thanks to combination of Kinoshita–Lee–Nauenberg theorem and factorization

Scattering

Scattering matrix element

$$ {}_{\text{out}}\langle p_1 p_2 \cdots \mid k_a k_b \rangle_{\text{in}} \equiv {}_{\text{asymp}}\langle p_1 p_2 \cdots \mid S \mid k_a k_b \rangle_{\text{asymp}} $$

Decompose it $S = 1 + iT$

Invariant matrix element $\mathscr{M}$

$$ \langle p_1 p_2 \cdots \mid iT \mid k_a k_b \rangle = (2\pi)^4 \delta(k_a + k_b - P) i\mathcal{M}(k_a, k_b \to \{p_f\}) $$

Differential cross section

$$ d\sigma \text{ `='} \; \frac{1}{4E_a E_b |v_a - v_b|} \prod_f \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} $$
$$ \times (2\pi)^4 \delta^4(k_a + k_b - P) |\mathcal{M}(k_a, k_b \to \{p_f\})|^2 $$

Lorentz-invariant phase-space measure

$$\int \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i}\,F(p_i) = \int \frac{d^4p_i}{(2\pi)^4}\,(2\pi)\,\Theta(p_i^0)\delta(p_i^2)$$

$$\Theta(p_i^0)\delta(p_i^2) \to \delta^{(+)}(p_i^2)$$

Compute invariant matrix element by crossing

$$\begin{aligned}\mathcal{M}(k_a,k_b \to \{p_f\}) &= \mathcal{M}(0 \to -k_a,-k_b,\{p_f\})\\ &= \mathcal{F}\,\langle\Omega \mid T\phi_a(x_a)\phi_b(x_b)\phi_1(x_1)\cdots \mid \Omega\rangle\end{aligned}$$

Lagrangian

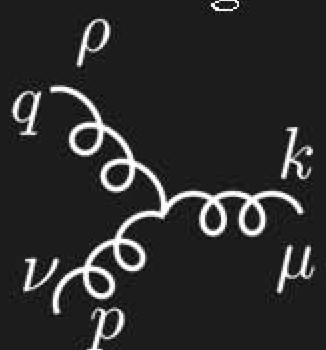
$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{4}\left(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu\right)\left(\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu}\right) \\
& \sum_f \overline{q}(i\not{\partial})q + \frac{g}{\sqrt{2}} G^a_\mu \sum_f \overline{q}\gamma^\mu T^a q \\
& - \frac{g}{2} f^{abc}\left(\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu}\right) G^b_\mu G^c_\nu \\
& - \frac{g^2}{4} f^{abe} f^{cde} G^{a\mu} G^{b\nu} G^c_\mu G^d_\nu
\end{aligned}
$$

Feynman Rules

Propagator (like QED)

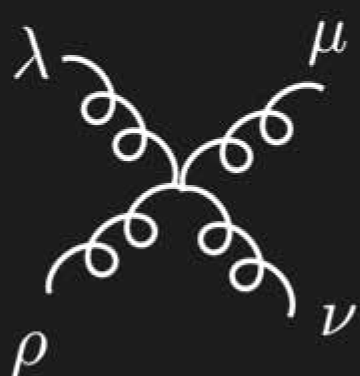
$$-\frac{i}{k^2+i\epsilon}\delta^{ab}\left(g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

Three-gluon vertex (unlike QED)



$$g f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right],$$

Four-gluon vertex (unlike QED)



$$\begin{aligned}-ig^2\Big[&f^{abe}f^{dce}(g^{\mu\rho}g^{\nu\lambda}-g^{\mu\lambda}g^{\nu\rho})\\ &+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\lambda}-g^{\mu\lambda}g^{\rho\nu})\\ &+f^{ace}f^{bde}(g^{\mu\nu}g^{\lambda\rho}-g^{\mu\rho}g^{\lambda\nu})\Big]\end{aligned}$$

From the Faddeev–Popov functional determinant

$$\det \partial \cdot D = \int [Dc][D\overline{c}] \exp\left\{ -i \int \overline{c}(\partial^\mu D_\mu) c \right\}$$

anticommuting scalars or *ghosts*

Propagator $\quad \dfrac{i}{k^2 + i\epsilon}\delta^{ab}$

coupling to gauge bosons $\quad g f^{abc} k^\mu$

So What's Wrong with Feynman Diagrams?

- Huge number of diagrams in calculations of interest
- But answers often turn out to be very simple
- Vertices and propagators involve gauge-variant off-shell states
- Each diagram is not gauge invariant — huge cancellations of gauge-noninvariant, redundant, parts in the sum over diagrams

Simple results should have a simple derivation — attr to Feynman

- Want approach in terms of physical states only

Light-Cone Gauge

Only physical (transverse) degrees of freedom propagate

$$-\frac{i}{k^2+i\epsilon}\delta^{ab}\left(g_{\mu\nu}-\frac{n^\mu k^\nu+n^\nu k^\mu}{n\cdot k}\right)$$

$n^2 = 0$

physical projector — two degrees of freedom

Color Decomposition

Standard Feynman rules ⇒ function of momenta, polarization vectors ε, and color indices

Color structure is predictable. Use representation

$$f^{abc} = -\frac{i}{\sqrt{2}}\,\mathrm{Tr}([T^a, T^b]T^c) \qquad \mathrm{Tr}(T^aT^b) = \delta^{ab}$$

to represent each term as a product of traces,

and the Fierz identity

$$(T^a)_{i_1}{}^{\bar{\imath}_1}(T^a)_{i_2}{}^{\bar{\imath}_2} = \delta_{i_1}{}^{\bar{\imath}_2}\delta_{i_2}{}^{\bar{\imath}_1} - \frac{1}{N}\delta_{i_1}{}^{\bar{\imath}_1}\delta_{i_2}{}^{\bar{\imath}_2}$$

To unwind traces

$$f^{abc}f^{cde} = -\frac{1}{2}\,\mathrm{Tr}([T^a,T^b]T^c)\,\mathrm{Tr}(T^c[T^d,T^e])$$
$$= -\frac{1}{2}\,\mathrm{Tr}([T^a,T^b][T^d,T^e]) + \frac{1}{2N}\,\mathrm{Tr}([T^a,T^b])\,\mathrm{Tr}([T^d,T^e])$$

Leads to tree-level representation in terms of single traces

$$\mathcal{A}_n^{\mathrm{tree}}(\{k_i,\varepsilon_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\cdots T^{a_{\sigma(n)}})$$
$$\times A_n^{\mathrm{tree}}(k_{\sigma(1)},\varepsilon_{\sigma(1)};k_{\sigma(2)},\varepsilon_{\sigma(2)};k_{\sigma(n)},\varepsilon_{\sigma(n)})$$

Color-ordered amplitude — function of momenta & polarizations alone; *not* Bose symmetric

Symmetry properties

- Cyclic symmetry

$$A_n^{\text{tree}}(1,\ldots,n) = A_n(2,\ldots,n,1)$$

- Reflection identity

$$A_n^{\text{tree}}(n,\ldots,1) = (-1)^n A_n(1,\ldots,n)$$

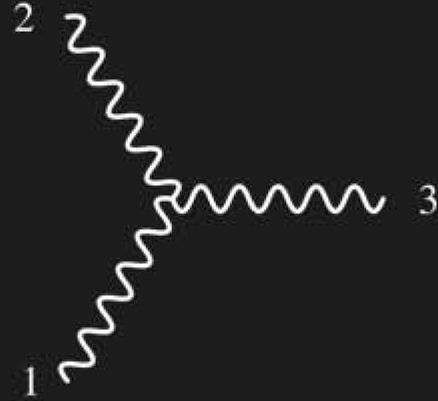
- Parity flips helicities

$$A_n^{\text{tree}}(1^{-\lambda_1},\ldots,n^{-\lambda_n}) = \left[A_n^{\text{tree}}(1^{\lambda_1},\ldots,n^{\lambda_n})\right]^\dagger$$

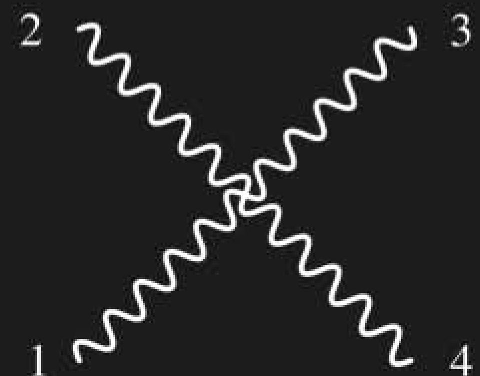
- Decoupling equation

$$A_{n+1}^{\text{tree}}(p,1,2,\ldots,n) + A_{n+1}^{\text{tree}}(1,p,2,\ldots,n) + A_{n+1}^{\text{tree}}(1,2,p,\ldots,n) + \cdots + A_{n+1}^{\text{tree}}(1,2,\ldots,p,n) = 0$$

Color-Ordered Feynman Rules



$$\frac{i}{\sqrt{2}}\left[\varepsilon_1 \cdot \varepsilon_2 \,(k_1 - k_2) \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 \,(k_2 - k_3) \cdot \varepsilon_1 + \varepsilon_3 \cdot \varepsilon_1 \,(k_3 - k_1) \cdot \varepsilon_2\right]$$



$$i\varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_2 \cdot \varepsilon_4 - \frac{i}{2}(\varepsilon_1 \cdot \varepsilon_2 \, \varepsilon_3 \cdot \varepsilon_4 + \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_1)$$

Amplitudes

Functions of momenta k, polarization vectors ε for gluons; momenta k, spinor wavefunctions u for fermions

Gauge invariance implies this is a redundant representation:

$\varepsilon \to k$: $A = 0$

Spinor Helicity

Spinor wavefunctions $|j^\pm\rangle \equiv u_\pm(k_j), \qquad \langle j^\pm| \equiv \overline{u_\pm}(k_j)\,.$

Introduce *spinor products*

$$\langle i\, j\rangle \equiv \langle i^-|j^+\rangle = \overline{u_-}(k_i)u_+(k_j)\,,$$

$$[i\, j] \equiv \langle i^+|j^-\rangle = \overline{u_+}(k_i)u_-(k_j)$$

Explicit representation

$$u_+(k) = \begin{pmatrix} \sqrt{k_+} \\ \sqrt{k_-}e^{i\phi_k} \end{pmatrix}, \quad u_-(k) = \begin{pmatrix} \sqrt{k_-}e^{-i\phi_k} \\ -\sqrt{k_+} \end{pmatrix}$$

where

$$e^{\pm i\phi_k} = \frac{k^1 \pm ik^2}{\sqrt{k_+k_-}}, \qquad k_\pm = k^0 \pm k^3$$

We then obtain the explicit formulæ

$$\langle i\, j\rangle = \sqrt{k_{i-}k_{j+}}e^{i\phi_{k_i}} - \sqrt{k_{i+}k_{j-}}e^{i\phi_{k_j}}\ ,$$

$$[i\, j] = \langle j\, i\rangle^* = \sqrt{k_{i+}k_{j-}}e^{-i\phi_{k_j}} - \sqrt{k_{i-}k_{j+}}e^{-i\phi_{k_i}} \qquad (k^0_{i,j} > 0)$$

otherwise $[j\, i] = \mathrm{sign}(k_i^0 k_j^0)\, \langle i\, j\rangle^*$

so that the identity $\langle i\, j\rangle\, [j\, i] = 2k_i \cdot k_j$ always holds

Introduce four-component representation

$$\begin{bmatrix} u_+(k) \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ u_-(k) \end{bmatrix}$$

corresponding to γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

in order to define spinor strings

$$\langle i^\pm|\gamma^\mu|j^\pm\rangle \leftrightarrow \langle i^\pm|\sigma^\mu|j^\pm\rangle \qquad \sigma^\mu = (1, \sigma^i)$$

Properties of the Spinor Product

- Antisymmetry $\langle j\, i\rangle = -\langle i\, j\rangle\,, \qquad [j\, i] = -[i\, j]$
- Gordon identity $\langle i^{\pm}|\gamma^{\mu}|i^{\pm}\rangle = 2k_i^{\mu}$
- Charge conjugation $\langle i^{-}|\,\gamma^{\mu}\,|j^{-}\rangle = \langle j^{+}|\,\gamma^{\mu}\,|i^{+}\rangle\,,$
- Fierz identity $\langle i^{-}|\,\gamma^{\mu}\,|j^{-}\rangle\,\langle p^{+}|\,\gamma_{\mu}\,|q^{+}\rangle = 2\,\langle i\, q\rangle\,[p\, j]$
- Projector representation $|i^{\pm}\rangle\langle i^{\pm}| = \frac{1}{2}(1\pm\gamma_5)\not{k}_i$
- Schouten identity $\langle i\, j\rangle\,\langle p\, q\rangle = \langle i\, q\rangle\,\langle p\, j\rangle + \langle i\, p\rangle\,\langle j\, q\rangle\,.$

Spinor-Helicity Representation for Gluons

Gauge bosons also have only ± physical polarizations

Elegant — and covariant — generalization of circular polarization

$$\varepsilon_\mu^+(k,q) = \frac{\langle q^- |\, \gamma_\mu \,| k^- \rangle}{\sqrt{2}\,\langle q\, k\rangle}\,, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q^+ |\, \gamma_\mu \,| k^+ \rangle}{\sqrt{2}\,[k\, q]}$$

Xu, Zhang, Chang (1984)

reference momentum q $\qquad q \cdot k \neq 0$

Transverse $\qquad k \cdot \varepsilon^\pm(k,q) = 0$

Normalized $\qquad \varepsilon^+ \cdot \varepsilon^- = -1, \qquad \varepsilon^+ \cdot \varepsilon^+ = 0$

What is the significance of q?

$$\varepsilon_\mu^+(k,q') = \frac{\langle q'^- |\, \gamma_\mu \,| k^- \rangle}{\sqrt{2}\,\langle q'\, k\rangle} = \frac{\langle q'^- |\, \gamma_\mu \not{k} \,| q^+ \rangle}{\sqrt{2}\,\langle q'\, k\rangle\,\langle k\, q\rangle}$$

$$= -\frac{\langle q'^- |\, \not{k}\gamma_\mu \,| q^+ \rangle}{\sqrt{2}\,\langle q'\, k\rangle\,\langle k\, q\rangle} + \frac{\sqrt{2}\,\langle q\, q'\rangle}{\langle q'\, k\rangle\,\langle k\, q\rangle}\, k_\mu$$

$$= \varepsilon_\mu^+(k,q) + \frac{\sqrt{2}\,\langle q\, q'\rangle}{\langle q'\, k\rangle\,\langle k\, q\rangle}\, k_\mu$$

Properties of the Spinor-Helicity Basis

Physical-state projector

$$\sum_{\sigma=\pm} \varepsilon_\mu^\sigma(k,q)\varepsilon_\nu^{\sigma*}(k,q) = \sum_{\sigma=\pm} \varepsilon_\mu^\sigma(k,q)\varepsilon_\nu^{-\sigma}(k,q) = -g_{\mu\nu} + \frac{q_\mu k_\nu + k_\mu q_\nu}{q\cdot k}$$

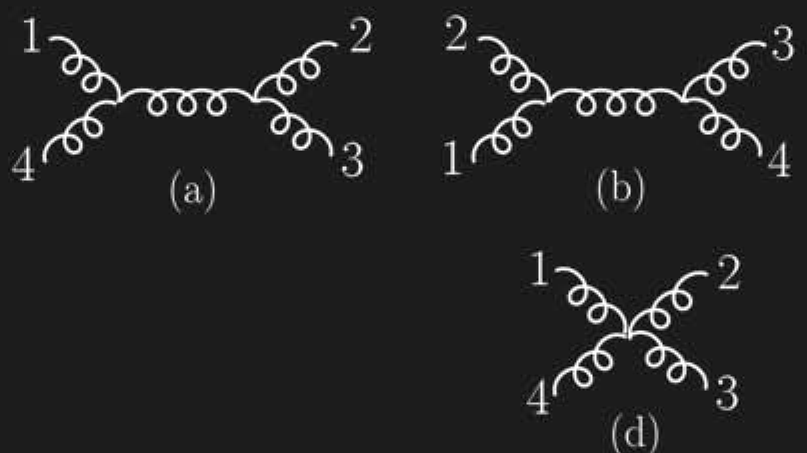
Simplifications

$$q\cdot\varepsilon^\pm(k,q) = 0\,,$$
$$\varepsilon^+(k_1,q)\cdot\varepsilon^+(k_2,q) = \varepsilon^-(k_1,q)\cdot\varepsilon^-(k_2,q) = 0\,,$$
$$\varepsilon^+(k_1,q)\cdot\varepsilon^-(k_2,k_1) = 0$$

Examples

By explicit calculation (or other arguments), every term in the gluon tree-level amplitude has at least one factor of $\varepsilon_i \cdot \varepsilon_j$

Look at four-point amplitude



Recall three-point color-ordered vertex

$$\frac{i}{\sqrt{2}}\left[\varepsilon_1 \cdot \varepsilon_2 \,(k_1 - k_2) \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 \,(k_2 - k_3) \cdot \varepsilon_1 + \varepsilon_3 \cdot \varepsilon_1 \,(k_3 - k_1) \cdot \varepsilon_2\right]$$

Calculate $A_4^{\text{tree}}(1^+, 2^+, 3^+, 4^+)$

choose identical reference momenta for all legs $\Rightarrow$ all $\varepsilon \cdot \varepsilon$ vanish

$\Rightarrow$ amplitude vanishes

Calculate $A_4^{\text{tree}}(1^-, 2^+, 3^+, 4^+)$

choose reference momenta 4,1,1,1 $\Rightarrow$ all $\varepsilon \cdot \varepsilon$ vanish

$\Rightarrow$ amplitude vanishes

Calculate $A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$

choose reference momenta 3,3,2,2

$\Rightarrow$ only nonvanishing $\varepsilon \cdot \varepsilon$ is $\varepsilon_1 \cdot \varepsilon_4$

$\Rightarrow$ only s_{12} channel contributes

$$
\begin{aligned}
&\left(\frac{i}{\sqrt{2}}\right)^2 \left(-\frac{ig^{\mu\nu}}{s_{12}}\right) \left[-2k_1 \cdot \varepsilon_2^- \varepsilon_{1\mu}^-\right] \left[2k_4 \cdot \varepsilon_3^+ \varepsilon_{4\nu}^+\right] \\
&\quad = -\frac{2i}{s_{12}} k_1 \cdot \varepsilon_2^- k_4 \cdot \varepsilon_3^+ \varepsilon_1^- \cdot \varepsilon_4^+ \\
&\quad = -\frac{i}{s_{12}} \left(\frac{[3\,1]\,\langle 1\,2\rangle}{[3\,2]}\right) \left(\frac{\langle 2\,4\rangle\,[4\,3]}{\langle 2\,3\rangle}\right) \left(\frac{\langle 2\,1\rangle\,[3\,4]}{\langle 2\,4\rangle\,[3\,1]}\right) \\
&\quad = -i\frac{\langle 1\,2\rangle^2\,[3\,4]^2}{s_{12}s_{23}} \\
&\quad = i\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}
\end{aligned}
$$

No diagrammatic calculation required for the last helicity amplitude,

$$A_4^{\text{tree}}(1^-,2^+,3^-,4^+)$$

Obtain it from the decoupling identity

$$
\begin{aligned}
&- A_4^{\text{tree}}(3^-,1^-,2^+,4^+) - A_4^{\text{tree}}(1^-,3^-,2^+,4^+)\\
&= i\frac{\langle 1\,3\rangle^3}{\langle 2\,4\rangle}\left(-\frac{1}{\langle 1\,2\rangle\,\langle 4\,3\rangle}+\frac{1}{\langle 3\,2\rangle\,\langle 4\,1\rangle}\right)\\
&= i\frac{\langle 1\,3\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}
\end{aligned}
$$

These forms hold more generally, for larger numbers of external legs:

$$A_n^{\rm tree}(1^+, 2^+, \ldots, n^+) = 0\,,$$
$$A_n^{\rm tree}(1^-, 2^+, \ldots, n^+) = 0$$

Parke-Taylor equations
Mangano, Xu, Parke (1986)

Maximally helicity-violating or 'MHV'

$$A_n^{\rm tree}(1^+, \ldots, m_1^-, (m_1+1)^+, \ldots, m_2^-, (m_2+1)^+, \ldots, n^+) =$$
$$i\frac{\langle m_1\, m_2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle n\,1\rangle}$$

Proven using the Berends–Giele recurrence relations

On-Shell Methods in Field Theory

David A. Kosower

International School of Theoretical Physics,
Parma,
September 10-15, 2006

Lecture II

Review of Lecture I

Color-ordered amplitude

$$\mathcal{A}_n^{\text{tree}}(\{k_i,\varepsilon_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\cdots T^{a_{\sigma(n)}})$$
$$\times A_n^{\text{tree}}(k_{\sigma(1)},\varepsilon_{\sigma(1)};k_{\sigma(2)},\varepsilon_{\sigma(2)};k_{\sigma(n)},\varepsilon_{\sigma(n)})$$

Color-ordered amplitude — function of momenta & polarizations alone; *not* Bose symmetric

Spinor-Helicity Representation for Gluons

Gauge bosons also have only ± physical polarizations

Elegant — and covariant — generalization of circular polarization

$$\varepsilon_\mu^+(k,q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2}\,\langle q\,k\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2}\,[k\,q]}$$

Xu, Zhang, Chang (preprint 1984); NPB291:392 (1987)

reference momentum q $\quad q \cdot k \neq 0$

Transverse $\quad k \cdot \varepsilon^\pm(k,q) = 0$

Normalized $\quad \varepsilon^+ \cdot \varepsilon^- = -1, \qquad \varepsilon^+ \cdot \varepsilon^+ = 0$

Parke-Taylor Equations

For any number of external legs:

$$A_n^{\text{tree}}(1^+, 2^+, \ldots, n^+) = 0\,,$$
$$A_n^{\text{tree}}(1^-, 2^+, \ldots, n^+) = 0$$

Parke & Taylor, PRL 56:2459 (1986)

Mangano, Xu, & Parke, NPB298:653 (1986)

Maximally helicity-violating or 'MHV'

$$A_n^{\text{tree}}(1^+, \ldots, m_1^-, (m_1+1)^+, \ldots, m_2^-, (m_2+1)^+, \ldots, n^+) =$$
$$i\frac{\langle m_1\, m_2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle (n-1)\, n\rangle\,\langle n\,1\rangle}$$

Proven using the Berends–Giele recurrence relations

Berends & Giele, NPB294:700 (1987)

Gauge-theory amplitude

$\downarrow$ Color decomposition & stripping

Color-ordered amplitude: function of k_i and ε_i

$\downarrow$ Spinor-helicity basis

Helicity amplitude: function of spinor products and helicities ± 1

Spinor products $\rightarrow$ spinors

Spinor Variables

From Lorentz vectors to bi-spinors

$$p_\mu \qquad \longleftrightarrow \qquad p_{a\dot{a}} \equiv p \cdot \sigma = \begin{pmatrix} p^0 + p^3 & p^1 + ip^2 \\ p^1 - ip^2 & p^0 - p^3 \end{pmatrix}$$

$$p^2 \qquad \longleftrightarrow \qquad \det(p)$$

$$p' = \Lambda p \qquad \longleftrightarrow \qquad p' = upu^\dagger, \qquad u \in SL(2, C)$$

2×2 complex matrices
with det = 1

Null momenta $p^2 = 0 \Longrightarrow \det(p) = 0$

can write it as a bispinor $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$

phase ambiguity in $\lambda_a, \tilde{\lambda}_{\dot{a}}$ (same as seen in spinor products)

For real Minkowski p, take $\tilde{\lambda} = \mathrm{sign}(p^0)\overline{\lambda}$

Invariant tensor ϵ_{ab}

$$u_{aa'} u_{bb'} \epsilon_{a'b'} = \det(u) = 1$$

gives spinor products

$$\langle \lambda_1, \lambda_2 \rangle = \epsilon_{ab} \lambda_1^a \lambda_2^b$$

$$[\tilde{\lambda}_1, \tilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$$

Connection to earlier spinor products

$$\langle \lambda_1, \lambda_2 \rangle = \langle 1\,2 \rangle$$
$$[\tilde\lambda_1, \tilde\lambda_2] = -[1\,2]$$

and spinor-helicity basis

$$+1: \qquad \varepsilon_{a\dot a} = \frac{\eta_a \tilde\lambda_{\dot a}}{\langle \eta, \lambda \rangle}$$
$$-1: \qquad \varepsilon_{a\dot a} = \frac{\lambda_a \tilde\eta_{\dot a}}{[\tilde\lambda, \tilde\eta]}$$

$\Rightarrow$ Amplitudes as functions of spinor variables $\lambda_a, \tilde\lambda_{\dot a}$ and helicities ± 1

Scaling of Amplitudes

Suppose we scale the spinors

also called 'phase weight'

$$\lambda_i \longmapsto \alpha_i \lambda_i,$$
$$\tilde{\lambda}_i \longmapsto \alpha_i^{-1} \tilde{\lambda}_i,$$

then by explicit computation we see that the MHV amplitude

$$A^{\mathrm{MHV}} \longmapsto i \frac{\alpha_{m_1}^2 \alpha_{m_2}^2}{\Pi_{j \neq m_1, m_2} \alpha_j^2} \frac{\langle m_1\, m_2 \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle (n-1)\, n \rangle \langle n\, 1 \rangle}$$

and that more generally

$$A \longmapsto \Pi_j \alpha_j^{-2h_j} A$$

For the non-trivial parts of the amplitude, we might as well use uniformly rescaled spinors ⇒ $\mathbb{CP}^1$ 'complex projective space'

Start with $\mathbb{C}^2$, and rescale all vectors by a common scale

$$\begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \equiv \tau \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}$$

the spinors are then 'homogeneous' coordinates on $\mathbb{CP}^1$

If we look at each factor in the MHV amplitude,

$$\frac{1}{\langle \lambda_1, \lambda_2 \rangle} = \frac{1}{\lambda_1^1 \lambda_2^1 (w_1 - w_2)} \qquad w_i = \lambda_i^2 / \lambda_i^1$$

we see that it is just a free-field correlator (Green function) on $\mathbb{CP}^1$

This is the essence of Nair's construction of MHV amplitudes as correlation functions on the 'line' = $\mathbb{CP}^1$

Gauge-theory amplitude

$\downarrow$ Color decomposition & stripping

Color-ordered amplitude: function of k_i and ε_i

$\downarrow$ Spinor-helicity basis

Helicity amplitude: function of spinor products and helicities ±1

$\downarrow$

Function of spinor variables and helicities ±1

$\downarrow$ Half-Fourier transform

Conjectured support on simple curves in twistor space

Let's Travel to Twistor Space!

