TWO-LOOP Renormalization in the Making

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Part I

Prolegomena

Motivations

After LEP

After the end of the Lep era it became evident that including estimates of higher order radiative corrections into one-loop calculations for physical (pseudo-)observables could not, anymore, satisfy the need of precision required by the new generation of experiments.

ILC vs LHC

Admittedly, LHC is an arena for discovery physics, more than anything else: high precision is certainly not needed, at least in its first phase. According to some predestinate design hadron machines are alternating with electron-positron ones and, hopefully, ILC will come into operation; at that moment the highest available theoretical precision will play a fundamental role.

Possible landscapes

As a matter of fact, it is not clear – at this moment – what kind of scenario will follow after the first few months of running at LHC; any evidence of new pysics will favor a striking search for new theoretical models, for their Born predictions, and the hearthquake could be so strong to remove any interest in quantum effects of the standard model. On the contrary, after few months of running, we could be back to the familiar landscape: effects of new physics hidden inside loops.

si vis pacem para bellum

We decided to build the envirovment that allows for a complete two-loop analysis of a spontaneously broken gauge field theory. This construction requires several steps, so it is difficult to caractherize the approach with a single achronimus; there are a lot of analytical aspects in what we are doing, yet the final step (computing arbitrary two-loop diagrams) can only be done with ‘the numerical way’: we call it the algebraic - numerical approach.

What's new?

If one thinks for a while, everything is in the old papers of 't Hooft and Veltman; however, translating few formal properties into a working scheme is far from trivial; most of the times it is not a question of *how do I do it?*, rather it is a question of bookkeeping, namely *can I do it without exhausting the memory of my computer?*, or, *is there any practical way of presenting my results besides making my codes public?*.

layout

First we will deal with general aspects of a spontaneously broken gauge theory; the treatment of tadpoles, everybody knows how to do it, yet general results are never presented in a way that everybody can use them. Secondly, there is the need for a proper diagonalization, order-by-order, of the neutral sector of a theory of fundamental interactions: once again, we need a comprehensive collections of results which allows for practical applications.

Counterterms?

Then, there is the perennial question, with or without counter-terms? In a way, it is a fake question. The two approaches are fully equivalent and we will discuss the transition from bare parameters to renormalized ones. Finally we discuss the ultimate step in any renormalization procedure: the transition from renormalized parameters to a set of physical (pseudo-)observables.

Perhaps, one should try to make a clear vocabulary of renormalization in QFT; a renormalization procedure is designed to bring you from a Lagrangian to theoretical predictions; it includes,

- regularization (nowadays dimensional regularization is easy to understand),
- a renormalization scheme and
- an input parameter set.

Comments

- The scheme, being a transitory step, is almost irrelevant; it can be on-mass-shell or $\overline{MS}$ or complex poles, but unless you do something illegal (resummations that are not allowed or similar things) it really does not matter.
- One can define $\overline{MS}$ quantities as convenient landmarks but it is the last step that matters, at least as long as we have a convenient subtraction point (which we miss in QCD). Renormalized quantities should always be expressed in terms of a set of physical quantities.
- One may indulge to the introduction of an $\overline{MS}$ running e.m. coupling constant (importing from QCD to QED, which sounds strange anyway) but, finally, only cross sectios matter.

Steps

- All the Green functions of the theory have to be made finite, up to two-loops, by introduction of counter-terms and all counter-terms are of non logarithmic nature, to respect unitarity.
- Renormalized Ward-Slavnov-Taylor identities must be satisfied.
- All ultraviolet finte parts must be classified and an algorithm has to be designed for their evaluation at any scale.

Of course, there are preliminar steps – not always the easy ones – but it is only the full control on the multi-scale level that pays off.