It turns out that the natural setting for amplitudes is not exactly spinor space, but something similar. The motivation comes from studying the representation of the conformal algebra.

Half-Fourier transform of spinors: transform $\tilde{\lambda}_{\dot{a}}$, leave alone λ_a

$\Rightarrow$ Penrose's original twistor space, real or complex

$$\tilde{\lambda}_{\dot{a}} \to i\frac{\partial}{\partial \mu^{\dot{a}}}, \qquad -i\frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \to \mu_{\dot{a}}$$

Study amplitudes of definite helicity: introduce homogeneous coordinates $Z_I = (\lambda_a, \mu_a)$

$\Rightarrow$ $\boxed{\mathbf{CP}^3 \text{ or } \mathbf{RP}^3 \text{ (projective) twistor space}}$

Back to momentum space by Fourier-transforming μ

MHV Amplitudes in Twistor Space

Write out the half-Fourier transform including the energy-momentum conserving δ function

$$\widetilde{A}(Z) = \int \prod_j \frac{d^2\tilde{\lambda}_j}{(2\pi)^2}\; \exp[i\mu_{j\dot{a}}\tilde{\lambda}_j^{\dot{a}}]\; (2\pi)^4\delta^4(\Sigma_r\lambda_r^a\tilde{\lambda}_r^{\dot{a}})\; A^{\mathrm{MHV}}(\lambda_j)$$

$$\int d^4x\; \exp[\Sigma_j x_{a\dot{a}}\lambda_j^a\tilde{\lambda}_j^{\dot{a}}]$$

$$\widetilde{A}(Z) = \int d^4x \prod_j \frac{d^2\tilde{\lambda}_j}{(2\pi)^2}\; \exp[\Sigma_j i(\mu_{j\dot{a}}+x_{a\dot{a}}\lambda_j^a)\tilde{\lambda}_j^{\dot{a}}]\; A^{\mathrm{MHV}}(\lambda_j)$$

Result

$$\widetilde{A}(Z) = \int d^4x \prod_j \delta^2\left(\mu_{j\dot{a}} + x_{a\dot{a}}\lambda_j^a\right) A^{\mathrm{MHV}}(\lambda_j)$$

equation for a line

MHV amplitudes live on lines in twistor space

Value of the twistor-space amplitude is given by a correlation function on the line

Analyzing Amplitudes in Twistor Space

Amplitudes in twistor space turn out to be hard to compute directly. Even with computations in momentum space, the Fourier transforms are hard to compute explicitly.

We need other tools to analyze the amplitudes.

Simple 'algebraic' properties in twistor space — support on $\mathbb{CP}^1$s or $\mathbb{CP}^2$s — become differential properties in momentum space.

Construct differential operators.

Equation for a line ($\mathbb{CP}^1$): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K = 0$

gives us a differential ('line') operator in terms of momentum-space spinors

$$F_{123} = \langle \lambda_1 \, \lambda_2 \rangle \frac{\partial}{\partial \tilde{\lambda}_3} + \langle \lambda_2 \, \lambda_3 \rangle \frac{\partial}{\partial \tilde{\lambda}_1} + \langle \lambda_3 \, \lambda_1 \rangle \frac{\partial}{\partial \tilde{\lambda}_2}.$$

Equation for a plane ($\mathbb{CP}^2$): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K Z_4^L = 0$

also gives us a differential ('plane') operator

$$K_{1234} = \langle \lambda_1 \, \lambda_2 \rangle \frac{\partial}{\partial \tilde{\lambda}_{3\dot{a}}} \frac{\partial}{\partial \tilde{\lambda}_4^{\dot{a}}} + \text{ perms}$$

Properties

$$F_{ijl}\, f(p_i + p_j + p_l) = 0$$

$$F_{ijl}\, f(\{\lambda_r\}) = 0$$

$$K_{ijlm}\, f(\{\lambda_r\}) = 0$$

Thus for example

$$F_{ijl} \frac{\langle m_1\, m_2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle (n-1)\, n\rangle\,\langle n\,1\rangle} = 0$$

Beyond MHV

Witten's proposal: ph/0312171

- Each external particle represented by a point in twistor space
- Amplitudes non-vanishing only when points lie on a curve of degree d and genus g, where
  - $d = \#$ negative helicities $- 1 + \#$ loops
  - $g \leq \#$ loops; $g = 0$ for tree amplitudes
- Integrand on curve supplied by a topological string theory
- Obtain amplitudes by integrating over all possible curves $\Rightarrow$ moduli space of curves
- Can be interpreted as D_1-instantons

Strings in Twistor Space

- String theory can be defined by a two-dimensional field theory whose fields take values in target space:
  - n-dimensional flat space
  - 5-dimensional Anti-de Sitter × 5-sphere
  - twistor space: intrinsically four-dimensional $\Rightarrow$ Topological String Theory
- Spectrum in Twistor space is $\mathcal{N} = 4$ supersymmetric multiplet (gluon, four fermions, six real scalars)
- Gluons and fermions each have two helicity states

A New Duality

- String Theory ⟷ Gauge Theory

  Topological *B*-model on $\mathrm{CP}^{3|4}$ ⟷ $\mathcal{N}=4$ SUSY

  'Twistor space'

  Witten (2003); Berkovits & Motl; Neitzke & Vafa; Siegel (2004)

weak–weak

Simple Cases

Amplitudes with all helicities '+' $\Rightarrow$ degree -1 curves.
No such curves exist, so the amplitudes should vanish.
Corresponds to the first Parke–Taylor equation.

Amplitudes with one '$-$' helicity $\Rightarrow$ degree-0 curves: points.
Generic external momenta, all external points won't coincide (singular configuration, all collinear), $\Rightarrow$ amplitudes must vanish.
Corresponds to the second Parke–Taylor equation.

Amplitudes with two '$-$' helicities (MHV) $\Rightarrow$ degree-1 curves: lines.
All F operators should annihilate them, and they do.

Other Cases

Amplitudes with three negative helicities (next-to-MHV) live on conic sections (quadratic curves)

Amplitudes with four negative helicities (next-to-next-to-MHV) live on twisted cubics

Fourier transform back to spinors $\Rightarrow$ differential equations in conjugate spinors

Even String Theorists Can Do Experiments

- Apply F operators to NMHV $(3-)$ amplitudes: products annihilate them! K annihilates them;
- Apply F operators to N^2MHV $(4-)$ amplitudes: longer products annihilate them! Products of K annihilate them;

$$F_{512}F_{234}F_{345}F_{451}A_5(1^-,2^-,3^-,4^+,5^+) =$$

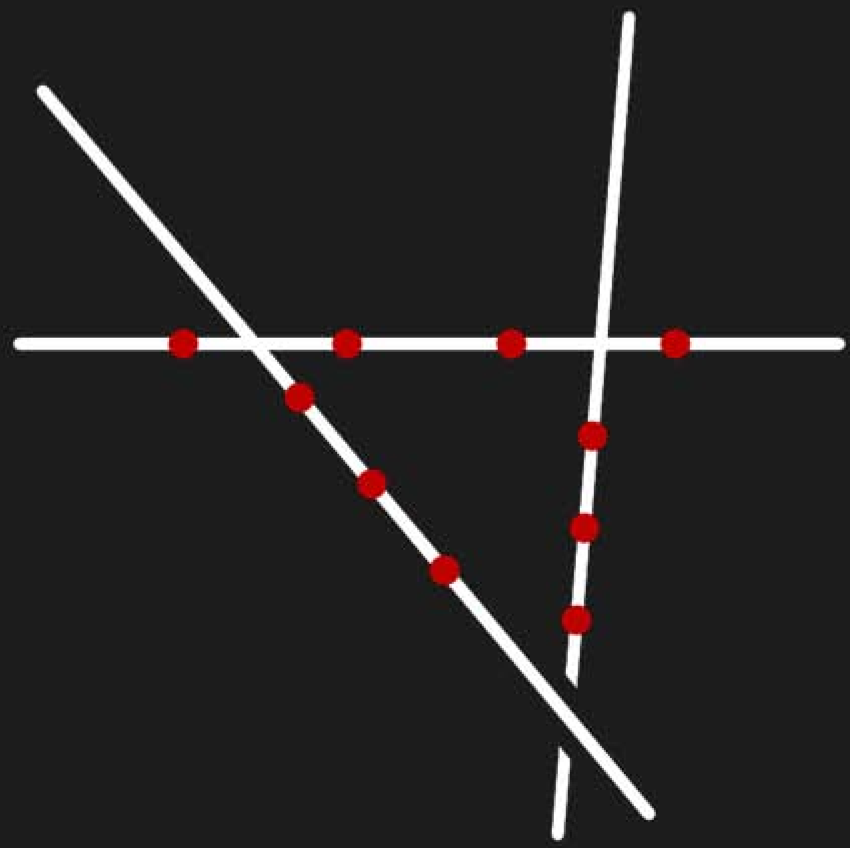
$$F_{512}F_{234}F_{345}F_{451}\frac{[4\,5]^4}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,1]} = 0$$

A more involved example

$$F_{612}F_{234}F_{345}F_{561}A_6(1^-,2^-,3^-,4^+,5^+,6^+)=0$$

Don't try this at home!

Interpretation: twistor-string amplitudes are supported on intersecting line segments

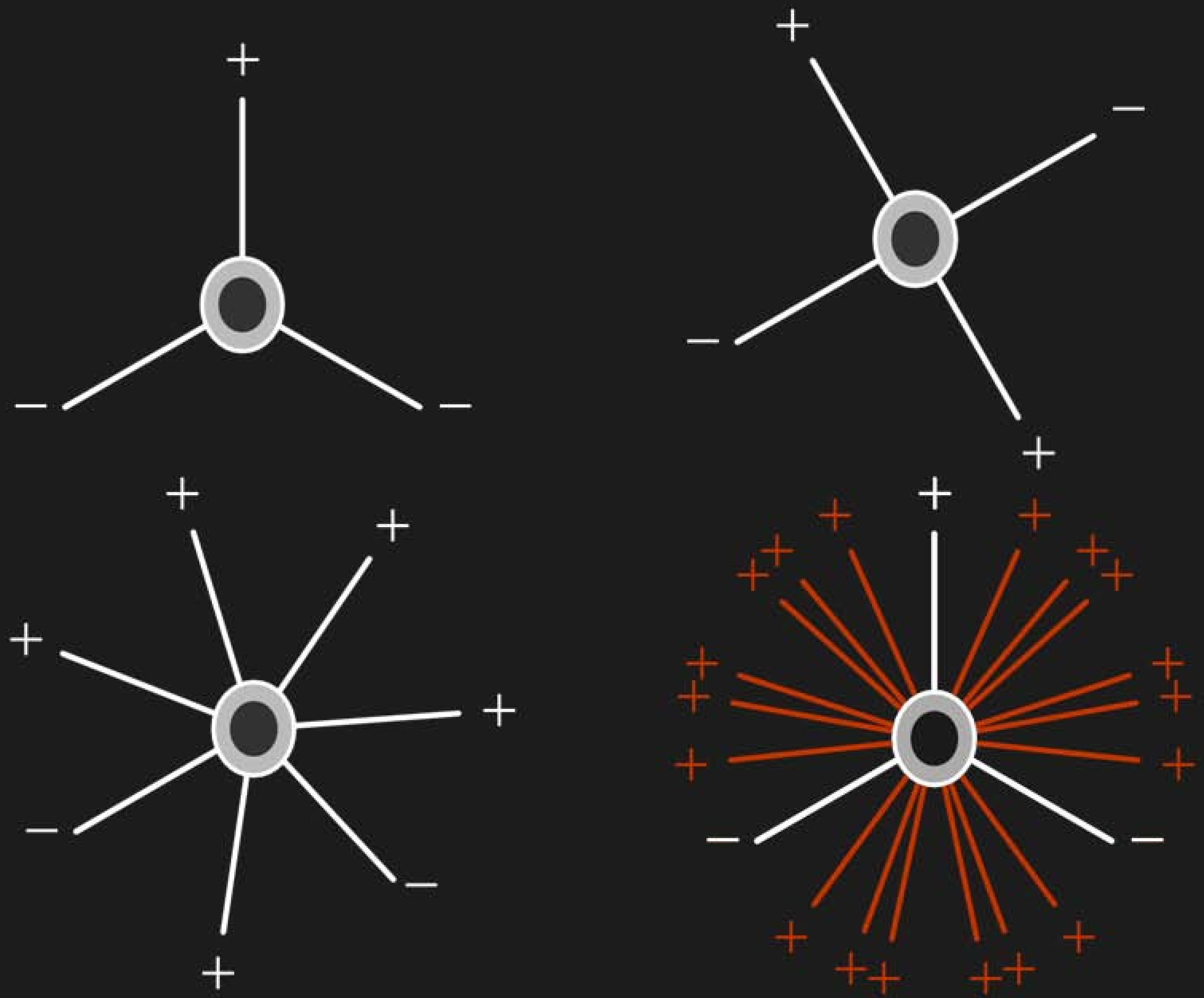


Simpler than expected: what does this mean in field theory?

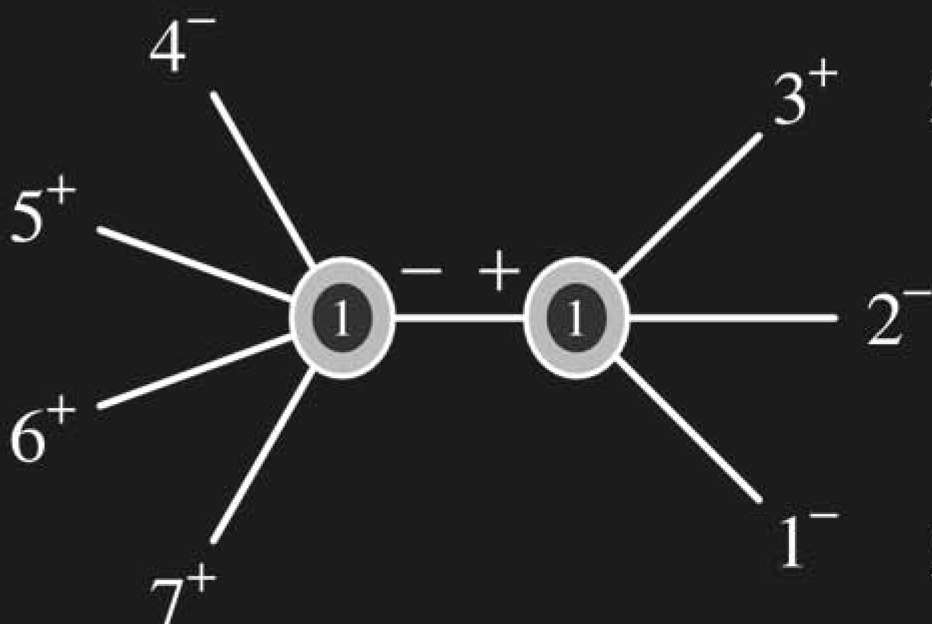
Cachazo–Svrček–Witten Construction

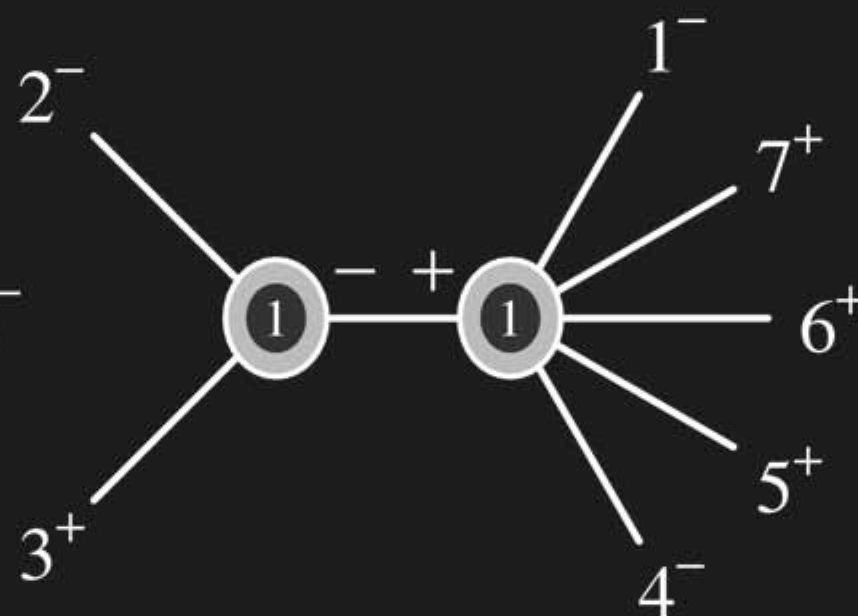
Cachazo, Svrček, & Witten, th/0403047

- Vertices are off-shell continuations of MHV amplitudes: every vertex has two '–' helicities, and one or more '+' helicities
- Includes a three-point vertex
- Propagators are scalar ones: i/K^2; helicity projector is in the vertices
- Draw all tree diagrams with these vertices and propagator
- Different sets of diagrams for different helicity configurations
- Corresponds to all multiparticle factorizations

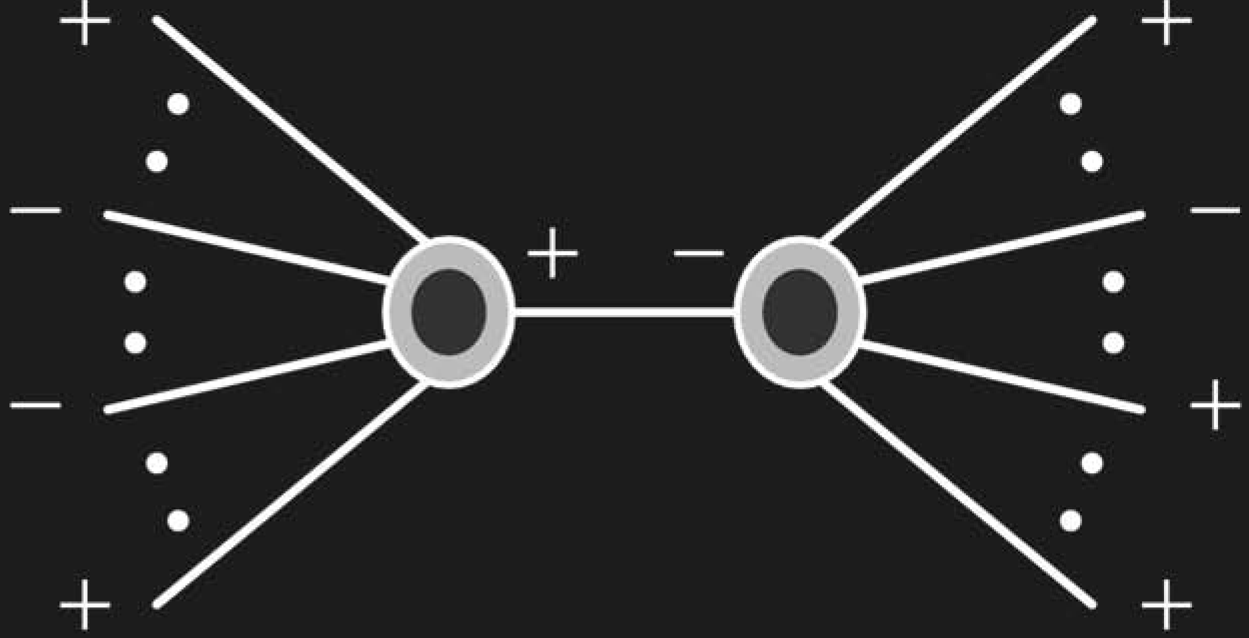


- Seven-point example with three negative helicities





Next-to-MHV



Factorization Properties of Amplitudes

- As sums of external momenta approach poles,

$$p^2 = (k_1 + k_2)^2 \to m_X^2$$

- amplitudes factorize

$$A(1{+}2 \to \ldots) \to A_L(1{+}2 \to X)\frac{i}{p^2 - m^2}A_R(X \to \ldots)$$

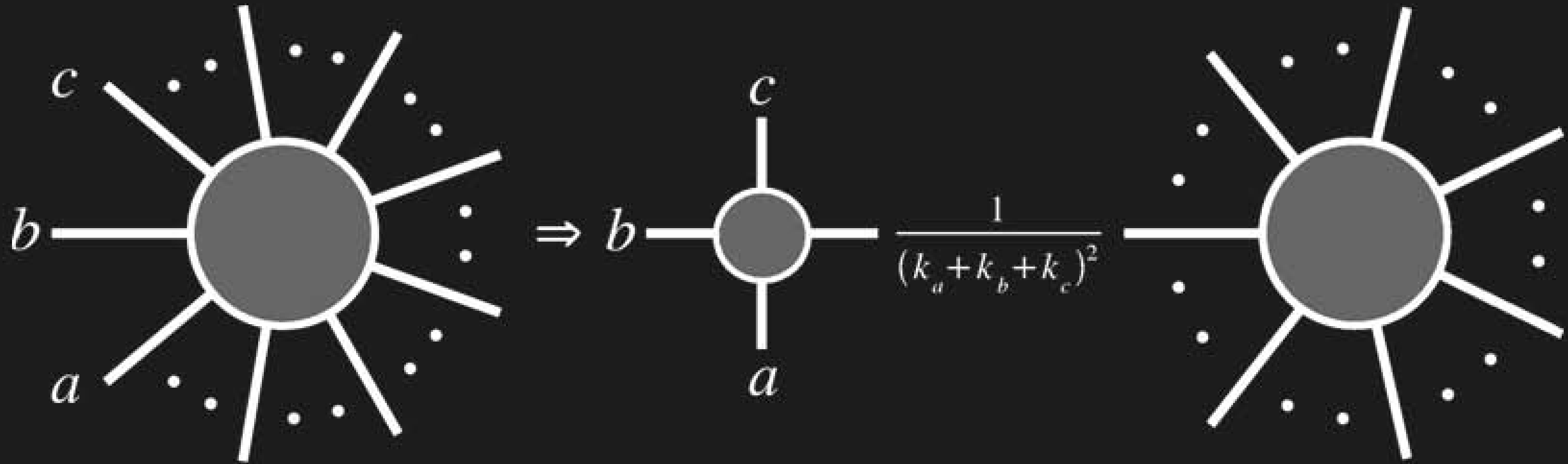
- More generally as

$$p^2 = (k_1 + \cdots + k_n)^2 \to m_X^2$$

$$A(1{+}\cdots{+}n \to \ldots) \to A_L(1{+}\cdots{+}n \to X)\frac{i}{p^2 - m^2}A_R(X \to \ldots)$$

Factorization in Gauge Theories

Tree level



As $(k_a + k_b + k_c)^2 \to 0$ but $s_{ab},\, s_{bc},\, s_{ac} \not\to 0$

Sum over helicities of intermediate leg

In massless theories beyond tree level, the situation is more complicated but at tree level it works in a standard way

What Happens in the Two-Particle Case?

We would get a three-gluon amplitude on the left-hand side

But $k_3^2 = 0 = (k_1 + k_2)^2 = 2k_1 \cdot k_2$

so all invariants vanish,

$$k_1 \cdot k_2 = k_2 \cdot k_3 = k_3 \cdot k_1 = 0$$

hence all spinor products vanish

$$\langle 12\rangle = 0, \quad \langle 23\rangle = 0, \quad \langle 31\rangle = 0$$
$$[12] = 0, \quad [23] = 0, \quad [31] = 0$$

hence the three-point amplitude vanishes

$$A_3(1, 2, 3) = 0$$

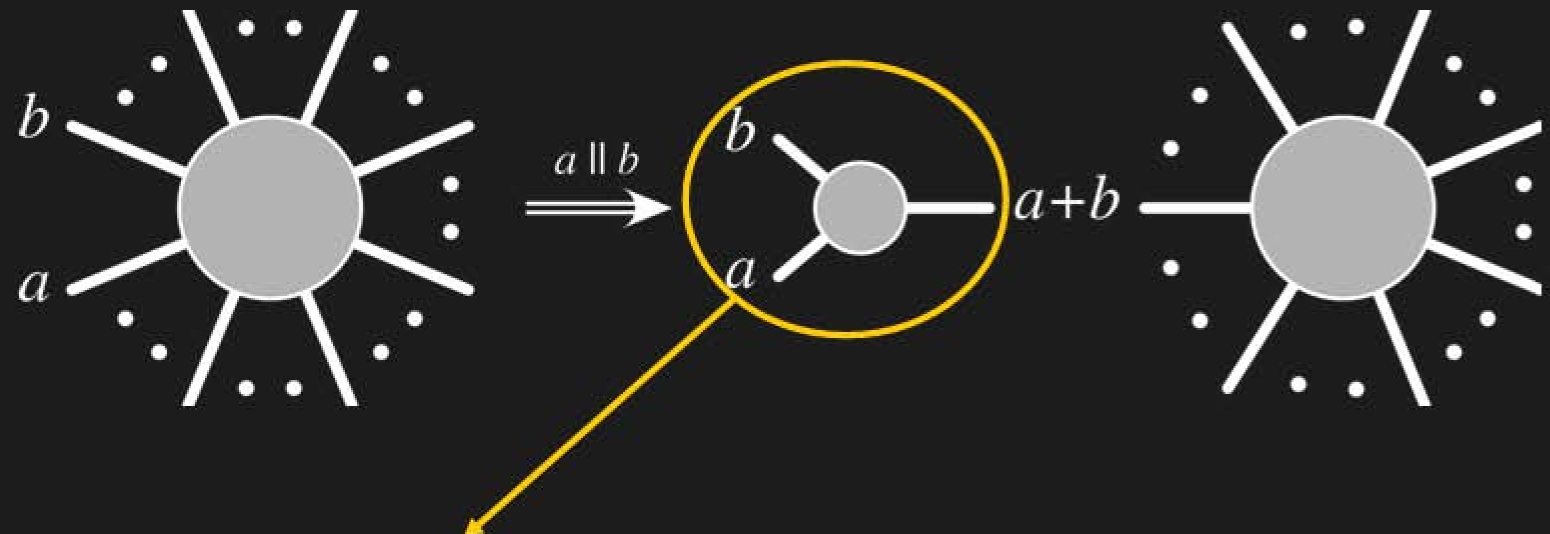
- In gauge theories, it holds (at tree level) for $n \geq 3$ but breaks down for $n = 2$: $A_3 = 0$ so we get $0/0$
- However A_3 only vanishes linearly, so the amplitude is not finite in this limit, but should $\sim 1/k$, that is $1/\sqrt{s_{12}}$
- This is a *collinear* limit

$$(k_1 + k_2)^2 = 2k_1 \cdot k_2 \to 0$$

$$\Longrightarrow k_1 \propto k_2, \;\; i.e., \; k_1 \parallel k_2$$

- Combine amplitude with propagator to get a non-vanishing object

Two-Particle Case



Collinear limit: *splitting* amplitude

Universal Factorization

- Amplitudes have a universal behavior in this limit

$$A_n^{\text{tree}}(\ldots, a^{h_a}, b^{h_b}, \ldots) \stackrel{k_a \| k_b}{\longrightarrow}$$

$$\sum_{h=\pm} \text{Split}_{-h}^{\text{tree}}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{tree}}(\ldots, (k_a+k_b)^h, \ldots)$$

$$+ \;\; \text{non-singular}$$

- Depend on a collinear momentum fraction z

$$k_a = z(k_a + k_b), \qquad k_b = (1-z)(k_a + k_b)$$

- In this form, a powerful tool for checking calculations
- As expressed in on-shell recursion relations, a powerful tool for computing amplitudes

Example: Three-Particle Factorization

Consider

$$-iA_6(1^-,2^-,3^-,4^+,5^+,6^+) =$$

$$\frac{(\langle 2\,3\rangle\,[5\,6]\,\langle 1^-|\,1+2+3\,|4^-\rangle)^2}{s_{234}s_{23}s_{34}s_{56}s_{61}}$$

$$+\frac{(\langle 1\,2\rangle\,[4\,5]\,\langle 3^-|\,1+2+3\,|6^-\rangle)^2}{s_{345}s_{34}s_{45}s_{61}s_{12}}$$

$$+\frac{s_{123}\,\langle 2\,3\rangle\,[5\,6]\,\langle 1^-|\,1+2+3\,|4^-\rangle\,\langle 1\,2\rangle\,[4\,5]\,\langle 3^-|\,1+2+3\,|6^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}$$

As $s_{123} \to 0$, it's finite: expected because $A_4(1^-, 2^-, 3^-, X^\pm) = 0$

As $s_{234} \to 0$, pick up the first term; with $K = k_2 + k_3 + k_4$

$$\frac{\langle 2\,3\rangle\,[K\,4]^2}{[2\,3]\,s_{34}} \times \frac{1}{s_{234}} \times \frac{[5\,6]\,\langle 1\,K\rangle^2}{\langle 5\,6\rangle\,s_{61}} =$$

$$\frac{[4\,K]^2\,[K\,2]\,\langle 2\,3\rangle}{[2\,3]\,[3\,4]\,\langle 4\,3\rangle\,[K\,2]} \times \frac{1}{s_{234}} \times \frac{\langle 1\,K\rangle^2\,\langle K\,5\rangle\,[5\,6]}{\langle K\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,[1\,6]}$$

$$= \frac{[4\,K]^3}{[2\,3]\,[3\,4]\,[K\,2]} \times \frac{1}{s_{234}} \times \frac{\langle 1\,K\rangle^3}{\langle K\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle}$$

$$= A_4(2^-, 3^-, 4^+, K^+) \times \frac{1}{s_{234}} \times A_4(1^-, (-K)^-, 5^+, 6^+)$$

Splitting Amplitudes

Compute it from the three-point vertex

$$\begin{aligned}
\mathrm{Split}_{-}^{\mathrm{tree}}(a^{+}, b^{+}) &= -\frac{\sqrt{2}}{s_{ab}} \left[k_b \cdot \varepsilon_a \, \varepsilon_b \cdot \varepsilon_{a+b} - k_a \cdot \varepsilon_b \, \varepsilon_a \cdot \varepsilon_{a+b}\right] \\
&= -\frac{1}{s_{ab}} \left[\frac{\langle q\, b\rangle\, [b\, a]}{\langle q\, a\rangle} \frac{\langle q\, (a+b)\rangle\, [q\, b]}{\langle q\, b\rangle\, [(a+b)\, q]} \right. \\
&\qquad \left. - \frac{\langle q\, a\rangle\, [a\, b]}{\langle q\, b\rangle} \frac{\langle q\, (a+b)\rangle\, [q\, a]}{\langle q\, a\rangle\, [(a+b)\, q]} \right] \\
&= \frac{1}{\langle a\, b\rangle} \left[\sqrt{\frac{1-z}{z}} + \sqrt{\frac{z}{1-z}} \right] \\
&= \frac{1}{\sqrt{z(1-z)}\, \langle a\, b\rangle} \, .
\end{aligned}$$

Explicit Values

$$\mathrm{Split}^{\mathrm{tree}}_{-}(a^-, b^-) = 0$$

$$\mathrm{Split}^{\mathrm{tree}}_{-}(a^+, b^+) = \frac{1}{\sqrt{z(1-z)}\,\langle a\, b\rangle}$$

$$\mathrm{Split}^{\mathrm{tree}}_{-}(a^+, b^-) = -\frac{z^2}{\sqrt{z(1-z)}\,[a\, b]}$$

$$\mathrm{Split}^{\mathrm{tree}}_{-}(a^-, b^+) = -\frac{(1-z)^2}{\sqrt{z(1-z)}\,[a\, b]}$$

Collinear Factorization at One Loop

$$A_n^{\text{1-loop; LC}}(\ldots, a^{h_a}, b^{h_b}, \ldots) \xrightarrow{k_a \| k_b}$$

$$\sum_{h=\pm}\Bigg(\text{Split}^{\text{tree}}_{-h}(a^{h_a}, b^{h_b})\, A_{n-1}^{\text{1-loop;LC}}(\ldots, (k_a+k_b)^h, \ldots)$$

$$+ \text{Split}^{\text{1-loop}}_{-h}(a^{h_a}, b^{h_b})\, A_{n-1}^{\text{tree}}(\ldots, (k_a+k_b)^h, \ldots)\Bigg)$$

$$+ \ \text{non-singular}$$

Anomalous Dimensions & Amplitudes

- In QCD, one-loop anomalous dimensions of twist-2 operators in the OPE are related to the tree-level Altarelli-Parisi function,

Twist-2 Anomalous Dimension	Mellin Transform $\Leftrightarrow$	Altarelli-Parisi function	=	Helicity-summed splitting amplitude

- Relation understood between two-loop anomalous dimensions & one-loop splitting amplitudes

DAK & Uwer (2003)

Recursion Relations

Considered color-ordered amplitude with one leg off-shell, amputate its polarization vector

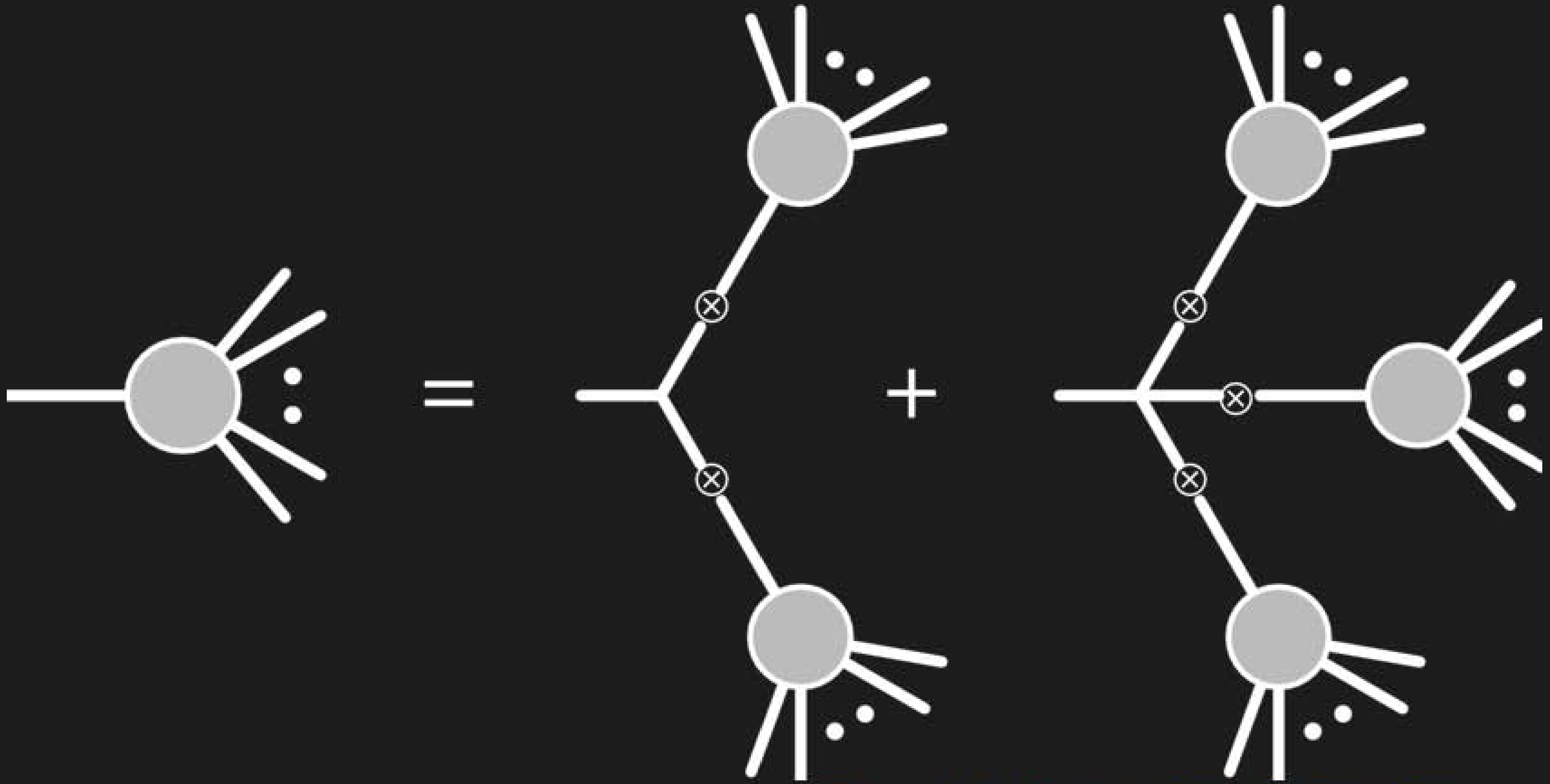
This is the Berends–Giele current $J^\mu(1,\ldots,n)$

Given by the sum of all (n+1)-point color-ordered diagrams with legs 1… n on shell

Follow the off-shell line into the sum of diagrams. It is attached to either a three- or four-point vertex.

Other lines attaching to that vertex are also sums of diagrams with one leg off-shell and other on shell, that is currents

Recursion Relations



Berends & Giele (1988); DAK (1989)

⇒ Polynomial complexity per helicity

$$
\begin{aligned}
J^\mu(1,\ldots,n) = \\
-\frac{i}{K_{1,n}^2}\Bigg[&\sum_{j=1}^{n-1} V_3^{\mu\nu\lambda}\, J_\nu(1,\ldots,j) J_\lambda(j+1,\ldots,n) \\
&+\sum_{j=1}^{n-2}\sum_{l=j+1}^{n-1} V_4^{\mu\nu\lambda\rho}\, J_\nu(1,\ldots,j) \\
&\times J_\lambda(j+1,\ldots,l) J_\rho(l+1,\ldots,n)\Bigg]
\end{aligned}
$$

Properties of the Current

- Decoupling identity
- Reflection identity
- Conservation $K^{\mu}_{1,n} J_{\mu}(1,\ldots,n) = 0$

Complex Momenta

For real momenta, $|k^+\rangle = \pm|k^-\rangle^*$

but we can choose these two spinors independently and still have $k^2 = 0$

Recall the polarization vector: $\varepsilon^+ \propto \langle q^-|\gamma^\mu|k^-\rangle$
but $\varepsilon \cdot \varepsilon = 0$

Now when two momenta are collinear $k \cdot k' = 0$
only one of the spinors has to be collinear

$\langle k\, k'\rangle = 0$ or $[k\, k'] = 0$ but not necessarily both

On-Shell Recursion Relations

Britto, Cachazo, Feng th/0412308; & Witten th/0501052

- Ingredients
  - Structure of factorization
  - Cauchy's theorem

Introducing Complex Momenta

- Define a shift $[j, l\rangle$ of spinors by a complex parameter z

$$
\begin{aligned}
|j^-\rangle &\to |j^-\rangle - z|l^-\rangle, \\
|l^+\rangle &\to |l^+\rangle + z|j^+\rangle
\end{aligned}
$$

- which induces a shift of the external momenta

$$
\begin{aligned}
k_j^\mu &\to k_j^\mu(z) = k_j^\mu - \frac{z}{2}\langle j^- | \gamma^\mu | l^- \rangle, \\
k_l^\mu &\to k_l^\mu(z) = k_l^\mu + \frac{z}{2}\langle j^- | \gamma^\mu | l^- \rangle
\end{aligned}
$$

- and defines a z-dependent continuation of the amplitude $A(z)$
- Assume that $A(z) \to 0$ as $z \to \infty$

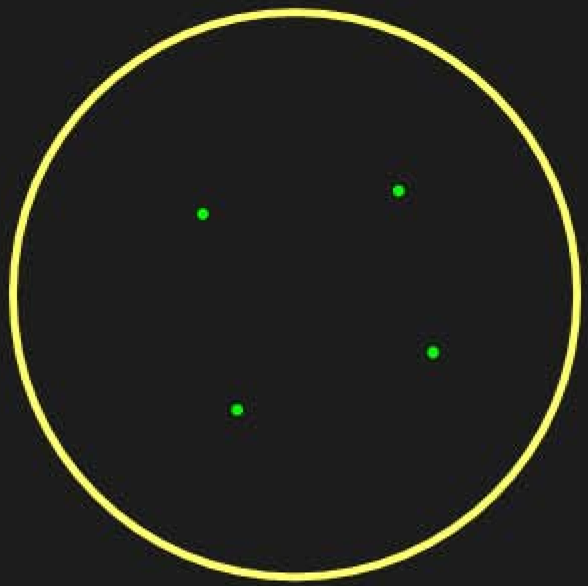
A Contour Integral

Consider the contour integral

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z}\,A(z)$$

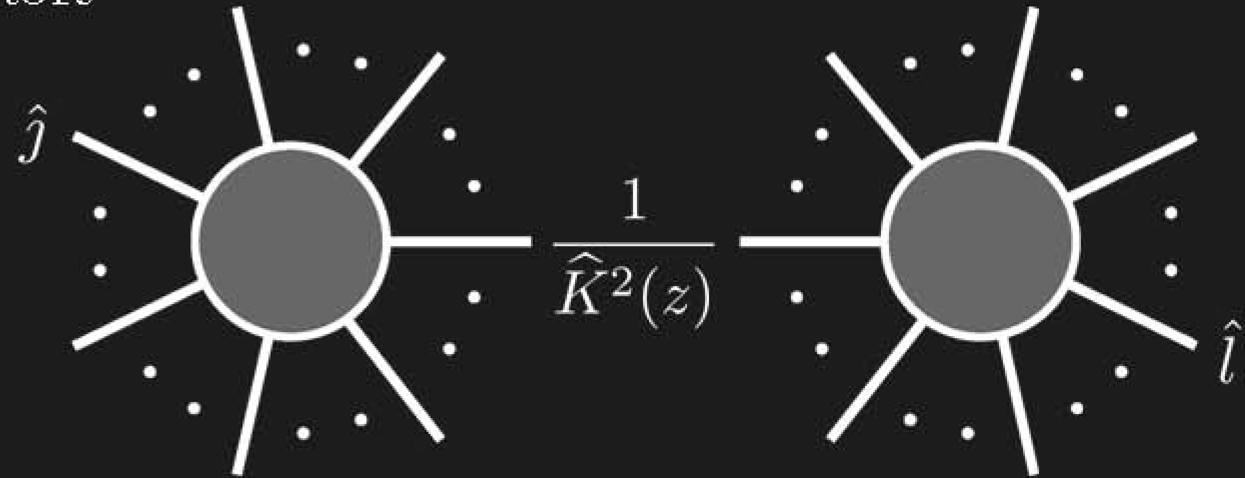
Determine A(0) in terms of other residues

$$A(0) = -\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{A(z)}{z}$$



Using Factorization

Other poles in z come from zeros of z-shifted propagator denominators



Splits diagram into two parts with z-dependent momentum flow

$$\longrightarrow \sum_{\text{partitions}} \begin{array}{c}\text{shifted legs on} \\ \text{opposite sides}\end{array}$$

Exactly factorization limit of z-dependent amplitude

poles from zeros of

$$K^2_{a\cdots j\cdots b}(z) = K^2_{a\cdots b} - z\,\langle j^- |\, \not{K}_{a\cdots b} \,| l^- \rangle$$

That is, a pole at

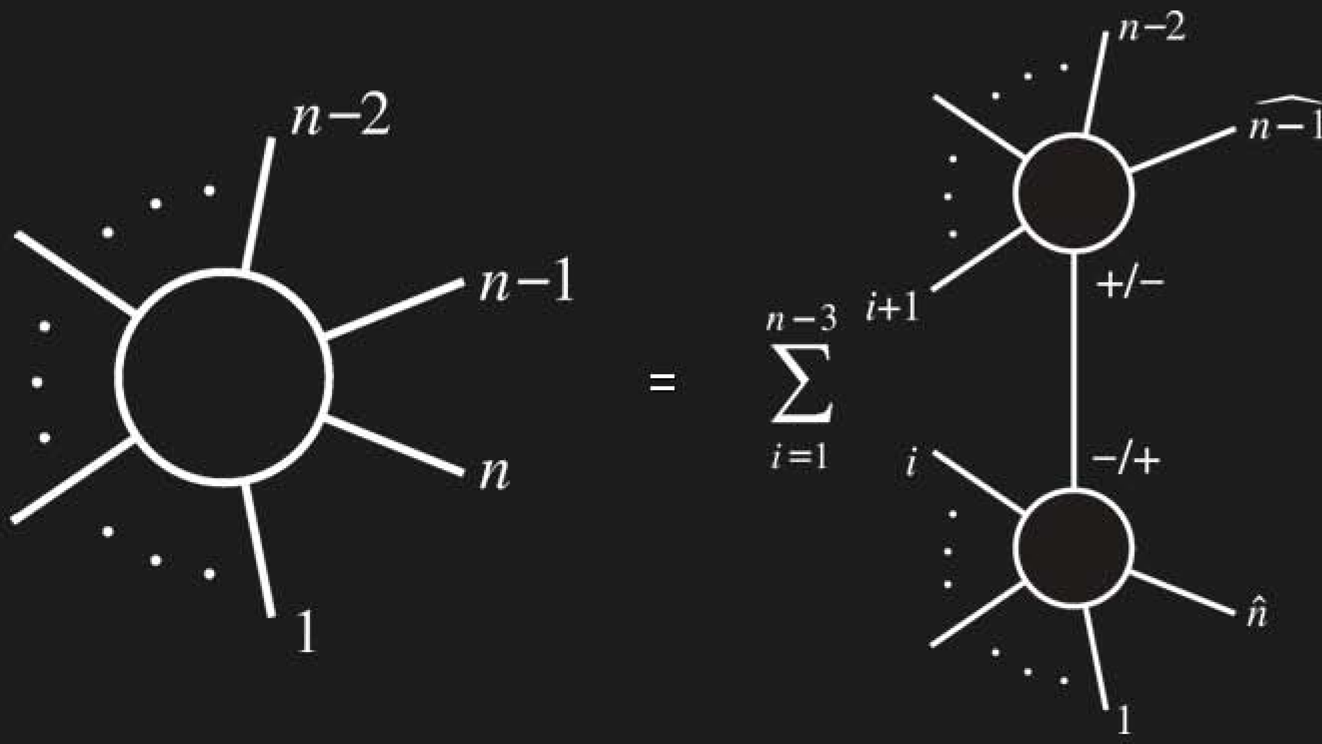
$$z_{ab} = \frac{K^2_{a\cdots b}}{\langle j^- |\, \not{K}_{a\cdots b} \,| l^- \rangle}$$

Residue

$$\operatorname*{Res}_{z=z_{ab}} \frac{f(z)}{z\, K^2_{a\cdots b}(z)} = A_L(z_{ab}) \times \frac{i}{K^2_{a\cdots b}} \times A_R(z_{ab})$$