Part II

Higgs tadpoles

the basics

The minimal Higgs sector of the Standard Model (SM) is given by the Lagrangian

$$\mathcal{L}_S = -(D_\mu K)^\dagger (D_\mu K) - \mu^2 K^\dagger K - (\lambda/2)(K^\dagger K)^2, \tag{1}$$

where the covariant derivative is given by

$$D_\mu K = \left(\partial_\mu - \frac{i}{2} g B^a_\mu \tau^a - \frac{i}{2} g' B^0_\mu\right) K, \tag{2}$$

$g'/g = -\sin\theta/\cos\theta$, τ^a are the standard Pauli matrices, B^a_μ is a triplet of vector gauge bosons and B^0_μ a singlet. For the theory to be stable we must require $\lambda > 0$. We choose $\mu^2 < 0$ in order to have spontaneous symmetry breaking (SSB). The scalar field in the minimal realization of the SM is

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta + i\phi_0 \\ -\phi_2 + i\phi_1 \end{pmatrix}, \tag{3}$$

The parameter v is *not* a new parameter of the model; its value must be fixed by the requirement that $\langle H\rangle_0 = 0$ [i.e. $\langle K\rangle_0 = (1/\sqrt{2})(v, 0)$], so that the vacuum doesn't absorb/create Higgs particles. To see how this works at the lowest order, consider the part of $\mathcal{L}_S$ containing the Higgs field:

$$-(1/2)(\partial_\mu H)^2 - (\mu^2/2)(H+v)^2 - (\lambda/8)(H+v)^4. \tag{4}$$

These terms generate vertices that imply absorption of H in the vacuum, namely those linear in H,

$$\left[-\mu^2 v - (\lambda/2)v^3\right] H, \tag{5}$$

which correspond to the vertex H  This vertex gives a non-zero value to the diagrams with one ingoing H line, and thus a non-zero VEV. We will set it to zero, i.e. $v = (-2\mu^2/\lambda)^{1/2}$ (or $v = 0$, but then, no SSB).

Definitions and Lagrangian

In h.o. of perturbation theory there are more complicated diagrams contributing to $\langle H \rangle_0$. The parameter v must then be readjusted to make $\langle H \rangle_0 = 0$. First of all, let's introduce

- the new *bare* parameters M (the W mass),
- $M_{_H}$, the mass of the physical Higgs particle and
- β_h (the tadpole constant) according to the following definitions:

$$\left\{ \begin{array}{lcl} M &=& gv/2 \\ M_{_H}^2 &=& \lambda v^2 \\ \beta_h &=& \mu^2 + \frac{\lambda}{2} v^2 \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} v &=& 2M/g \\ \lambda &=& (gM_{_H}/2M)^2 \\ \mu^2 &=& \beta_h - \frac{1}{2} M_{_H}^2 \end{array} \right. \quad (6)$$

The new set of (bare) parameters is therefore

$$\{g, g', M, M_H, \beta_h\} \quad \text{(instead of } \{g, g', \mu, \lambda, v\}\text{)}. \tag{7}$$

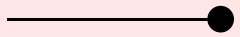
Remember that β_h (like v) is not an independent parameter. In terms of these parameters the interaction part of the scalar Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_S^I &= -\mu^2 K^\dagger K - (\lambda/2)(K^\dagger K)^2 = -\beta_h \Big[\frac{2M^2}{g^2} + \frac{2M}{g} H \\
&+ \frac{1}{2}\Big(H^2 + \phi_0^2 + 2\phi_+\phi_-\Big)\Big] \\
&+ \frac{M_H^2 M^2}{2g^2} - \frac{1}{2} M_H^2 H^2 - g\frac{M_H^2}{4M} H \Big(H^2 + \phi_0^2 + 2\phi_+\phi_-\Big) \\
&- g^2 \frac{M_H^2}{32M^2}\Big(H^2 + \phi_0^2 + 2\phi_+\phi_-\Big)^2,
\end{aligned} \tag{8}$$

with $\phi_\pm = (\phi_1 \mp i\phi_2)/\sqrt{2}$.

β_h setting at the lowest order

Let's now set β_h such that the VEV of H remains zero to each order of PT. At the lowest order, the only diagram contributing to $\langle H \rangle_0$ is

H  (9)

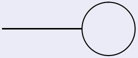
originated by the term in $\mathcal{L}_S^I$ linear in H, $-(2\beta_h M/g)H$. Therefore, at the lowest order we will simply set $\beta_h = 0$.