Notation $\hat{k} = k(z_{ab})$

On-Shell Recursion Relation



- Partition P: two or more cyclicly-consecutive momenta containing j, such that complementary set $\overline{P}$ contains l,

$$\begin{aligned} P &\equiv \{P_1, P_2, \ldots, j, \ldots, P_{-1}\}, \\ \overline{P} &\equiv \{\overline{P}_1, \overline{P}_2, \ldots, l, \ldots, \overline{P}_{-1}\}, \\ P \cup \overline{P} &= \{1, 2, \ldots, n\} \end{aligned}$$

On shell

- The recursion relations are then

$$\begin{aligned} A_n(1, \ldots, n) &= \sum_{\substack{\text{partitions P} \\ \text{h}=\pm}} A_{\#P+1}(k_{P_1}, \ldots, \hat{j}, \ldots, k_{P_{-1}}, -\widehat{P}^{\,h}) \\ &\quad \times \frac{i}{P^2} \times A_{\#\overline{P}+1}(k_{\overline{P}_1}, \ldots, \hat{l}, \ldots, k_{\overline{P}_{-1}}, \widehat{P}^{\,-h}) \end{aligned}$$

Number of terms ~ $|l-j| \times (n-3)$

so best to choose l and j nearby

Complexity still exponential, because shift changes as we descend the recursion

Applications

- Very general: relies only on complex analysis + factorization
- Fermionic amplitudes
- Applied to gravity

Bedford, Brandhuber, Spence, & Travaglini (2/2005)

Cachazo & Svrček (2/2005)

- Massive amplitudes

Badger, Glover, Khoze, Svrček (4/2005, 7/2005)

Forde & DAK (7/2005)

- Other rational functions

Bern, Bjerrum-Bohr, Dunbar, & Ita (7/2005)

- Connection to Cachazo–Svrček–Witten construction

Risager (8/2005)

- CSW construction for gravity

Bjerrum-Bohr, Dunbar, Ita, Perkins, & Risager (9/2005)

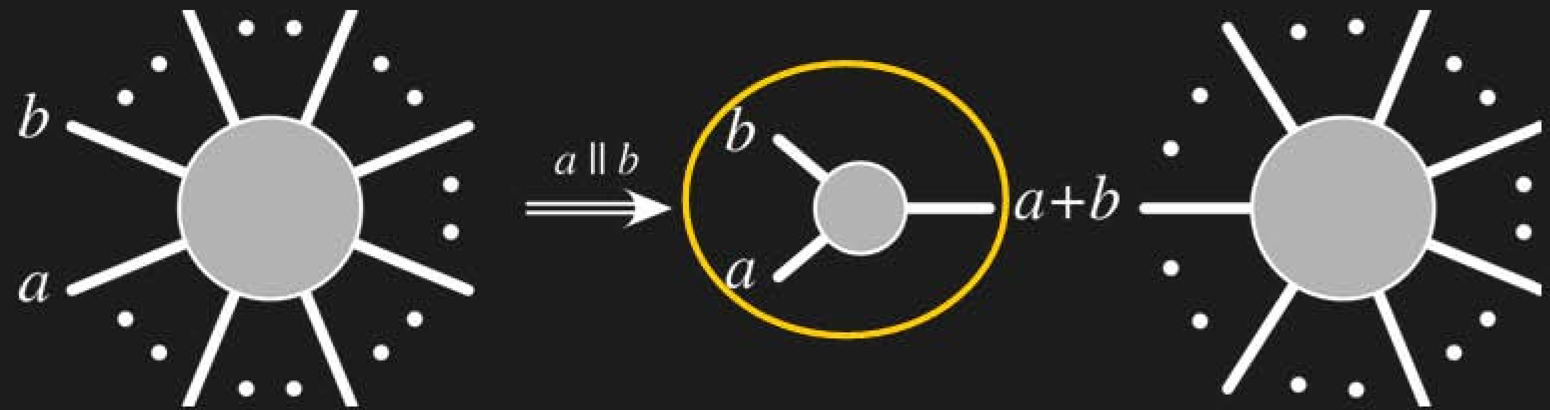
On-Shell Methods in Field Theory

David A. Kosower

International School of Theoretical Physics,
Parma,
September 10-15, 2006

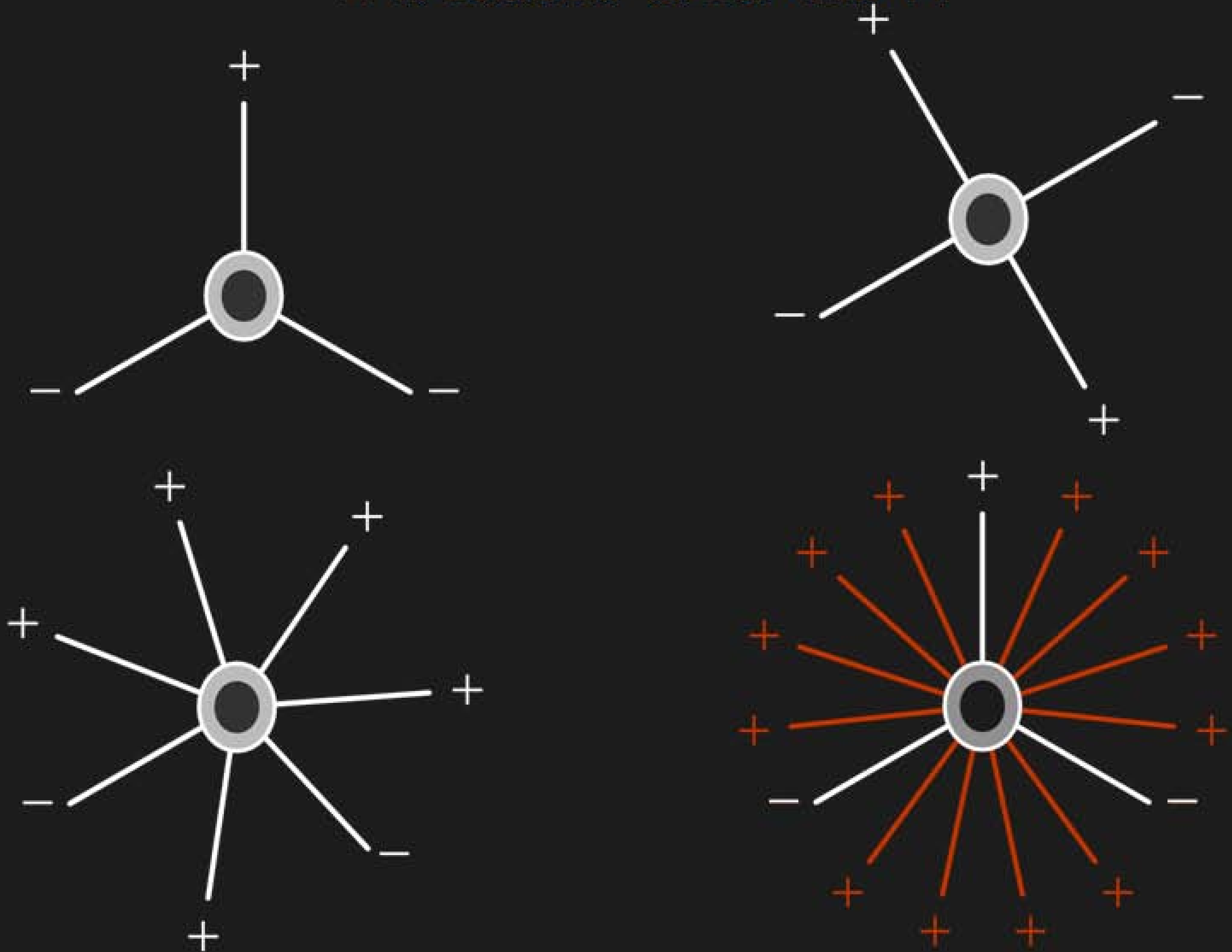
Lecture III

Factorization in Gauge Theory

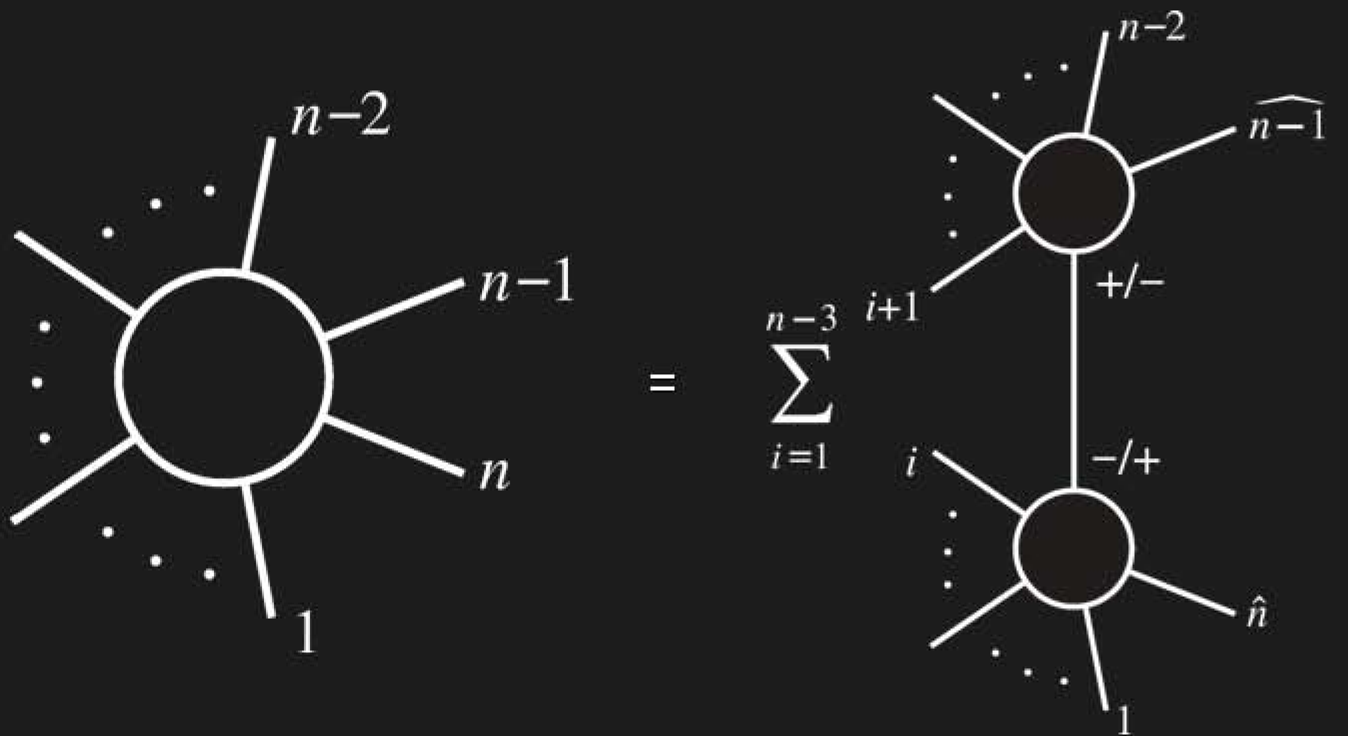


Collinear limits with splitting amplitudes

Twistors and CSW



On-Shell Recursion Relation



Three-Gluon Amplitude Revisited

Let's compute it with complex momenta chosen so that

$$|1^-\rangle \propto |2^-\rangle \propto |3^-\rangle$$

that is,

$$[1\,2] = [2\,3] = [3\,1] = 0$$

but

$$\langle 1\,2\rangle \neq 0,\ \langle 2\,3\rangle \neq 0,\ \langle 3\,1\rangle \neq 0$$

compute $A_3(1^-, 2^-, 3^+)$

$A_3(1^-, 2^-, 3^+)$

Choose common reference momentum q

$\varepsilon_1 \cdot \varepsilon_2 = 0$ so we have to compute

$$\sqrt{2}i\left[\varepsilon_2 \cdot \varepsilon_3\, k_2 \cdot \varepsilon_1 + \varepsilon_3 \cdot \varepsilon_1\, k_3 \cdot \varepsilon_2\right]$$

$$= i\frac{\langle q\,2\rangle\,[q\,3]}{[2\,q]\,\langle q\,3\rangle}\frac{[q\,2]\,\langle 2\,1\rangle}{[1\,q]} + i\frac{\langle q\,1\rangle\,[q\,3]}{[1\,q]\,\langle q\,3\rangle}\frac{[q\,3]\,\langle 3\,2\rangle}{[2\,q]}$$

$$= i\frac{[q\,3]}{\langle q\,3\rangle}\left[\frac{\langle q\,2\rangle\,\langle 2\,1\rangle}{[1\,q]} - \frac{\langle q\,1\rangle\,\langle 1\,2\rangle}{[2\,q]}\right]$$

$$= -2i\frac{\langle 1\,2\rangle\,[q\,3]\,q\cdot(k_1+k_2)}{\langle q\,3\rangle\,[q\,1]\,[q\,2]}$$

$$= i\frac{\langle 1\,2\rangle\,[q\,3]^2}{[q\,1]\,[q\,2]}$$

Not manifestly gauge invariant

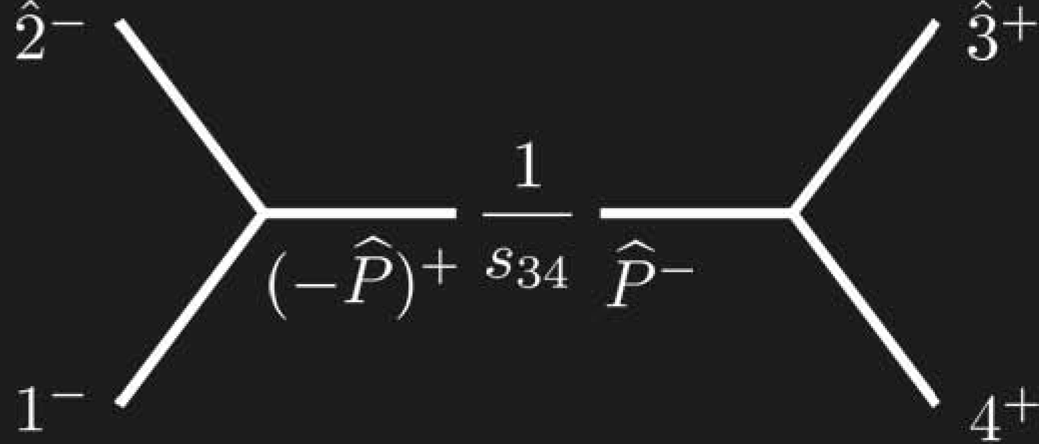
$$i\frac{\langle 1\,2\rangle\,[q\,3]^2}{[q\,1]\,[q\,2]}$$
$$= -i\frac{\langle 1\,2\rangle\,[q\,3]\,\langle 3\,2\rangle\,[q\,3]\,\langle 3\,1\rangle}{\langle 2\,3\rangle\,\langle 3\,1\rangle\,[q\,1]\,[q\,2]}$$
$$= i\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,1\rangle}$$

but gauge invariant nonetheless,

and exactly the $n=3$ case of the general Parke–Taylor formula!

Four-Point Example

Pick a $[2,3\rangle$ shift, giving one diagram



$$i\frac{\langle 1\,2\rangle^3}{\langle 2\,(-\widehat{P}\,)\rangle\,\langle(-\widehat{P}\,)\,1\rangle} \times \frac{i}{s_{34}} \times (-i)\frac{[3\,4]^3}{[\widehat{P}\;3]\,[4\,\widehat{P}\,]}$$

$$= -i \frac{\langle 1\,2\rangle^3\,[3\,4]^2}{\langle 4\,3\rangle\,\langle 1^-|\,\widehat{P}\,|3^-\rangle\,\langle 2^-|\,\widehat{P}\,|4^-\rangle}$$

$$= -i \frac{\langle 1\,2\rangle^3\,[3\,4]^2}{\langle 4\,3\rangle\,\langle 1^-|\,P\,|3^-\rangle\,\langle 2^-|\,P\,|4^-\rangle}$$

$$\langle 2^-|\,\gamma^\mu\,|3^-\rangle$$

$$= i \frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}$$

Choosing Shift Momenta

- What are legitimate choices?
- Need to ensure that $A(z) \to 0$ as $z \to \infty$
- At tree level, legitimate choices

$$[-,+\rangle,\ [+,+\rangle,\ [-,-\rangle$$

- Power counting argument in Feynman diagrams for $[-,+\rangle$

$$\varepsilon_{\hat{j}}^{(-)} = \frac{\langle j^- | \gamma^\mu | q^- \rangle}{\sqrt{2}\left[\hat{j}\, q\right]} \sim \frac{1}{z}$$

$$\varepsilon_{\hat{l}}^{(+)} = \frac{\langle q^- | \gamma^\mu | l^- \rangle}{\sqrt{2}\left\langle q\, \hat{l}\right\rangle} \sim \frac{1}{z}$$

Three-point vertices with z-dependent momentum flow ~ z

Four-point vertices with z-dependent momentum flow ~ 1

Propagators with z-dependent momentum flow ~ $1/z$

⇒ Leading contributions from diagrams with only three-point vertices and propagators connecting j to l: ~ $1/z$

(one more vertex than propagators & two εs)

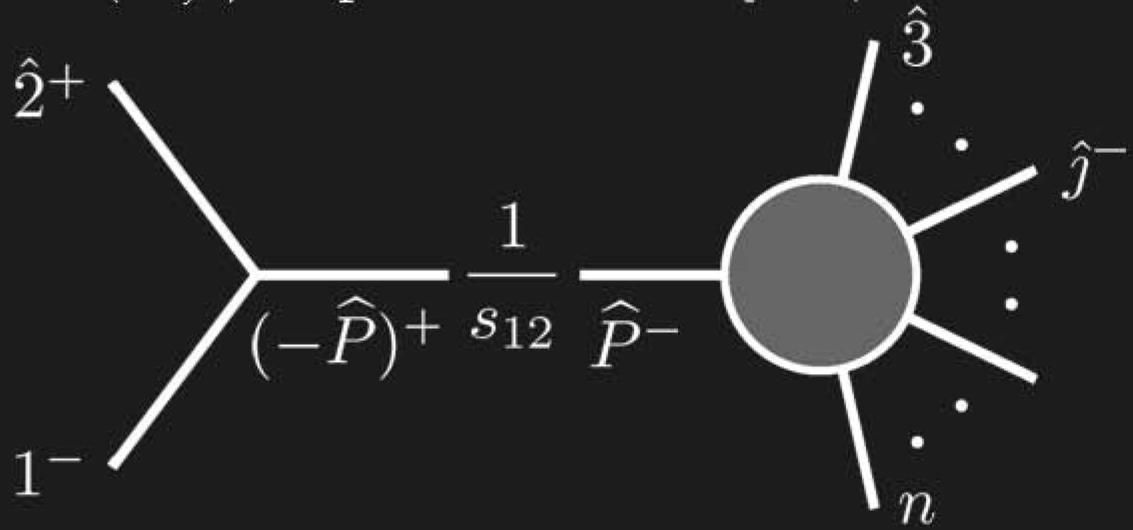
Factorization in Complex Momenta

- Factorization theorems derived for real momenta
- For multiparticle poles, hold for complex momenta as well

- At tree level, collinear factorization holds for complex momenta as well, because splitting amplitudes only involve 1/spinor product, so we only get pure single poles
- Double poles cannot arise because each propagator can only give rise to a single invariant in the denominator

MHV Amplitudes

Compute the $(1^-, j^-)$ amplitude: choose $[3,2\rangle$ shift



Other diagrams vanish because $A_n(+\cdots+\pm) = 0$ or $A_3(\hat{3}^+, 4^+, -\widehat{P}^{\pm}) = 0$

$$A_3(1^-, \hat{2}^+, -\widehat{P}^{\,+}) \frac{i}{s_{12}} A_{n-1}(\widehat{P}^{\,-}, \hat{3}^+, \ldots, j^-, \ldots, n^+)$$

$$= i \frac{[2\,(-\widehat{P}\,)]^3}{[(-\widehat{P}\,)\,1]\,[1\,2]} \frac{1}{s_{12}} \frac{\langle \widehat{P}\; j\rangle^4}{\langle \widehat{P}\; 3\rangle\,\langle 3\,4\rangle \cdots \langle (n-1)\,n\rangle\,\langle n\,\widehat{P}\,\rangle}$$

$$= i \frac{\left\langle j^- \middle| \,\widehat{P}\, \middle| 2^- \right\rangle^4 \langle 2\,3\rangle\,\langle n\,1\rangle}{[1\,2]^2 \left\langle 3^- \middle| \,\widehat{P}\, \middle| 1^- \right\rangle \left\langle n^- \middle| \,\widehat{P}\, \middle| 2^- \right\rangle} \frac{1}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle (n-1)\,n\rangle\,\langle n\,1\rangle}$$

$$= i \frac{\langle 1\,j\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle (n-1)\,n\rangle\,\langle n\,1\rangle}$$

- Prove Parke–Taylor equation by induction

CSW From Recursion

Risager, th/0508206

Consider NMHV amplitude: 3 negative helicities m_1, m_2, m_3, any number of positive helicities

Choose shift

$$
\begin{aligned}
|m_1^-\rangle &\rightarrow |m_1^-\rangle + z\,\langle m_2\, m_3\rangle\, |q^-\rangle \\
|m_2^-\rangle &\rightarrow |m_2^-\rangle + z\,\langle m_3\, m_1\rangle\, |q^-\rangle \\
|m_3^-\rangle &\rightarrow |m_3^-\rangle + z\,\langle m_1\, m_2\rangle\, |q^-\rangle
\end{aligned}
$$

Momenta are still on shell, and

$$\delta(k_1 + k_2 + k_3)^\mu =$$

$$\big(\langle m_2\, m_3\rangle\, \langle m_1^-| + \langle m_3\, m_1\rangle\, \langle m_2^-| + \langle m_1\, m_2\rangle\, \langle m_3^-|\big)\gamma^\mu |q^-\rangle = 0$$

because of the Schouten identity

- *z*-**dependent** momentum flow comes from configurations with one minus helicity on one amplitude, two on the other

$$A(1^+,\ldots,m_1^-,\ldots,m_2^-,\ldots,-\widehat{P}^{\,+})\frac{1}{P^2}A(\widehat{P}^{\,-},\ldots,m_3^-,\ldots,n^+)$$

- MHV × MHV

- For more negative helicities, proceed recursively or solve globally for shifts using Schouten identity that yield a complete factorization ⇒ CSW construction

- Can be applied to gravity too!

Bjerrum-Bohr, Dunbar, Ita, Perkins & Risager, th/0509016

Singularity Structure

- On-shell recursion relations lead to compact analytic expression
- Different form than Feynman-diagram computation
- Appearance of spurious singularities

$$A_6(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{1}{\langle 5^- | 3+4 | 2^- \rangle}\left(\frac{\langle 1^- | 2+3 | 4^- \rangle^3}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\, s_{234}} + \frac{\langle 3^- | 4+5 | 6^- \rangle^3}{[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\, s_{345}}\right)$$

unphysical singularity —
cancels between terms

physical singularities

Review of Supersymmetry

- Equal number of bosonic and fermionic degrees of freedom
- Only local extension possible of Poincaré invariance
- Extended supersymmetry: only way to combine Poincaré invariance with internal symmetry
- Poincaré algebra

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[P_\rho, M_{\mu\nu}] &= \eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu \\
[M_{\mu\nu}, M_{\rho\lambda}] &= -\eta_{\mu\rho} M_{\nu\lambda} - \eta_{\nu\lambda} M_{\mu\rho} + \eta_{\mu\lambda} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\lambda}
\end{aligned}
$$

- Supersymmetry algebra is graded, that is uses both commutators and anticommutators. For **N**=1, there is one supercharge Q, in a spin-½ representation (and its conjugate)

$$\{Q_a, \overline{Q}_{\dot{a}}\} = 2\sigma^{\mu}_{a\dot{a}} P_\mu$$

$$[P_\mu, Q_a] = [P_\mu, \overline{Q}_{\dot{a}}] = 0$$

$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu})_a{}^b Q_b$$

$$[\overline{Q}_{\dot{a}}, M_{\mu\nu}] = (\overline{\sigma}_{\mu\nu})_{\dot{a}}{}^{\dot{b}} \overline{Q}_{\dot{b}}$$

$$\{Q_a, Q_b\} = \{\overline{Q}_{\dot{a}}, \overline{Q}_{\dot{b}}\} = 0$$

- There is also an R symmetry, a U(1) charge that distinguishes between particles and superpartners

Supersymmetric Gauge Theories

- $\mathcal{N}=1$: gauge bosons + Majorana fermions, all transforming under the adjoint representation
- $\mathcal{N}=4$: gauge bosons + 4 Majorana fermions + 6 real scalars, all transforming under the adjoint representation

Supersymmetry Ward Identities

- Color-ordered amplitudes don't distinguish between quarks and gluinos ⇒ same for QCD and **N**=1 SUSY
- Supersymmetry should relate amplitudes for different particles in a supermultiplet, such as gluons and gluinos
- Supercharge annihilates vacuum

$$\langle 0|[Q,\Phi_1\Phi_2\cdots\Phi_n]|0\rangle = 0 = \sum_{i=1}^{n}\langle 0|\Phi_1\cdots[Q,\Phi_i]\cdots\Phi_n|0\rangle$$

Grisaru, Pendleton & van Nieuwenhuizen (1977)

- Use a practical representation of the action of supersymmetry on the fields. Multiply by a spinor wavefunction & Grassman parameter θ

$$[Q, G^{\pm}(k)] = \pm\Gamma^{\pm}(k,q)\Lambda^{\pm}(k)$$
$$[Q, \Lambda^{\pm}(k)] = \mp\Gamma^{\mp}(k,q)G^{\pm}(k)$$

- where $\Gamma^{-} = \theta\,[q\,k]\,,\ \Gamma^{+} = \theta\,\langle q\,k\rangle$
- With explicit helicity choices, we can use this to obtain equations relating different amplitudes
- Typically start with Q acting on an 'amplitude' with an *odd* number of fermion lines (overall a bosonic object)

Supersymmetry WI in Action

- All helicities positive:

$$
\begin{aligned}
0 &= \langle 0|[Q_q, \Lambda_1^+ G_2^+ \cdots G_n^+]|0\rangle \\
&= -\Gamma^-(k_1,q) A_n^{\rm tree}(1^+,\ldots,n^+) \\
&\quad + \Gamma^+(k_2,q) A_n^{\rm tree}(1_\Lambda^+, 2_\Lambda^+, 3^+,\ldots,n^+) \\
&\quad + \cdots + \Gamma^+(k_n,q) A_n^{\rm tree}(1_\Lambda^+, 2^+,\ldots,(n-1)^+, n_\Lambda^+)
\end{aligned}
$$

- Helicity conservation implies that the fermionic amplitudes vanish

$$0 = -\Gamma^-(k_1,q) A_n^{\rm tree}(1^+,\ldots,n^+)$$

- so that we obtain the first Parke–Taylor equation

- With two negative helicity legs, we get a non-vanishing relation

$$
\begin{aligned}
0 &= \langle 0|[Q_q, \Lambda_1^+ G_2^- G_3^+ \cdots G_j^- G_{j+1}^+ \cdots G_n^+]|0\rangle \\
&= -\Gamma^-(k_1, q) A_n^{\rm tree}(1^+, 2^-, \ldots, n^+) \\
&\quad - \Gamma^+(k_2, q) A_n^{\rm tree}(1_\Lambda^+, 2_\Lambda^-, 3^+, \ldots, j^-, \ldots, n^+) \\
&\quad - \Gamma^+(k_j, q) A_n^{\rm tree}(1_\Lambda^+, 2^-, 3^+, \ldots, j_\Lambda^-, \ldots, n^+)
\end{aligned}
$$

- Choosing $q = k_2$

$$
\begin{aligned}
&A_n^{\rm tree}(1_\Lambda^+, 2^-, 3^+, \ldots, j_\Lambda^-, \ldots, n^+) = \\
&\qquad -\frac{\langle 1\,2\rangle}{\langle 2\,j\rangle} A_n^{\rm tree}(1^+, 2^-, 3^+, \ldots j^-, \ldots, n^+)
\end{aligned}
$$

- Tree-level amplitudes with external gluons or one external fermion pair are given by supersymmetry even in QCD.

- Beyond tree level, there are additional contributions, but the Ward identities are still useful.

- For supersymmetric theories, they hold to all orders in perturbation theory

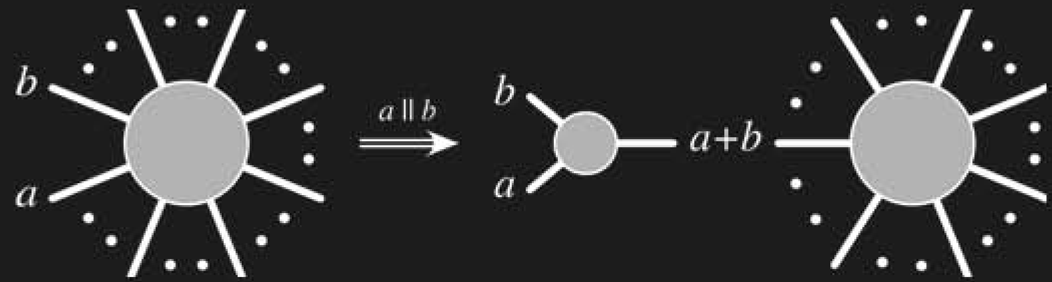
On-Shell Methods in Field Theory

David A. Kosower

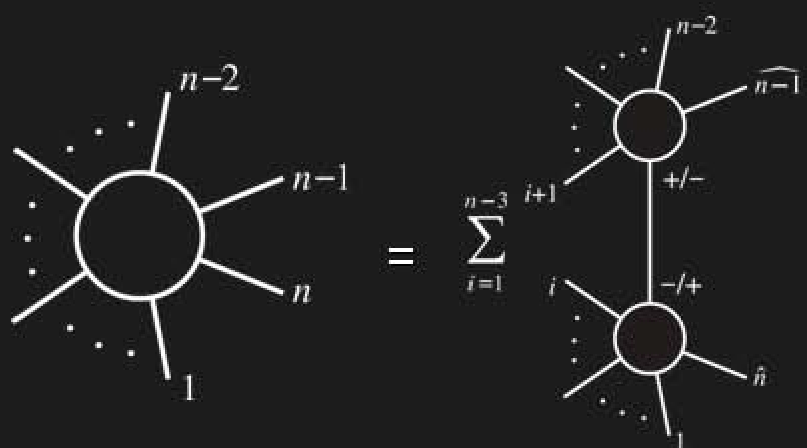
International School of Theoretical Physics,
Parma,
September 10-15, 2006

Lecture IV

- Property: tree-level factorization



⇒ Computational tool at tree level: on-shell recursion relations



Loop Calculations: Textbook Approach

- Sew together vertices and propagators into loop diagrams
- Obtain a sum over [2,n]-point [0,n]-tensor integrals, multiplied by coefficients which are functions of k and ε
- Reduce tensor integrals using Brown-Feynman & Passarino-Veltman brute-force reduction, or perhaps Vermaseren-van Neerven method
- Reduce higher-point integrals to bubbles, triangles, and boxes

- Can apply this to color-ordered amplitudes, using color-ordered Feynman rules
- Can use spinor-helicity method at the end to obtain helicity amplitudes

BUT

- This fails to take advantage of gauge cancellations early in the calculation, so a lot of calculational effort is just wasted.

Can We Take Advantage?

- Of tree-level techniques for reducing computational effort?
- Of any other property of the amplitude?

Unitarity

- Basic property of any quantum field theory: conservation of probability. In terms of the scattering matrix,

$$S^\dagger S = 1$$

In terms of the transfer matrix $iT = S - 1$ we get,

$$-i(T - T^\dagger) = T^\dagger T$$

or

$$2\ \text{“Im”}\ T_{fi} = (T^\dagger T)_{fi}$$

with the Feynman $i\varepsilon$

$$\mathrm{Disc}\, T = T^\dagger T$$

This has a direct translation into Feynman diagrams, using the *Cutkosky* rules. If we have a Feynman integral,

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2 + i\delta} \cdots \frac{1}{(\ell - K)^2 + i\delta}$$

and we want the discontinuity in the K^2 channel, we should replace

$$\frac{1}{\ell^2 + i\delta} \longrightarrow -2\pi i \delta^{(+)}(\ell^2)$$

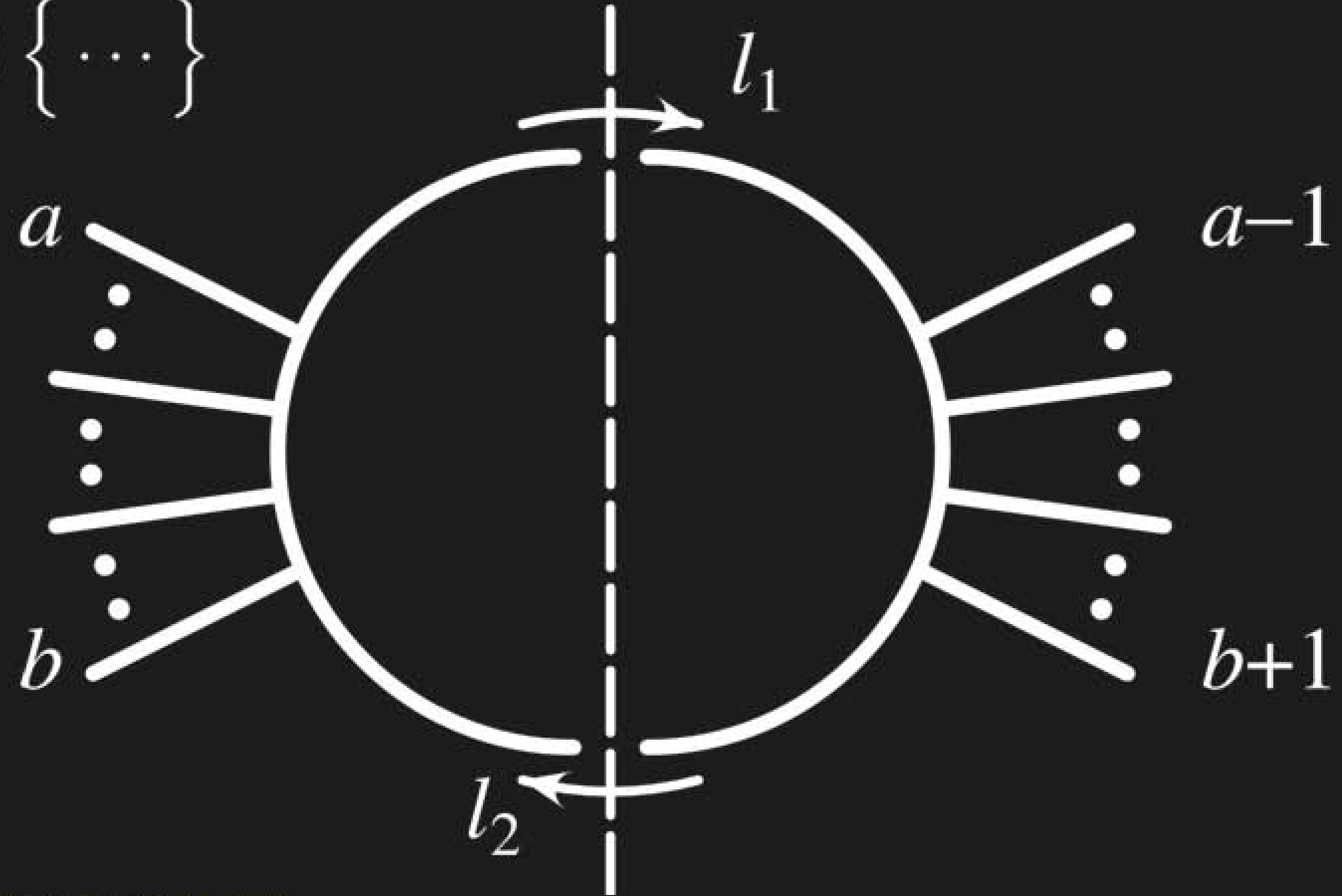
$$\frac{1}{(\ell - K)^2 + i\delta} \longrightarrow -2\pi i \delta^{(+)}((\ell - K)^2)$$

$$\delta^{(+)}(k^2) = \Theta(k^0)\delta(k^2)$$

- When we do this, we obtain a phase-space integral

$$\int \frac{d^D\ell}{(2\pi)^{D-1}}\, \delta^{(+)}(\ell^2)\delta^{(+)}((\ell-K)^2)\left\{\cdots\right\} =$$

$$\int d^D\mathrm{LIPS}\left\{\cdots\right\}$$



In the Bad Old Days of Dispersion Relations

- To recover the full integral, we could perform a dispersion integral

$$\mathrm{Re}\, f(s) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dw\, \frac{\mathrm{Im}\, f(w)}{w-s} + \mathrm{Re}\, C_\infty$$

in which $C_\infty = 0$ so long as $f(w) \to 0$ when $w \to \infty$

- If this condition isn't satisfied, there are 'subtraction' ambiguities corresponding to terms in the full amplitude which have no discontinuities

- But it's better to obtain the full integral by identifying which Feynman integral(s) the cut came from.
- Allows us to take advantage of sophisticated techniques for evaluating Feynman integrals: identities, modern reduction techniques, differential equations, reduction to master integrals, etc.

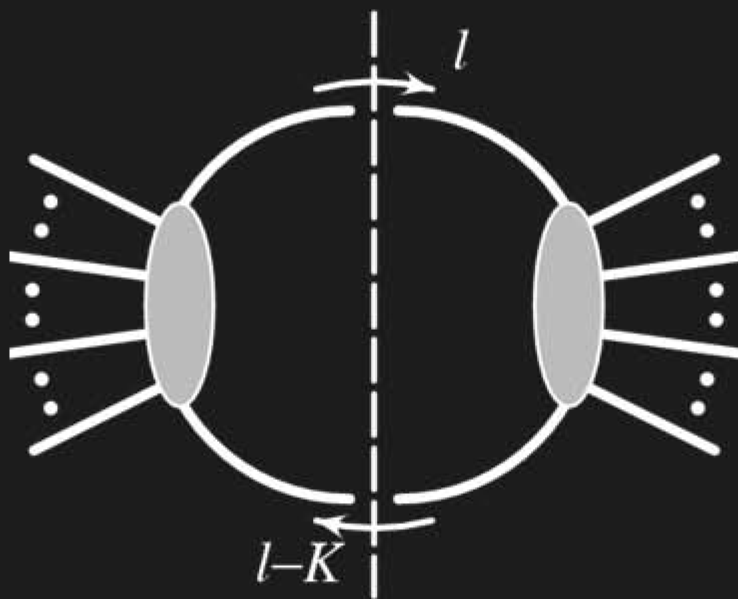
Computing Amplitudes *Not* Diagrams

- The cutting relation can also be applied to sums of diagrams, in addition to single diagrams
- Looking at the cut in a given channel s of the sum of all diagrams for a given process throws away diagrams with no cut — that is diagrams with one or both of the required propagators missing — and yields the sum of all diagrams on each side of the cut.
- Each of those sums is an on-shell tree amplitude, so we can take advantage of all the advanced techniques we've seen for computing them.

Unitarity-Based Method at One Loop

- Compute cuts in a set of channels
- Compute required tree amplitudes
- Form the phase-space integrals
- Reconstruct corresponding Feynman integrals
- Perform integral reductions to a set of master integrals
- Assemble the answer

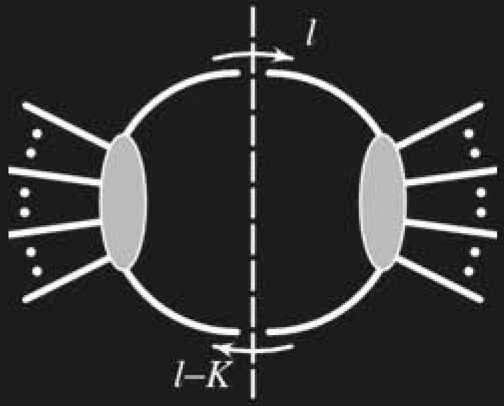
Unitarity-Based Calculations



Bern, Dixon, Dunbar, & DAK,
ph/9403226, ph/9409265

$$A^{\text{1-loop}} = \sum_{\text{cuts } K^2} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} A^{\text{tree}}_{\text{left}} \frac{i}{(\ell-K)^2} A^{\text{tree}}_{\text{right}}$$

Unitarity-Based Calculations



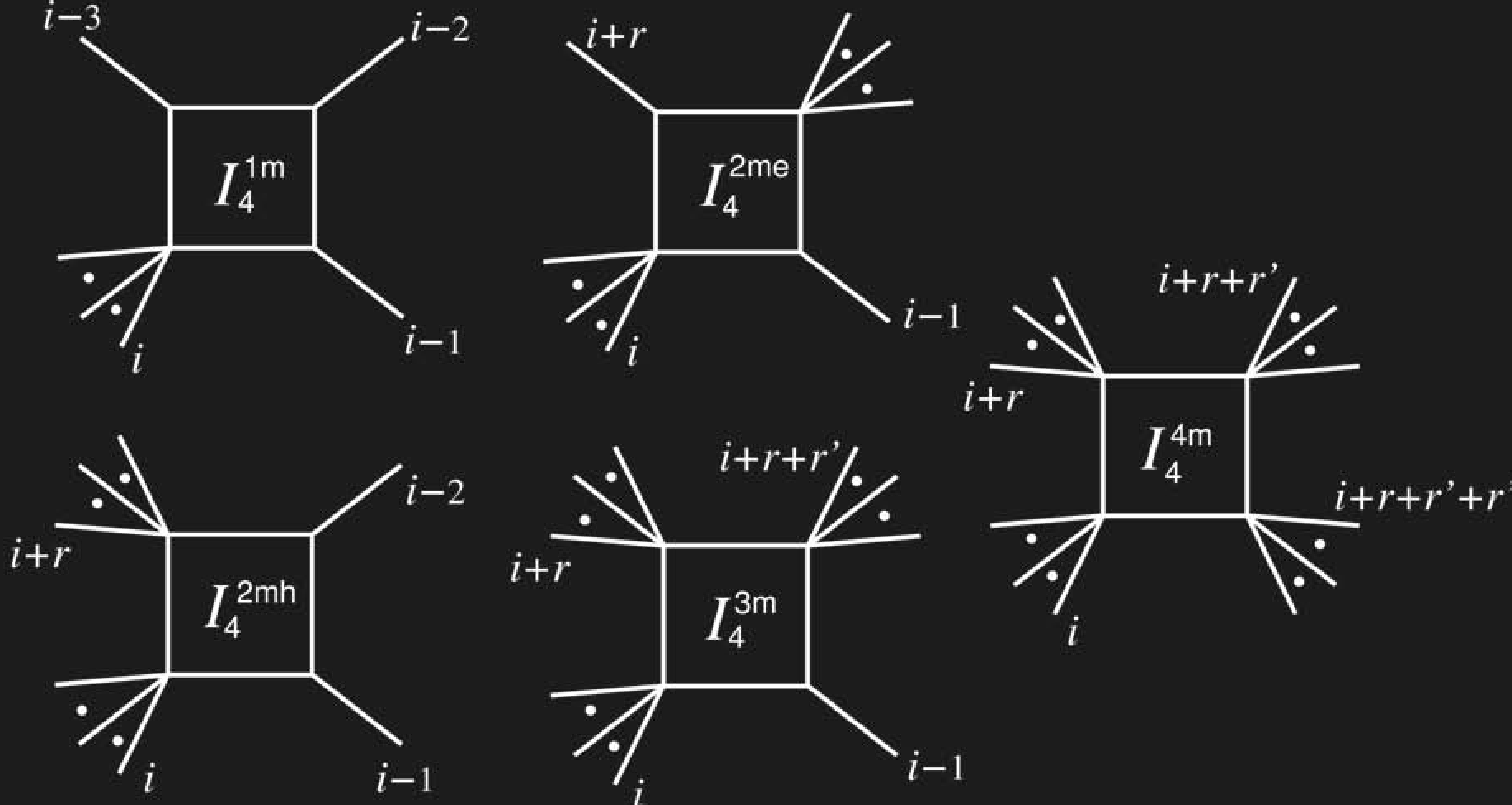
- In general, work in $D=4-2\epsilon \Rightarrow$ full answer
  van Neerven (1986): dispersion relations converge
- At one loop in $D=4$ for SUSY $\Rightarrow$ full answer
- Merge channels rather than blindly summing: find function w/given cuts in all channels

The Three Roles of Dimensional Regularization

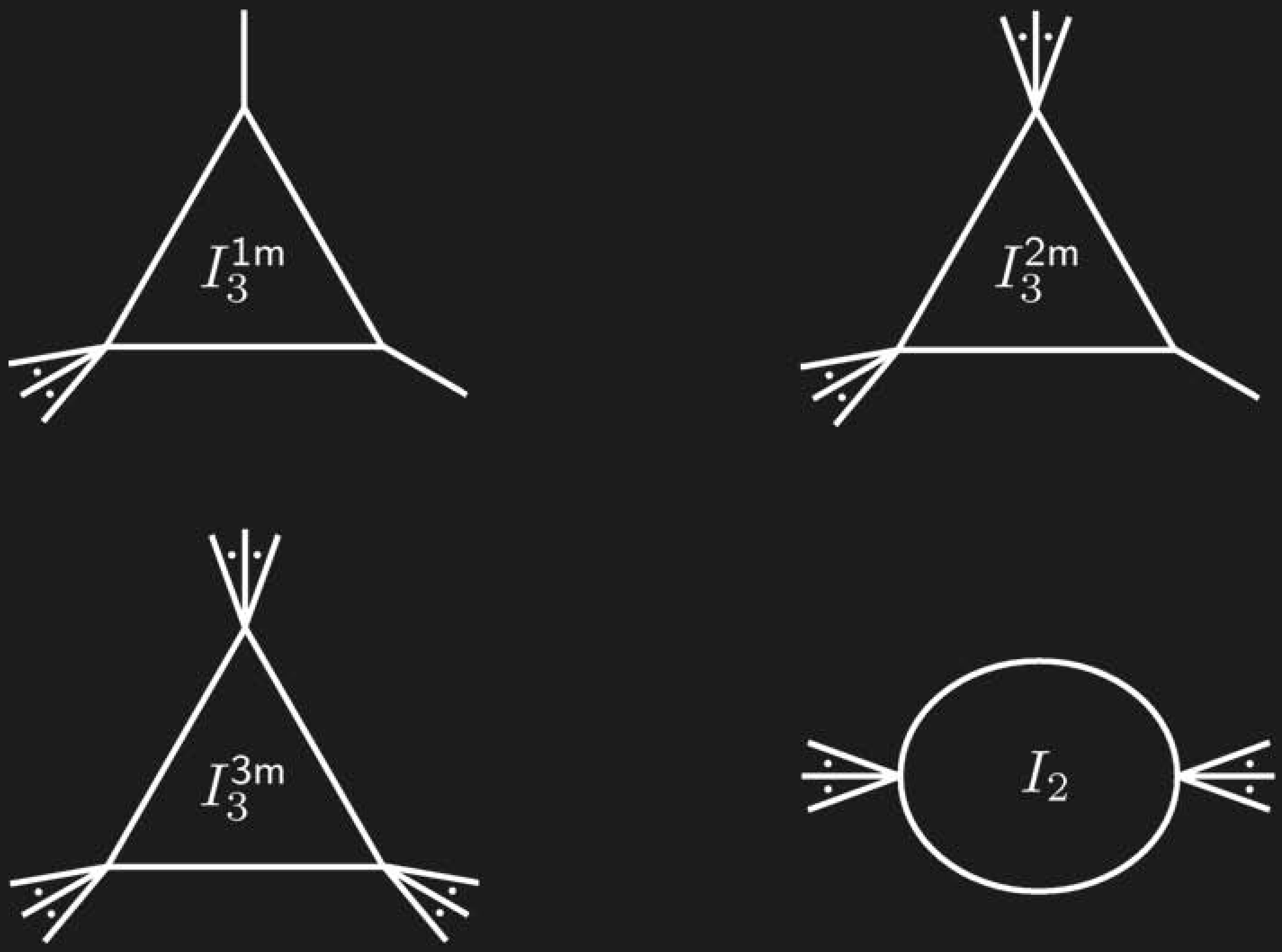
- Ultraviolet regulator;
- Infrared regulator;
- Handle on rational terms.
- Dimensional regularization effectively removes the ultraviolet divergence, rendering integrals convergent, and so removing the need for a subtraction in the dispersion relation
- Pedestrian viewpoint: dimensionally, there is always a factor of $(-s)^{-\epsilon}$, so at higher order in ϵ, even rational terms will have a factor of $\ln(-s)$, which has a discontinuity

Integral Reductions

- At one loop, all $n \geq 5$-point amplitudes in a massless theory can be written in terms of nine different types of scalar integrals:
- boxes (one-mass, 'easy' two-mass, 'hard' two-mass, three-mass, and four-mass);
- triangles (one-mass, two-mass, and three-mass);
- bubbles
- In an $\mathcal{N}=4$ supersymmetric theory, only boxes are needed.