β_h setting up to one loop

Define

$$\beta_h = \beta_{h_0} + \beta_{h_1} g^2 + \beta_{h_2} g^4 + \cdots . \tag{10}$$

The lowest-order β_h setting of the previous section amounts to $\beta_{h_0} = 0$.
At the one-loop level, two types of diagrams contribute to the Higgs VEV up to $\mathcal{O}(g)$:

T_0 :  + T_1 :  (11)

where the empty blob on the r.h.s. symbolically indicates all the one-loop diagrams containing a scalar field (H, $\phi_\pm$, ϕ_0), a gauge field (Z, $W_\pm$), a Faddeev–Popov ghost field (X_+, X_-, X_z), or a fermionic field.

As an example, consider only the r.h.s. diagram containing the H field: if this were the only T_1 diagram, in order to have $\langle H \rangle_0 = 0$ it should cancel with thel.h.s. one (T_0), i.e.

$$(2\pi)^4 i \left(-\beta_h \frac{2M}{g} \right) - g \frac{3M_H^2}{4M} i\pi^2 A_0(M_H) = 0, \qquad (12)$$

where $i\pi^2 A_0(m) = \mu^{4-n} \int d^n q / (q^2 + m^2 - i\epsilon)$. The solution of this equation is $\beta_{h_0} = 0$ and

$$\beta_{h_1}^{(H)} = \frac{1}{(2\pi)^4 i} \left(\frac{T_1}{2Mg} \right) = -\frac{1}{16\pi^2} \left[\frac{3M_H^2}{8M^2} A_0(M_H) \right]. \qquad (13)$$

Of course, $\beta_{h_1}^{(H)}$ is just the contribution to β_{h_1} arising from the one-loop tadpole diagram containing the H field.

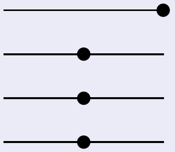
The complete expression for β_{h_1} in the R_ξ gauge is

$$\beta_{h_1} = -\frac{1}{16\pi^2}\Big[\,\frac{3}{2}A_0(M) + \frac{3}{4c^2}A_0(M_0) + M^2 + \frac{M_0^2}{2c^2} + \\ + \;\frac{M_H^2}{8M^2}\Big(A_0(\xi_z M_0) + 2A_0(\xi_W M)\Big) + + \;\frac{3M_H^2}{8M^2}A_0(M_H) \\ - \sum_f \frac{m_f^2}{M^2}A_0(m_f)\Big], \tag{14}$$

where $M_0 = M/c$ and m_f are the Z and fermion masses, and $c = \cos\theta$.

β_h vertices in one-loop calculations

Beyond the lowest order, β_h is *not* zero and the Lagrangian $\mathcal{L}^I_S$ contains the following vertices involving a β_h factor:



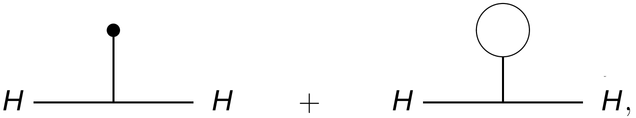
H		$(2\pi)^4 i\,(-2M\beta_h/g)$	(15)
H	H	$(2\pi)^4 i\,(-\beta_h)$	(16)
ϕ_0	ϕ_0	$(2\pi)^4 i\,(-\beta_h)$	(17)
ϕ_+	ϕ_-	$(2\pi)^4 i\,(-\beta_h)$	(18)

(as usual, the combinatorial factorials for identical fields are included.