$i-3$
$i-2$
$I_4^{\rm 1m}$
i
$i-1$
$i+r$
$I_4^{\rm 2me}$
i
$i-1$
$i+r+r'$
$i+r$
$I_4^{\rm 4m}$
$i+r+r'+r''$
i
$i+r$
$i-2$
$I_4^{\rm 2mh}$
i
$i-1$
$i+r+r'$
$i+r$
$I_4^{\rm 3m}$
i
$i-1$


I_3^{1m}
I_3^{2m}
I_3^{3m}
I_2

The Easy Two-Mass Box

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2(\ell-k_1)^2(\ell-K_{12})^2(\ell-K_{123})^2} =$$

$$\frac{c_\Gamma(\epsilon)}{st-m_2^2m_4^2}\left\{\frac{2}{\epsilon^2}\left[(-s)^{-\epsilon}+(-t)^{-\epsilon}-(-m_2^2)^{-\epsilon}-(-m_4^2)^{-\epsilon}\right]\right.$$

$$-2\,\mathrm{Li}_2\left(1-\frac{m_2^2}{s}\right)-2\,\mathrm{Li}_2\left(1-\frac{m_2^2}{t}\right)$$

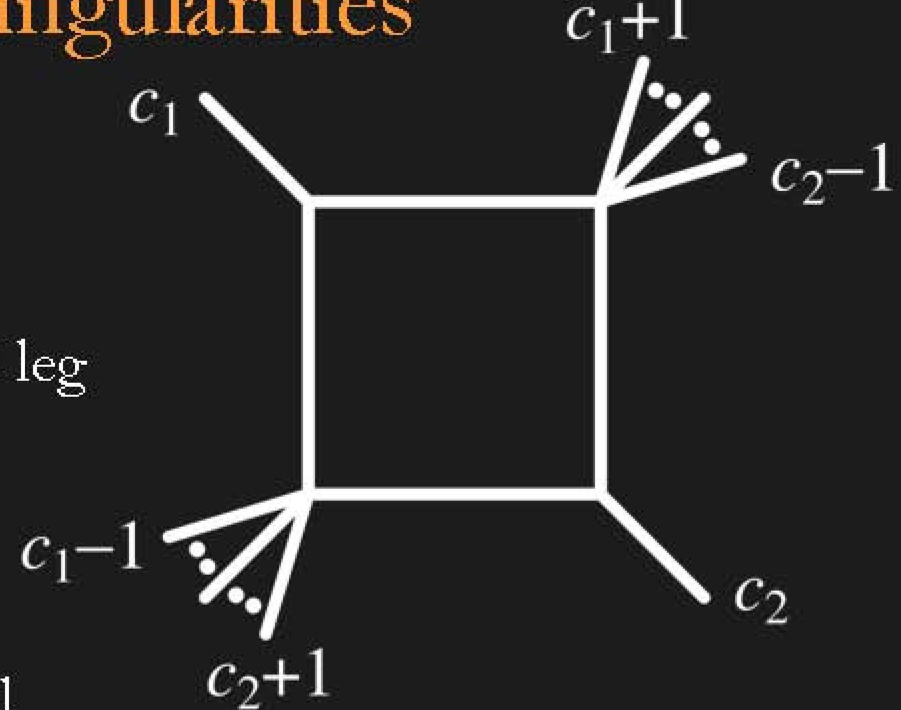
$$-2\,\mathrm{Li}_2\left(1-\frac{m_4^2}{s}\right)-2\,\mathrm{Li}_2\left(1-\frac{m_4^2}{t}\right)$$

$$\left.+2\,\mathrm{Li}_2\left(1-\frac{m_2^2m_4^2}{st}\right)-\ln^2\left(\frac{s}{t}\right)\right\}+\mathcal{O}(\epsilon)$$

Dilogarithm $\mathrm{Li}_2(x) = -\int_0^x dt\, \frac{\ln(1-t)}{t}$

Infrared Singularities

Loop momentum nearly on shell
and soft
or collinear with massless external leg
or both

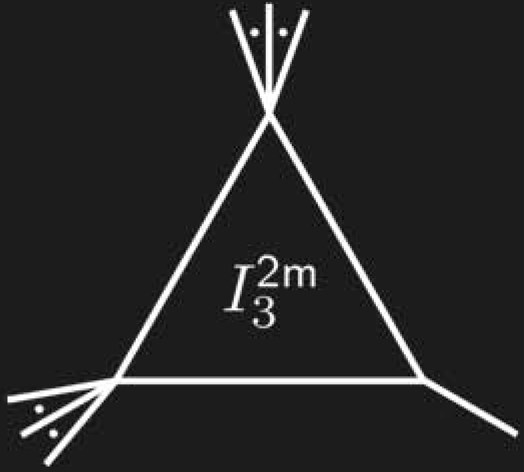


Coefficients of infrared poles and double poles must be proportional to the tree amplitude for cancellations to happen

Spurious Singularities

- When evaluating the two-mass triangle, we will obtain functions like

$$\frac{\ln(s_1/s_2)}{s_1 - s_2}$$



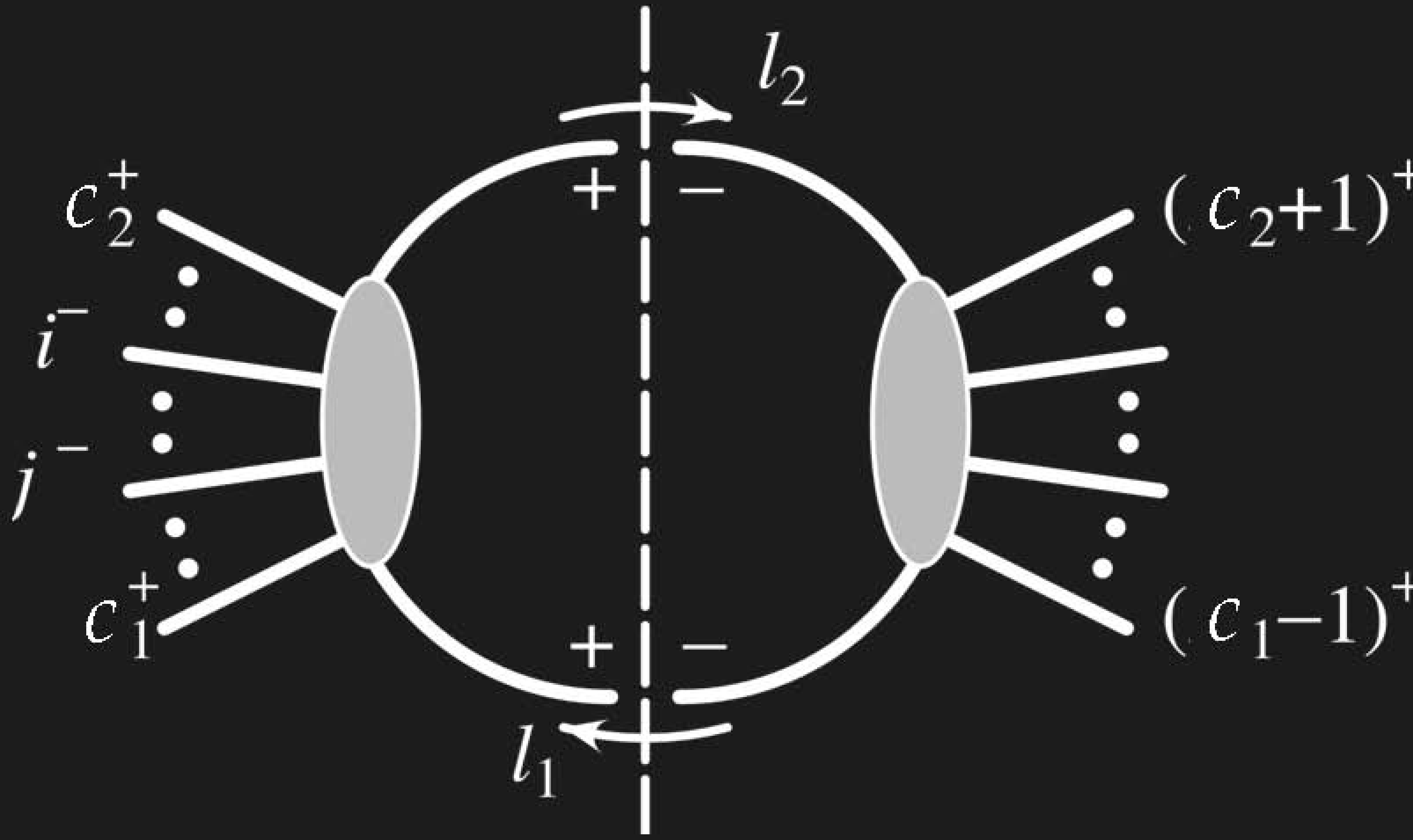
and $$\frac{\ln(s_1/s_2)}{(s_1 - s_2)^2} + \frac{1}{s_1 - s_2}$$

There can be no physical singularity as $s_1 \to s_2$

and there isn't

but cancellation happens non-trivially

Example: MHV at One Loop



- Start with the cut

$$\int d^4\,\mathrm{LIPS}(\ell_1,-\ell_2)\; A^{\rm tree}(-\ell_2, c_2{+}1,\ldots,c_1{-}1,\ell_1)$$
$$\times\, A^{\rm tree}(-\ell_1, c_1,\ldots,c_2,\ell_2)$$

- Use the known expressions for the MHV amplitudes

$$-\int d^4\,\mathrm{LIPS}(\ell_1,-\ell_2)\;\frac{\langle(-\ell_1)\,\ell_2\rangle^3}{\langle(-\ell_1)\,c_1\rangle\,\langle\!\langle c_1\cdots c_2\rangle\!\rangle\,\langle c_2\,\ell_2\rangle}$$
$$\times\,\frac{\langle i\,j\rangle^4}{\langle(-\ell_2)\,(c_2{+}1)\rangle\,\langle\!\langle(c_2{+}1)\cdots(c_1{-}1)\rangle\!\rangle\,\langle(c_1{-}1)\,\ell_1\rangle\,\langle\ell_1\,(-\ell_2)\rangle}$$

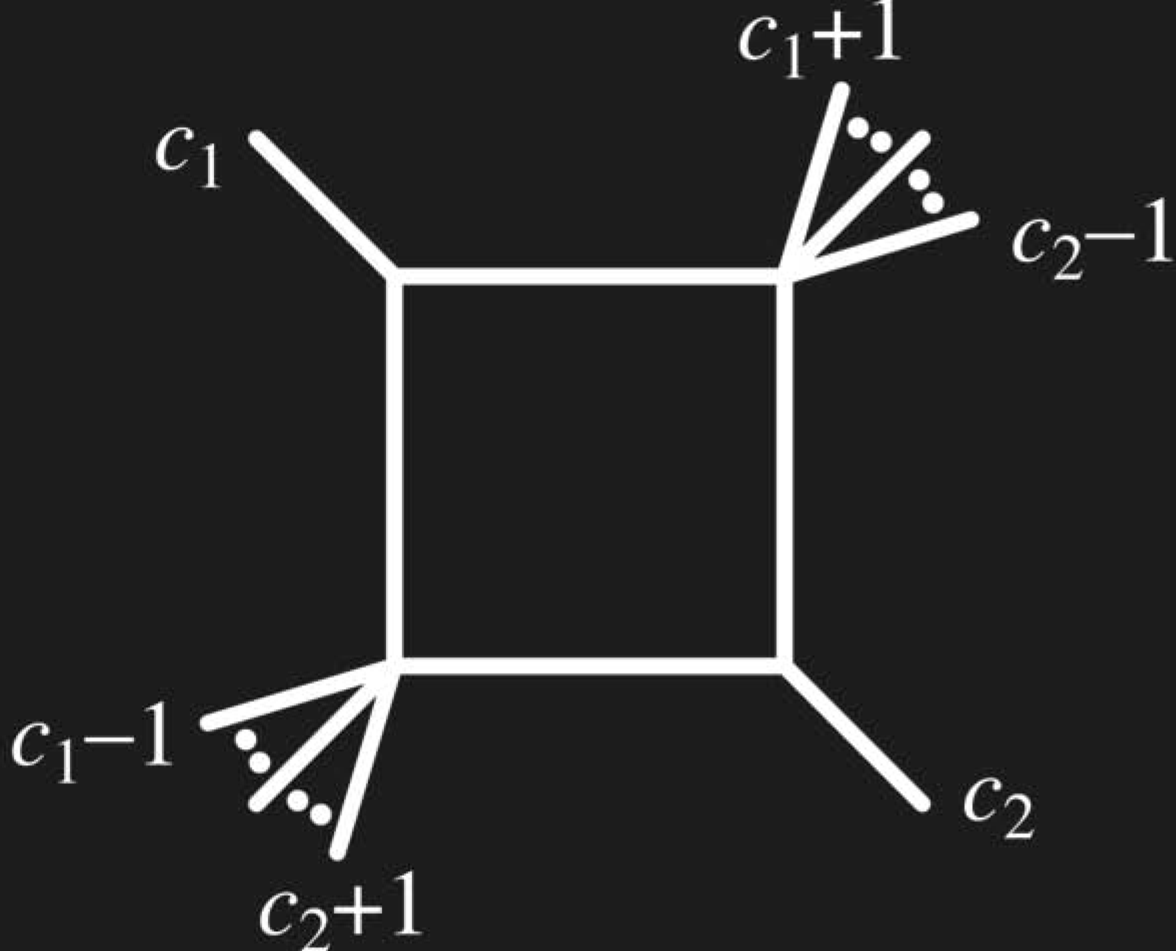
- Most factors are independent of the integration momentum

$$iA^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$
$$\times \int d^4\,{\rm LIPS}(\ell_1,-\ell_2)\;\frac{\langle (c_1-1)\,c_1\rangle\,\langle c_2\,(c_2+1)\rangle\,\langle \ell_1\,\ell_2\rangle^2}{\langle \ell_1\,c_1\rangle\,\langle c_2\,\ell_2\rangle\,\langle (c_1-1)\,\ell_1\rangle\,\langle \ell_2\,(c_2+1)\rangle}$$
$$= iA^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$
$$\times \int d^4\,{\rm LIPS}(\ell_1,-\ell_2)\;\langle (c_1-1)\,c_1\rangle\,\langle \ell_1\,\ell_2\rangle^2\,\langle c_2\,(c_2+1)\rangle$$
$$\times\;\frac{[c_1\,\ell_1]\,[\ell_2\,c_2]\,[(c_1-1)\,\ell_1]\,[\ell_2\,(c_2+1)]}{(\ell_1-k_{c_1})^2(\ell_2+k_{c_2})^2(\ell_1+k_{c_1-1})^2(\ell_2-k_{c_2+1})^2}$$

- We can use the Schouten identity to rewrite the remaining parts of the integrand,

$$
\begin{aligned}
&(\ell_1 + k_{c_1-1})^2 (\ell_2 - k_{c_2+1})^2 \, \tfrac{1}{2} \,\mathrm{Tr}\left[(1+\gamma_5)\not{\ell}_1 \not{k}_{c_2} \not{\ell}_2 \not{k}_{c_1}\right] \\
&\quad -\{k_{c_1-1} \leftrightarrow -k_{c_1}\} - \{k_{c_2+1} \leftrightarrow -k_{c_2}\} \\
&\quad +\{k_{c_1-1} \leftrightarrow -k_{c_1}, k_{c_2+1} \leftrightarrow -k_{c_2}\}
\end{aligned}
$$

- Two propagators cancel, so after a lot of algebra, and cancellation of triangles, we're left with a box — the γ_5 leads to a Levi-Civita tensor which vanishes upon integration
- What's left over is the same function which appears in the denominator of the box: $-st + m_2^2 m_4^2$


c_1+1
c_1
c_2-1
c_1-1
c_2
c_2+1

- We obtain the result,

$$-A^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$
$$\times \sum_{\text{easy 2 mass}} \text{Box}\cdot\tfrac{1}{2}(\text{its denominator})$$

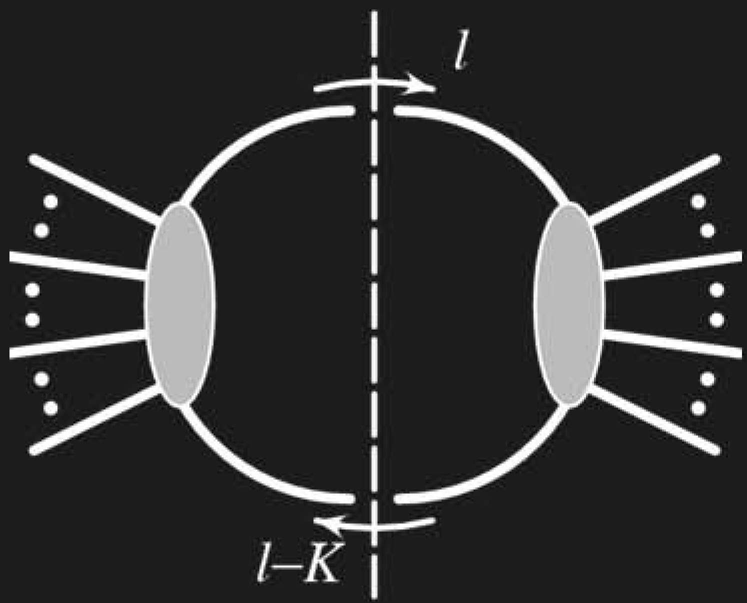
On-Shell Methods in Field Theory

David A. Kosower

International School of Theoretical Physics,
Parma,
September 10-15, 2006

Lecture V

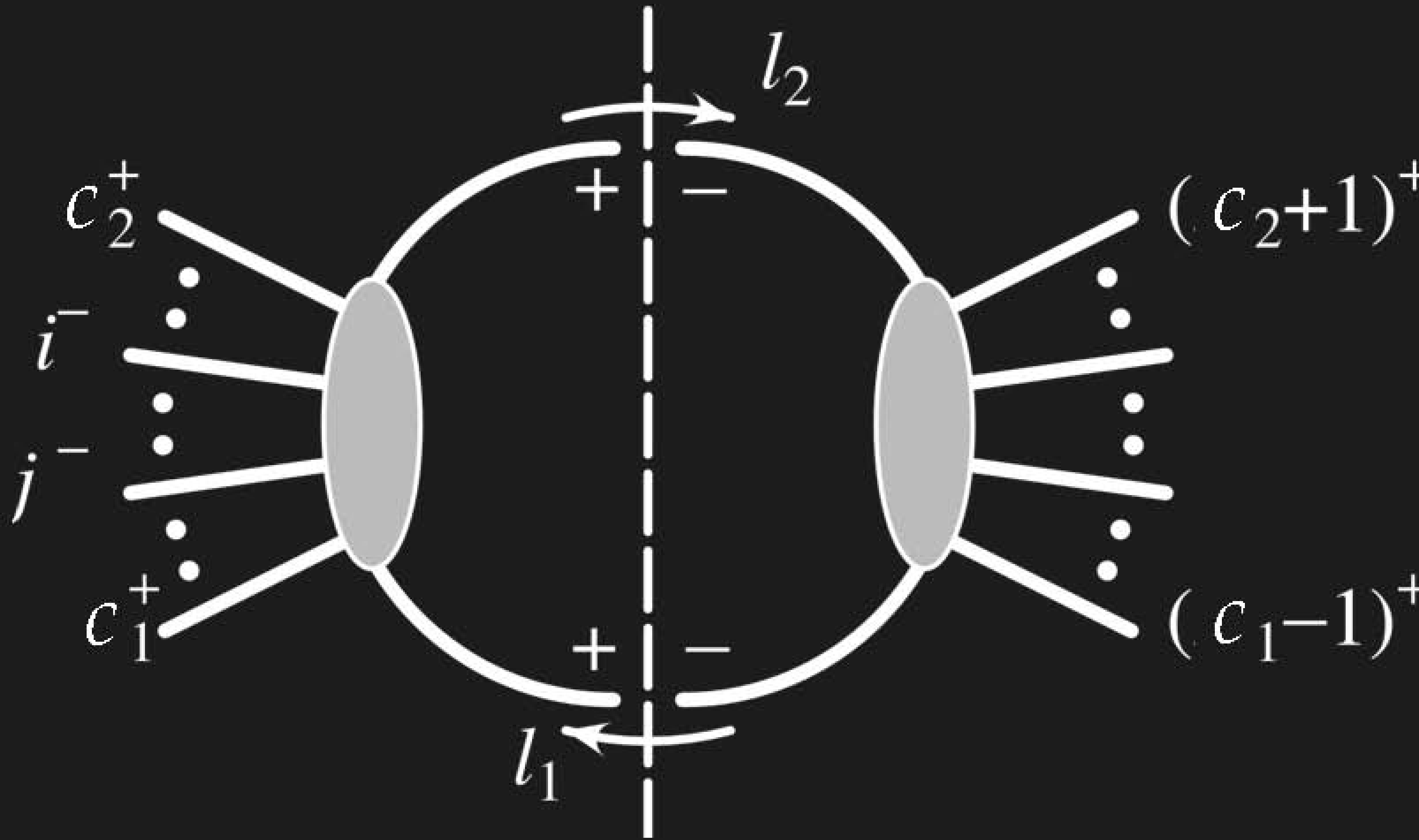
Unitarity-Based Method for Loops



Bern, Dixon, Dunbar, & DAK,
ph/9403226, ph/9409265

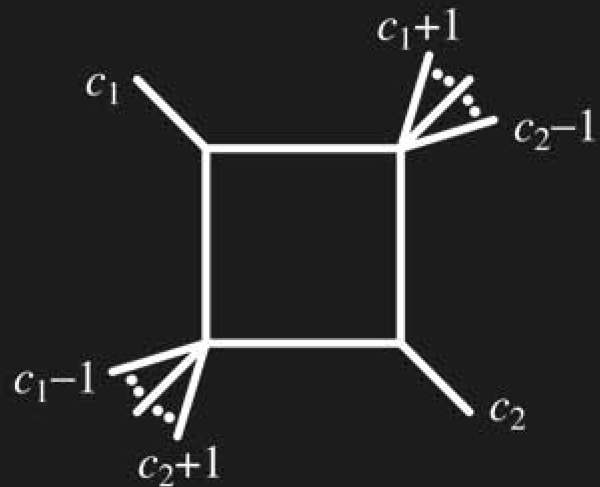
$$A^{\text{1-loop}} = \sum_{\text{cuts } K^2} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} A^{\text{tree}}_{\text{left}} \frac{i}{(\ell - K)^2} A^{\text{tree}}_{\text{right}}$$

Example: MHV at One Loop



The result,

$$-A^{\mathrm{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$
$$\times \sum_{\text{easy 2 mass}} \text{Box}\cdot\tfrac{1}{2}(\text{its denominator})$$

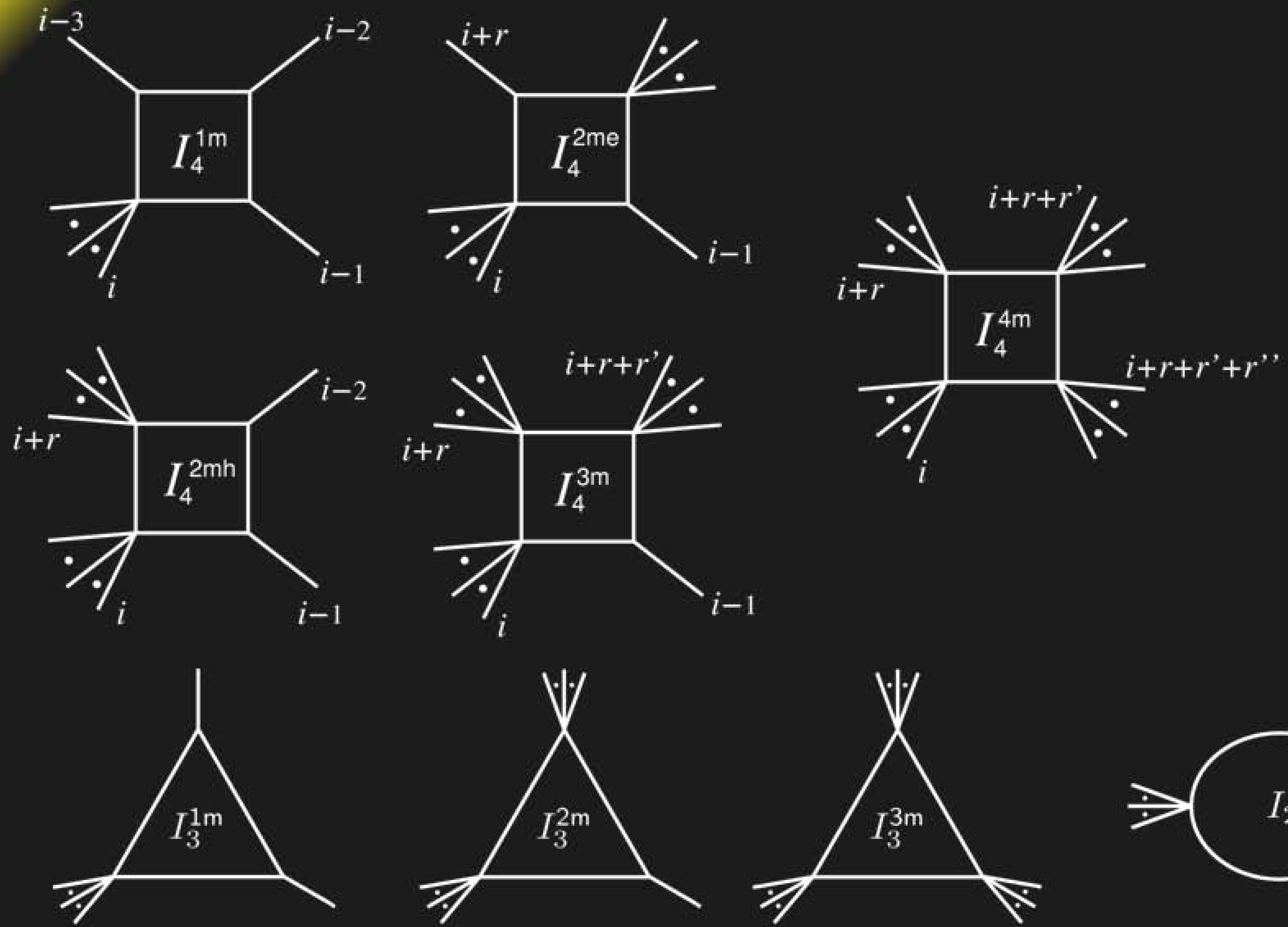


Have We Seen This Denominator Before?

$$\langle 5^- | 3+4 | 2^- \rangle$$

Consider

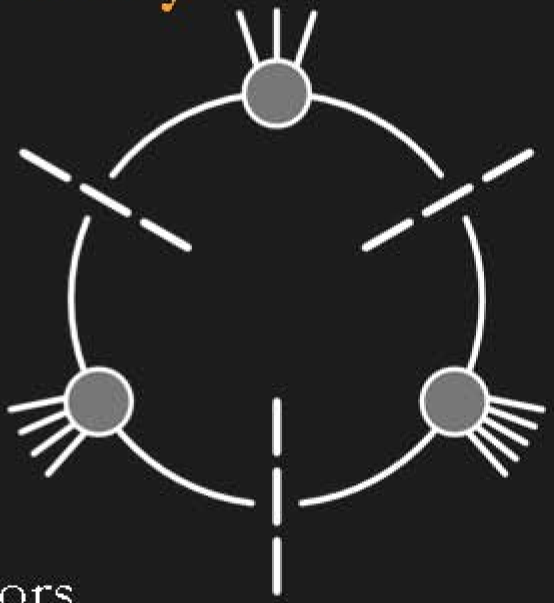
$$\langle c_1{}^- | \not{K}_{c_1\ldots c_2} | c_2{}^- \rangle \langle c_2{}^- | \not{K}_{c_1\ldots c_2} | c_1{}^- \rangle =$$



i−3
i−2
I_4^{1m}
i−1
i
i+r
I_4^{2me}
i−1
i
i+r+r'
i+r
I_4^{4m}
i+r+r'+r''
i
i−2
i+r
I_4^{2mh}
i
i−1
i+r+r'
i+r
I_4^{3m}
i
i−1
I_3^{1m}
I_3^{2m}
I_3^{3m}
I_2

$$A = \sum_j c_j I_j$$

Generalized Unitarity

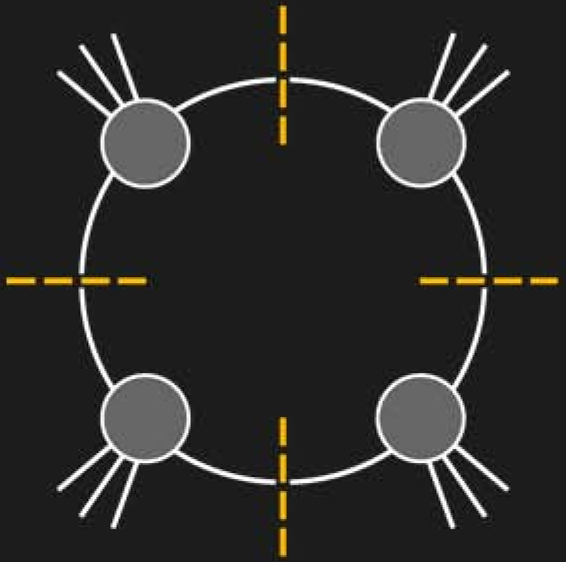


- Can sew together more than two tree amplitudes
- Corresponds to 'leading singularities'
- Isolates contributions of a smaller set of integrals: only integrals with propagators corresponding to cuts will show up

Bern, Dixon, DAK (1997)

- Example: in triple cut, only boxes and triangles will contribute

- Can we isolate a single integral?



- Quadruple cuts would isolate a single box

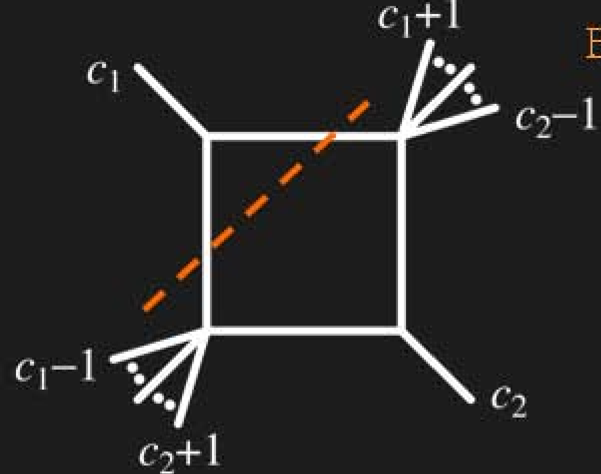
- Can't do this for one-mass, two-mass, or three-mass boxes because that would isolate a three-point amplitude

- Unless…

Cuts in Massless Channels

- With complex momenta, can form cuts using three-vertices too

Britto, Cachazo, & Feng, th/0412103



$\Rightarrow$ all box coefficients can be computed directly and algebraically, with no reduction or integration

- $\mathcal{N}=1$ and non-supersymmetric theories need triangles and bubbles, for which integration is still needed

Quadruple Cuts

Work in D=4 for the algebra

$$\int \frac{d^4\ell}{(2\pi)^4}\, \delta^{(+)}(\ell^2)\,\delta^{(+)}((\ell-k_1)^2)\,\delta^{(+)}((\ell-K_{12})^2)\,\delta^{(+)}((\ell-K_{123})^2)$$

$$\int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2(\ell-k_1)^2(\ell-K_{12})^2(\ell-K_{123})^2}$$

Four degrees of freedom & four delta functions

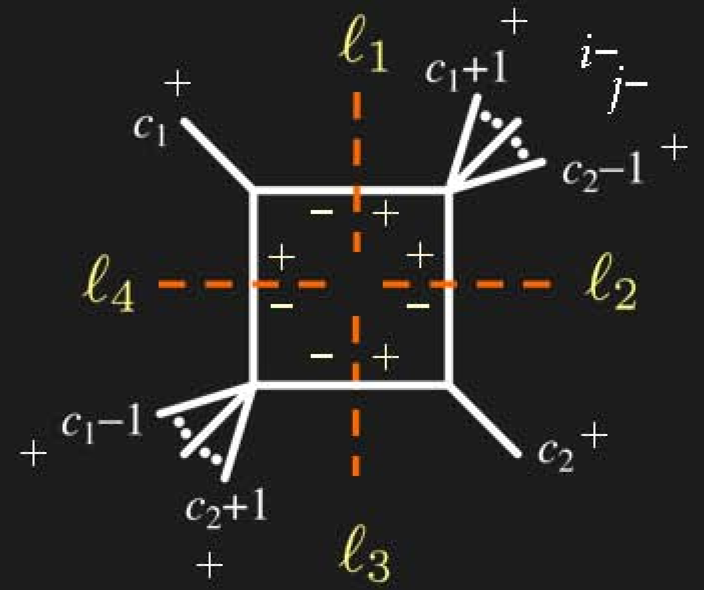
$\Rightarrow$ no integrals left, only algebra

$$\frac{1}{\#\text{ solutions}} \sum_{\substack{\text{solutions}\\ \text{helicities}}} A_a^{\text{tree}} A_b^{\text{tree}} A_c^{\text{tree}} A_d^{\text{tree}}$$

2

MHV

- Coefficient of a specific easy two-mass box: only one solution will contribute



$$\langle \ell_1\, c_1\rangle = 0 = \langle \ell_4\, c_1\rangle\,,$$
$$\langle \ell_2\, c_2\rangle = 0 = \langle \ell_3\, c_2\rangle$$

$$\tfrac{1}{2}A_3((-\ell_4)^+, c_1^+, \ell_1^-)A((-\ell_1)^+, c_1{+}1, \ldots, i^-, \ldots, j^-, \ldots, c_2{-}1, \ell_2^+)$$
$$\times A_3((-\ell_2)^-, c_2^+, \ell_3^+)A((-\ell_3)^-, c_2{+}1, \ldots, c_1{-}1, \ell_4^-)$$
$$= \frac{1}{2}\left(\frac{[\ell_4\, c_1]^3}{[c_1\, \ell_1]\,[\ell_1\, \ell_4]}\right)\left(\frac{\langle i\, j\rangle^4}{\langle \ell_1\,(c_1{+}1)\rangle\,\langle\!\langle (c_1{+}1)\cdots(c_2{-}1)\rangle\!\rangle\,\langle (c_2{-}1)\,\ell_2\rangle\,\langle \ell_2\,\ell_1\rangle}\right)$$
$$\times\left(\frac{[c_2\,\ell_3]^3}{[\ell_2\, c_2]\,[\ell_3\,\ell_2]}\right)\left(\frac{\langle \ell_4\,\ell_3\rangle^3}{\langle \ell_3\,(c_2{+}1)\rangle\,\langle\!\langle (c_2{+}1)\cdots(c_1{-}1)\rangle\!\rangle\,\langle (c_1{-}1)\,\ell_4\rangle}\right)$$
$$= \frac{1}{2}A_n^{\rm tree}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)$$
$$\times\frac{\langle (c_1{-}1)\,c_1\rangle\,\langle c_1\,(c_1{+}1)\rangle\,\langle (c_2{-}1)\,c_2\rangle\,\langle c_2\,(c_2{+}1)\rangle}{\langle (c_1{-}1)\,\ell_4\rangle\,\langle \ell_1\,(c_1{+}1)\rangle\,\langle (c_2{-}1)\,\ell_2\rangle\,\langle \ell_3\,(c_2{+}1)\rangle}$$
$$\times\frac{\left\langle c_1^{\,+}\right|\not\ell_4\not\ell_3\left|c_2^{\,-}\right\rangle^3}{\left\langle c_1^{\,+}\right|\not\ell_1\not\ell_2\left|c_2^{\,-}\right\rangle\,[\ell_1\,\ell_4]\,[\ell_2\,\ell_3]}$$

Using momentum conservation

$$\begin{aligned} \langle c_1{}^+|\,\not\ell_4\not\ell_3\,|c_2{}^-\rangle &= \langle c_1{}^+|\,\not\ell_1\not\ell_2\,|c_2{}^-\rangle \\ \frac{\langle (c_1-1)\,c_1\rangle}{\langle (c_1-1)\,\ell_4\rangle\,[\ell_1\,\ell_4]} &= \frac{1}{[\ell_1\,c_1]} \end{aligned}$$

our expression becomes

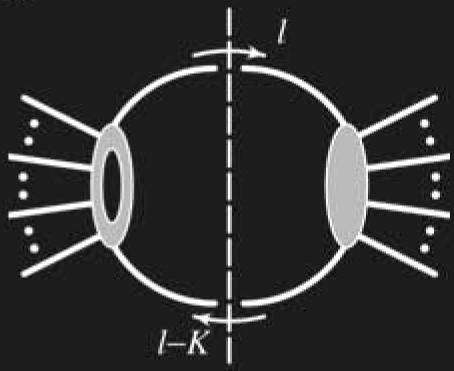
$$\begin{aligned} &\frac{1}{2}A_n^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)\,\frac{\langle c_1{}^+|\,\not\ell_4\not\ell_3\,|c_2{}^-\rangle^2\,[\ell_1\,\ell_4]\,[\ell_2\,\ell_3]}{[c_1\,\ell_1]\,[c_1\,\ell_4]\,[\ell_3\,c_2]\,[\ell_2\,c_2]} \\ &= \frac{1}{2}A_n^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)\,[\ell_1\,\ell_4]\,\langle \ell_4\,\ell_3\rangle\,[\ell_3\,\ell_2]\,\langle \ell_2\,\ell_1\rangle \\ &= A_n^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) \\ &\qquad \times \langle c_1{}^-|\,\ell_1+\ell_2\,|c_2{}^-\rangle\,\langle c_2{}^-|\,\ell_3+\ell_4\,|c_1{}^-\rangle \\ &= -A_n^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) \\ &\qquad \times \langle c_1{}^-|\,K_{(c_1+1)\cdots(c_2-1)}\,|c_2{}^-\rangle\,\langle c_2{}^-|\,K_{(c_1+1)\cdots(c_2-1)}\,|c_1{}^-\rangle \\ &= \frac{1}{2}A_n^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)\,(m_2^2 m_4^2 - st) \end{aligned}$$

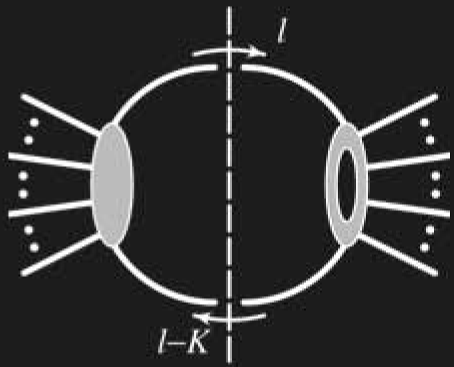
Higher Loops

- Technology for finding a set of *master* integrals for any given process: "integration by parts", solved using Laporta algorithm
- But no general basis is known
- So we may have to reconstruct the integrals in addition to computing their coefficients

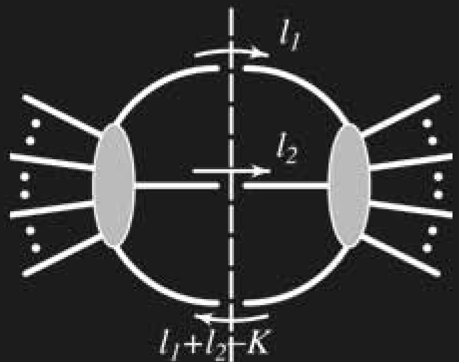
Unitarity-Based Method at Higher Loops

- Loop amplitudes on either side of the cut





- Multi-particle cuts in addition to two-particle cuts



- Find integrand/integral with given cuts in all channels

Generalized Cuts

- In practice, replace loop amplitudes by their cuts too

Computing QCD Amplitudes

$\mathcal{N}=4 \;=$ pure QCD + 4 fermions + 3 complex scalars

$\text{QCD} = \;\mathcal{N}=4 \;+\; \delta\mathcal{N}=1 \;+\; \delta\mathcal{N}=0$

	chiral multiplet	scalar	
cuts + rational	cuts + rational	cuts	+ rational
D=4 unitarity	D=4 unitarity	D=4 unitarity	bootstrap *or* on-shell recursion relations; $D=4-2\varepsilon$ unitarity

Rational Terms

- At tree level, we used on-shell recursion relations
- We want to do the same thing here
- Need to confront
  - Presence of branch cuts
  - Structure of factorization

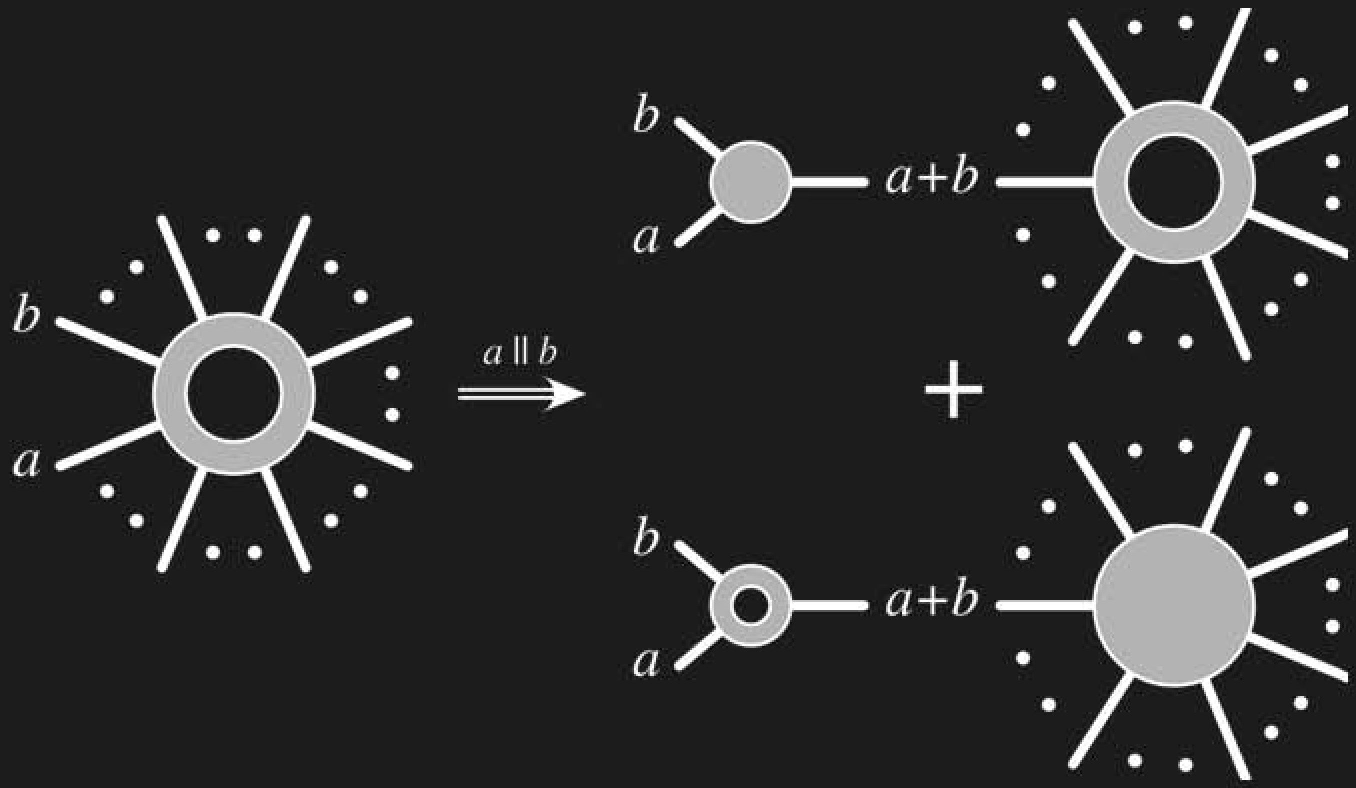
- At tree level,

$$A_n(1^\pm, 2^+, \ldots, n^+) = 0$$

- True in supersymmetric theories at all loop orders
- Non-vanishing at one loop in QCD
- but finite: no possible UV or IR singularities
- Separate V and F terms

$$A_n = V_n A_n^{\text{tree}} + F_n$$

Factorization at One Loop



Collinear Factorization at One Loop

- Most general form we can get is antisymmetric + nonsingular: two independent tensors for splitting amplitude

$$f_1(s_{ab}, z)\, \frac{1}{s_{ab}} \left(\varepsilon_a^\mu \varepsilon_b \cdot k_a - \varepsilon_b^\mu \varepsilon_a \cdot k_b + \tfrac{1}{2}(k_b^\mu - k_a^\mu)\varepsilon_a \cdot \varepsilon_b\right)$$

$$f_2(s_{ab}, z)\, \frac{(k_a^\mu - k_b^\mu)}{s_{ab}} \left(\varepsilon_a \cdot \varepsilon_b - \frac{\varepsilon_a \cdot k_b \varepsilon_b \cdot k_a}{k_a \cdot k_b}\right)$$

- Second tensor arises only beyond tree level, and only for like helicities

- Explicit form of +++ splitting amplitude

$$\text{Split}_{+}^{1-\text{loop, scalar}}(z;a^+,b^+) = -\frac{1}{48\pi^2}\sqrt{z(1-z)}\frac{[a\,b]}{\langle a\,b\rangle^2}$$

- No general theorems about factorization in complex momenta
- Just proceed
- Look at $-+\ldots++$

- Amplitudes contain factors like $\frac{[a\,b]}{\langle a\,b\rangle^2}$ known from collinear limits
- Expect also $\frac{[a\,b]}{\langle a\,b\rangle}$ as 'subleading' contributions, seen in explicit results

'unreal' poles

- Double poles with vertex $V_3(+++)$
- Non-conventional single pole: one finds the double-pole, multiplied by

$$s_{ab}\ \mathrm{Soft}(\hat{a}, (-\hat{P}_{ab})^-, b)\ \mathrm{Soft}(b+1, \hat{P}_{ab}^+, a-1)$$

On-Shell Recursion at Loop Level

Bern, Dixon, DAK (1–7/2005)

- Finite amplitudes are purely rational
- We can obtain simpler forms for known finite amplitudes (Chalmers, Bern, Dixon, DAK; Mahlon)
- These again involve spurious singularities
- Obtained last of the finite amplitudes: $f^- f^+ g^+ \ldots g^+$

On-Shell Recursion at Loop Level

Bern, Dixon, DAK (1–7/2005)

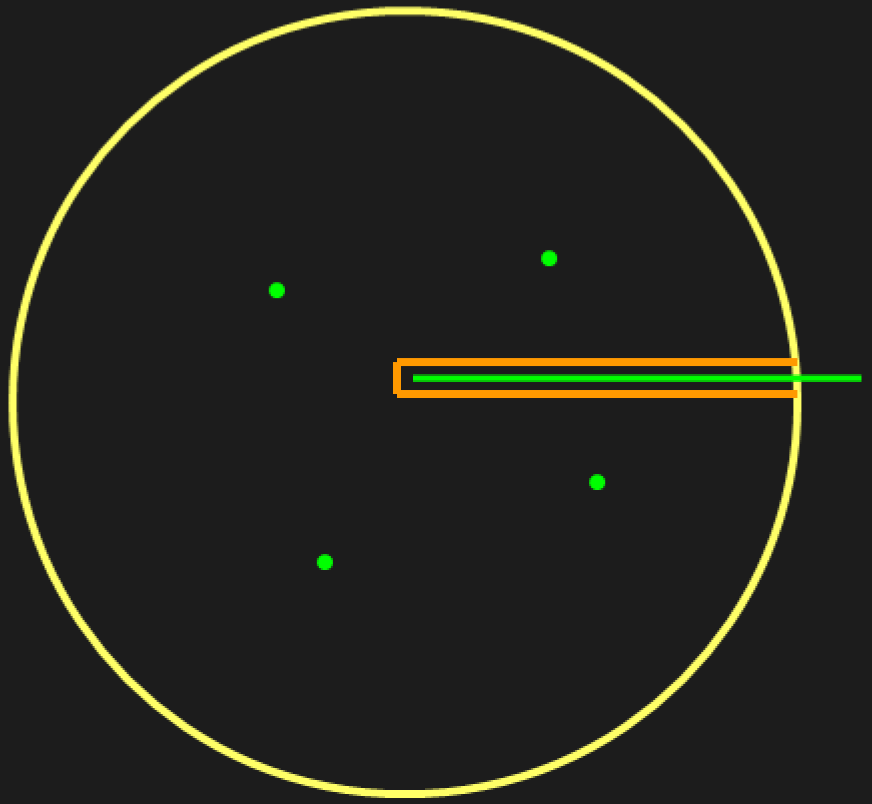
- Complex shift of momenta $[j, l\rangle$

$$p_j^\mu \to p_j^\mu(z) = p_j^\mu - \frac{z}{2}\langle j^- |\, \gamma^\mu \,| l^- \rangle,$$
$$p_l^\mu \to p_l^\mu(z) = p_l^\mu + \frac{z}{2}\langle j^- |\, \gamma^\mu \,| l^- \rangle.$$

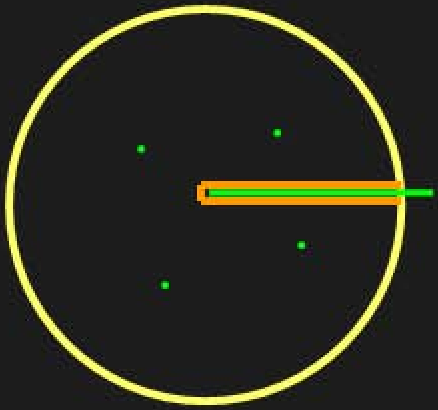
- Behavior as z $\to \infty$: require $A(\mathrm{z}) \to 0$
- Basic complex analysis: treat branch cuts
- Knowledge of *complex* factorization:
  - at tree level, tracks known factorization for real momenta
  - at loop level, same for multiparticle channels; and $- \to -+$
  - Avoid $\pm \to ++$

Rational Parts of QCD Amplitudes

- Start with cut-containing parts obtained from unitarity method, consider same contour integral



Derivation



- Consider the contour integral

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z}\, A(z)$$

- Determine A(0) in terms of other poles and branch cuts

$$A(0) = -\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{A(z)}{z} - \int_{\text{Branch}} \frac{dz}{z}\, \text{Disc}_B A(z)$$

Rational terms

Cut terms

- Cut terms have spurious singularities $\Rightarrow$ rational terms do too

$$\frac{\ln(s_1/s_2)}{(s_1 - s_2)^2}$$

- $\Rightarrow$ the sum over residues includes spurious singularities, for which there is no factorization theorem at all

Completing the Cut

- To solve this problem, define a modified 'completed' cut, adding in rational functions to cancel spurious singularities

$$\frac{\ln(s_1/s_2)}{(s_1-s_2)^2}+\frac{1}{s_1-s_2}$$

- We know these have to be there, because they are generated together by integral reductions
- Spurious singularity is unique
- Rational term is not, but difference is free of spurious singularities

- This eliminates residues of spurious poles

- $\widehat{C}$ entirely known from four-dimensional unitarity method
- Assume $\widehat{C}(z) \to 0$ as $z \to \infty$
- Modified separation

$$A_n(z) = c_\Gamma \left[\hat{C}(z) + \hat{R}(z)\right]$$

so

$$\begin{aligned} A(0) &= -\sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{C}(z)}{z} - \int_{\text{Branch}} \frac{dz}{z} \text{ Disc}_B \hat{C}(z) \\ &\quad - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{R}(z)}{z} \end{aligned}$$

- Perform integral & residue sum for $\widehat{C}$

$$\hat{C}(0) \quad = \quad - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{C}(z)}{z} - \int_{\text{Branch}} \frac{dz}{z} \; \text{Disc}_{\,B} \hat{C}(z)$$

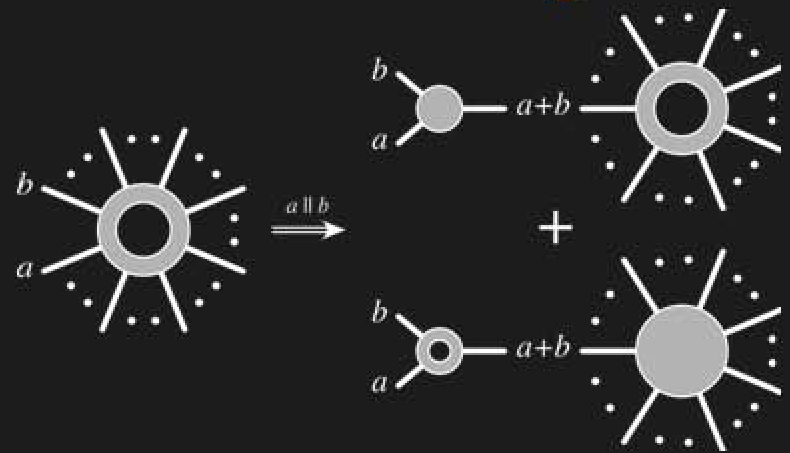
so

$$A(0) = c_\Gamma \left[\hat{C}(0) - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{R}(z)}{z} \right]$$

Unitarity Method

???

A Closer Look at Loop Factorization



- Only single poles in splitting amplitudes with cuts (like tree)
- Cut terms $\longrightarrow$ cut terms
- Rational terms $\longrightarrow$ rational terms
- Build up the latter using recursion, analogous to tree level

- Recursion on rational pieces would build up rational terms R, not $\hat{R}$

- Recursion gives

$$\text{Recursive} = -\sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\text{Rational}[\hat{C}(z)]}{z} - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{R}(z)}{z}$$

Double-counted: 'overlap'

- Subtract off overlap terms

$$A(0) = c_\Gamma \left[\hat{C}(0) + \text{Recursive} + \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\text{Rational}[\hat{C}(z)]}{z} \right]$$

Compute explicitly from known $\hat{C}$:
also have a diagrammatic expression

Tree-level On-Shell Recursion Relations

- Partition P: two or more cyclicly-consecutive momenta containing j, such that complementary set $\overline{P}$ contains l,

$$
\begin{aligned}
P &\equiv \{P_1, P_2, \ldots, j, \ldots, P_{-1}\}, \\
\overline{P} &\equiv \{\overline{P}_1, \overline{P}_2, \ldots, l, \ldots, \overline{P}_{-1}\}, \\
P \cup \overline{P} &= \{1, 2, \ldots, n\}
\end{aligned}
$$

On shell

- The recursion relations are

$$
\begin{aligned}
A_n(1,\ldots,n) &= \sum_{\substack{\text{partitions P} \\ h=\pm}} A_{\#P+1}(k_{P_1},\ldots,\hat{j},\ldots,k_{P_{-1}},-\widehat{P}^{h}) \\
&\quad \times \frac{i}{P^2} \times A_{\#\overline{P}+1}(k_{\overline{P}_1},\ldots,\hat{l},\ldots,k_{\overline{P}_{-1}},\widehat{P}^{-h})
\end{aligned}
$$

Recursive Diagrams

$$-\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{R_n(z)}{z} \equiv \text{Recursive Diagrams}$$

$$\begin{aligned}
= \sum_{\substack{\text{partitions } P \\ h=\pm}} \Big\{ & R_n^{(1)}(k_{P_1},\ldots,\hat{k}_j,\ldots,k_{P_{-1}},-\widehat{P}^h) \\
& \times \frac{i}{P^2} \times A_n^{\text{tree}}(k_{\overline{P}_1},\ldots,\hat{k}_l,\ldots,k_{\overline{P}_{-1}},\widehat{P}^{-h}) \\
& + A_n^{\text{tree}}(k_{P_1},\ldots,\hat{k}_j,\ldots,k_{P_{-1}},-\widehat{P}^h) \\
& \times \frac{i}{P^2} \times R_n^{(1)}(k_{\overline{P}_1},\ldots,\hat{k}_l,\ldots,k_{\overline{P}_{-1}},\widehat{P}^{-h}) \\
& + A_n^{\text{tree}}(k_{P_1},\ldots,\hat{k}_j,\ldots,k_{P_{-1}},-\widehat{P}^h) \\
& \times \frac{iR^{\text{Fact}}}{P^2} \times A_n^{\text{tree}}(k_{\overline{P}_1},\ldots,\hat{k}_l,\ldots,k_{\overline{P}_{-1}},\widehat{P}^{-h}) \Big\}
\end{aligned}$$

Five-Point Example

- Look at $F_5^s(1^-,2^-,3^+,4^+,5^+)$

- Cut terms

$$\frac{1}{6}\frac{\langle 1\,2\rangle^2\left(\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle+\langle 2\,3\rangle\,[4\,5]\,\langle 5\,1\rangle\right)}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\frac{\mathrm{L}_0\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}}$$

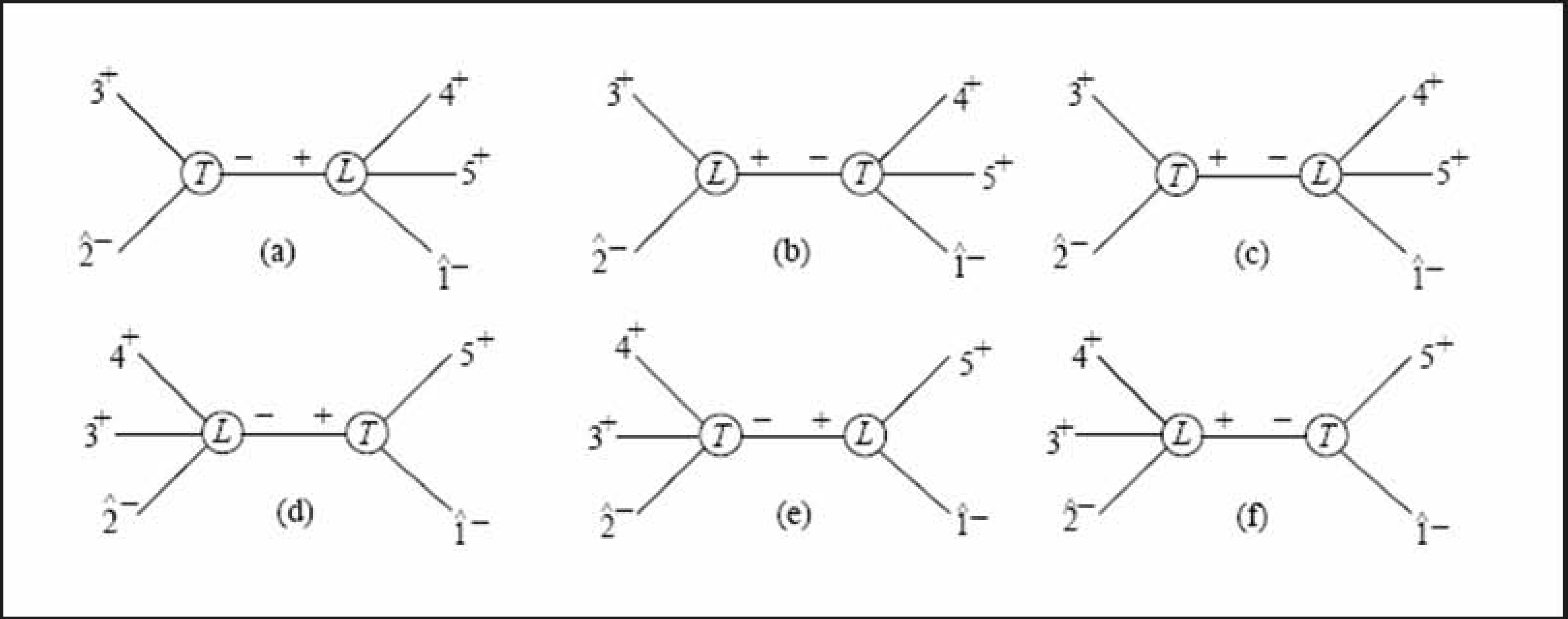
$$-\frac{1}{3}\frac{[3\,4]\,\langle 4\,1\rangle\,\langle 2\,4\rangle\,[4\,5]\left(\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle+\langle 2\,3\rangle\,[4\,5]\,\langle 5\,1\rangle\right)}{\langle 3\,4\rangle\,\langle 4\,5\rangle}\frac{\mathrm{L}_2\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}^3}$$

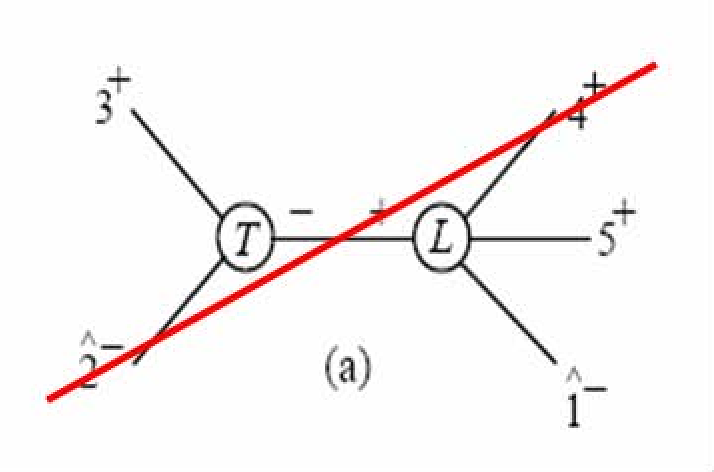
- have required large-z behavior

$$\mathrm{L}_0(r)=\frac{\ln r}{1-r},\qquad \mathrm{L}_2(r)=\frac{\ln r-(r-1/r)/2}{(1-r)^3}$$

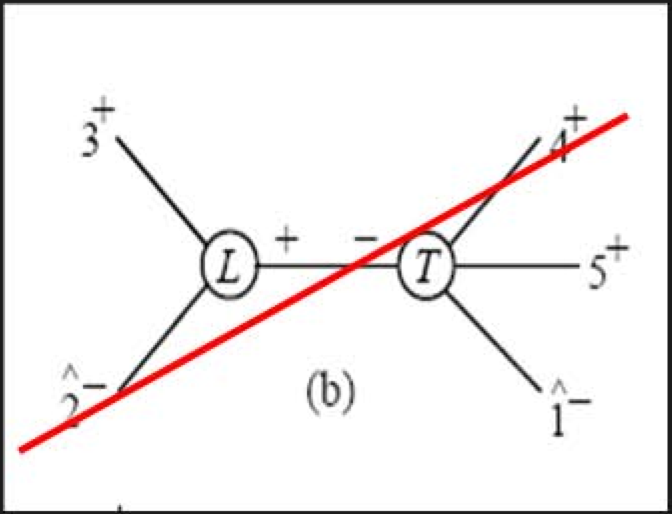
Five-Point Example

- Look at $F_5^s(1^-, 2^-, 3^+, 4^+, 5^+)$
- Recursive diagrams: use $[1, 2\rangle$ shift

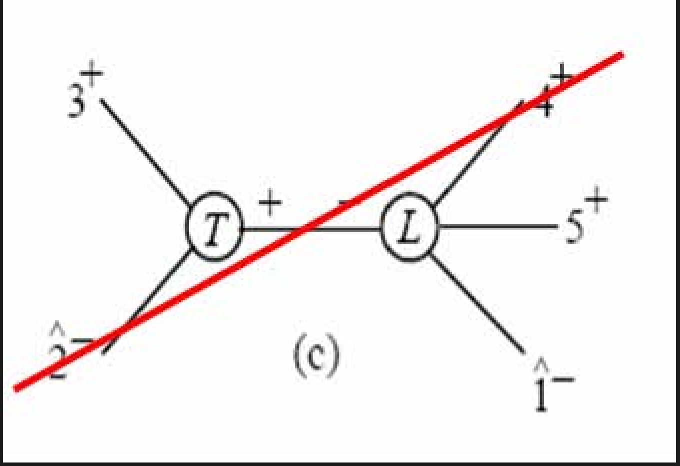




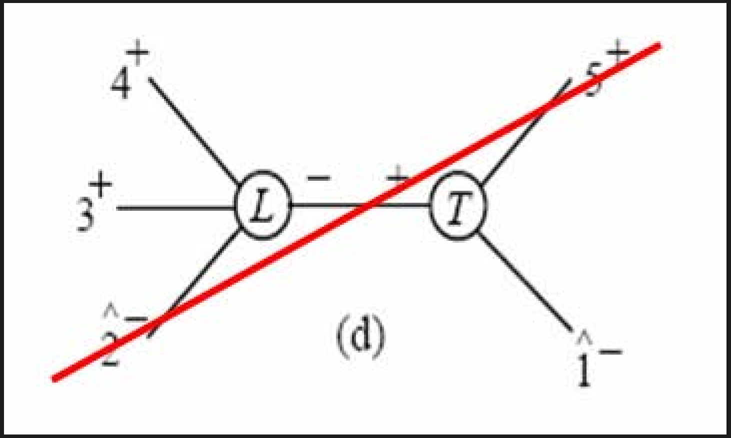
(a) Tree vertex $A(\hat{2}^-, 3^+, -\widehat{P}^-)$ vanishes



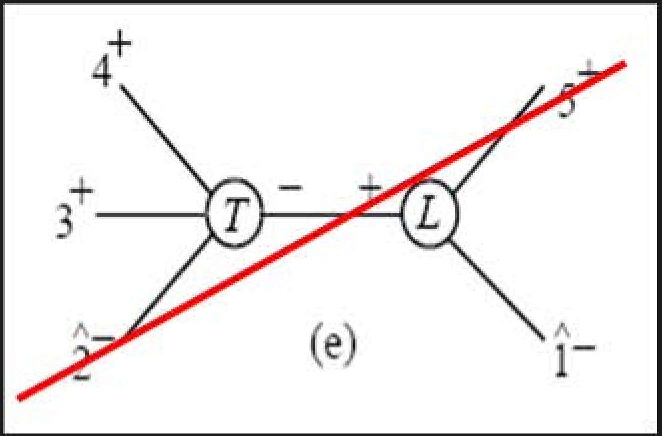
(b) Loop vertex $R(\hat{2}^-, 3^+, -\widehat{P}^-)$ vanishes



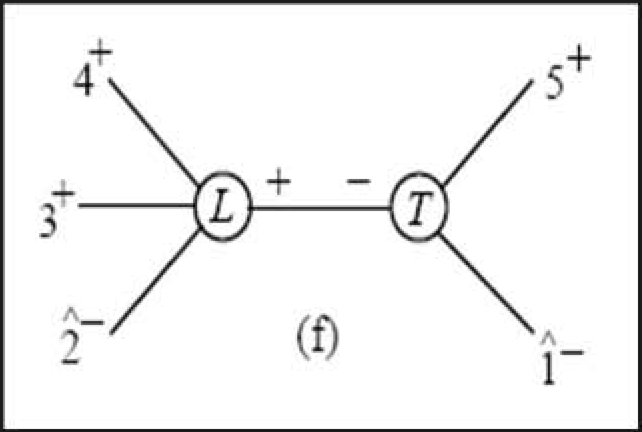
(c) Loop vertex $R_4(\hat{1}^-, \widehat{P}^-, 4^+, 5^+)$ vanishes



(d) Tree vertex $A(\hat{1}^-, 5^+, \widehat{P}^+)$ vanishes



(e) Loop vertex $R(\hat{1}^-, 5^+, \widehat{P}^+)$ vanishes



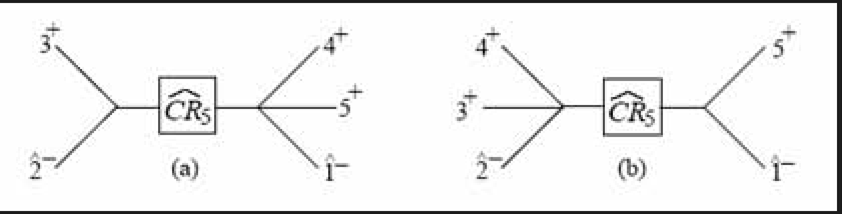
(f) Diagram doesn't vanish

$$D_5^{(\mathrm{f})} = A_3^{\mathrm{tree}}(5^+, \hat{1}^-, -\widehat{P}^-) \times \frac{i}{s_{51}} \times R_4(\hat{2}^-, 3^+, 4^+, \widehat{P}^+)$$

$$
\begin{aligned}
D_5^{(\mathrm{f})} &= -\frac{1}{3}\,\frac{\langle \hat{1}\,(-\widehat{P})\rangle^3}{\langle 5\,\hat{1}\rangle\;\langle(-\widehat{P})\,5\rangle}\,\frac{1}{s_{51}}\,\frac{\langle 3\,\widehat{P}\rangle\;[3\,\widehat{P}]^3}{[\hat{2}\,3]\,\langle 3\,4\rangle\;\langle 4\,\widehat{P}\rangle\;[\widehat{P}\,\hat{2}]} \\
&= \frac{1}{3}\,\frac{\left\langle 1^-\middle|\,5\,\middle|2^-\right\rangle^3}{\langle 5\,1\rangle\left\langle 5^-\middle|\,1\,\middle|2^-\right\rangle}\,\frac{1}{\langle 5\,1\rangle\,[1\,5]\left\langle 1^-\middle|\,5\,\middle|2^-\right\rangle^2} \\
&\qquad \times\frac{\left\langle 3^-\middle|\,4\,\middle|2^-\right\rangle\left\langle 1^-\middle|\,5\,\middle|3^-\right\rangle^3}{[2\,3]\,\langle 3\,4\rangle\left\langle 4^-\middle|\,3\,\middle|2^-\right\rangle\left\langle 1^-\middle|\,5\,\middle|2^-\right\rangle} \\
&= -\frac{1}{3}\,\frac{[2\,4]\,[3\,5]^3}{\langle 3\,4\rangle\,[1\,2]\,[1\,5]\,[2\,3]^2}
\end{aligned}
$$

Five-Point Example (cont.)

- 'Overlap' contributions $F_5^s(1^-,2^-,3^+,4^+,5^+)$



- Take rational terms in $\hat{C}$

$$\widehat{CR}_5 = -\frac{1}{6}\frac{s_{51}+s_{23}}{s_{23}s_{51}(s_{51}-s_{23})^2} \times \frac{[3\,4]\langle 4\,1\rangle\langle 2\,4\rangle[4\,5]\left(\langle 2\,3\rangle[3\,4]\langle 4\,1\rangle+\langle 2\,4\rangle[4\,5]\langle 5\,1\rangle\right)}{\langle 3\,4\rangle\langle 5\,1\rangle}$$

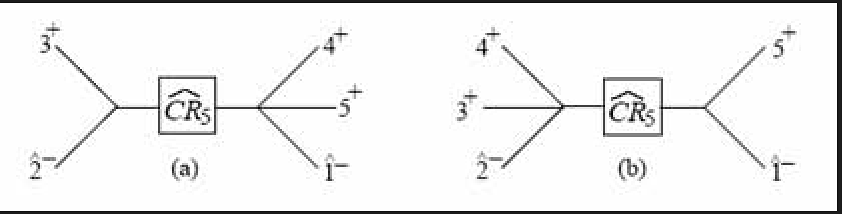
- Apply shift, extract residues in each "channel"

$$\widehat{CR}_5(z) = -\frac{1}{6}\frac{[3\,4]\,\langle 4\,1\rangle\,(\langle 2\,4\rangle + z\,\langle 1\,4\rangle)\,[4\,5]}{\langle 3\,4\rangle\,\langle 4\,5\rangle}$$

$$\times\frac{\Big((\langle 2\,3\rangle + z\,\langle 1\,3\rangle)\,[3\,4]\,\langle 4\,1\rangle + (\langle 2\,4\rangle + z\,\langle 1\,4\rangle)\,[4\,5]\,\langle 5\,1\rangle\Big)}{\big(s_{51} - s_{23} - z\,\langle 1^-|\,(3+5)\,|2^-\rangle\big)^2}$$

$$\times\frac{s_{51} + s_{23} - z\,\langle 1^-|\,5\,|2^-\rangle + z\,\langle 1^-|\,3\,|2^-\rangle}{(\langle 2\,3\rangle + z\,\langle 1\,3\rangle)\,[3\,2]\,\langle 1\,5\rangle\,([5\,1] - z\,[5\,2])}$$

- ‘Overlap’ contributions $F_5^s(1^-, 2^-, 3^+, 4^+, 5^+)$



$$O_5^{(\mathrm{a})} = -\frac{1}{6}\frac{\langle 1\,2\rangle^2 \langle 1\,4\rangle [3\,4]}{\langle 1\,5\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle [2\,3]}$$

$$O_5^{(\mathrm{b})} = \frac{1}{6}\frac{\langle 1\,4\rangle [3\,4] [3\,5] \left(\langle 1\,4\rangle [3\,4] - \langle 1\,5\rangle [3\,5]\right)}{\langle 1\,5\rangle \langle 3\,4\rangle \langle 4\,5\rangle [1\,5] [2\,3]^2}$$

On-Shell Methods

- Physical states
- Use of properties of amplitudes as calculational tools
- Kinematics: Spinor Helicity Basis ⇔ Twistor space
- Tree Amplitudes: On-shell Recursion Relations ⇐ Factorization
- Loop Amplitudes: Unitarity (SUSY)

  Unitarity + On-shell Recursion QCD