Note that only scalar fields appear in the β_h vertices. These β_h vertices must be included in the relevant one-loop calculations. Consider, for example, the Higgs self-energy at the one-loop level. The diagrams contributing to this $\mathcal{O}(g^2)$ quantity are

(19)

where the empty blob on the r.h.s. represents all the one-loop contributions (two possible topologies). The l.h.s. diagram containing a two-leg β_h vertex shouldn't be forgotten and plays an important role in the Ward identities (see later). One should also include diagrams containing tadpoles:



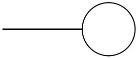
(20)

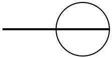
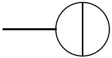
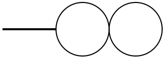
but these diagrams add up to zero as a consequence of our choice for β_h.

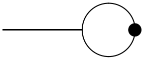
β_h setting up to two loops

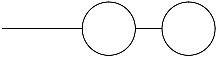
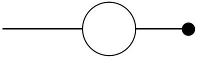
Up to terms of $\mathcal{O}(g^3)$, $\langle H \rangle_0$ gets contributions from the following diagrams:

T_0 :  (1) +

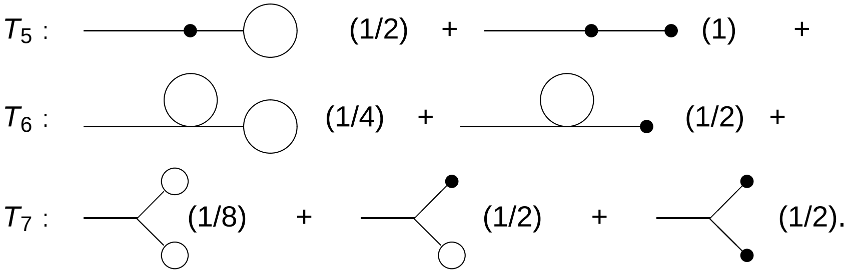
T_1 :  (1/2) +

T_2 :  (1/6) +  (1/4) +  (1/4) +

T_3 :  (1/2) +

T_4 :  (1/4) +  (1/2) +

(21)


T_5 : (1/2) + (1) +
T_6 : (1/4) + (1/2) +
T_7 : (1/8) + (1/2) + (1/2).

The coefficients in parentheses indicate the combinatorial factors of each diagram when all fields are identical. By virtue of our previous choice for β_{h_0} and β_{h_1}, all the *reducible* diagrams add up to zero: $T_4 = T_5 = T_6 = T_7 = 0$. The equation

$$\sum_{i=0}^{3} T_i = 0 \tag{22}$$

provides then β_{h_2}:

$$\beta_{h_2} = \frac{1}{(2\pi)^4 i} \left(\frac{T_2 + T_3}{2Mg^3} \right). \tag{23}$$

β_h vertices in two-loop calculations

As we described for calculations at the one-loop level, two-leg β_h vertices Eq.(16), Eq.(17),Eq.(18)) should be included in all the appropriate diagrams at the two-loop level, while all graphs (up to two loops) containing tadpoles will add up to zero as a consequence of our choice for β_{h_0}, β_{h_1} and β_{h_2}. Note that two-leg β_h vertices will also appear in $\mathcal{O}(g^4)$ self-energies of fields which do not belong to the Higgs sector; for example, in diagrams like these

which are representative of the only two irreducible $\mathcal{O}(g^4)$ Z self-energy topologies containing β_h vertices (excluding tadpoles, of course).

Definitions and Lagrangian

We will now consider a slightly different strategy to set the Higgs VEV to zero. Instead of using Eq.(6), the "β_h scheme", we will define the new bare parameters M' (the W mass), M'_H (the mass of the physical Higgs particle) and β_t (the tadpole constant) according to the following "β_t scheme":

$$
\left\{\begin{array}{lcl} M'(1+\beta_t) & = & gv/2 \\ (M'_H)^2 & = & \lambda\,(2M'/g)^2 \\ 0 & = & \mu^2 + \frac{\lambda}{2}\,(2M'/g)^2 \end{array}\right. \Longrightarrow \left\{\begin{array}{lcl} v & = & 2M'(1+\beta_t)/g \\ \lambda & = & \left(gM'_H/2M'\right)^2 \\ \mu^2 & = & -\frac{1}{2}(M'_H)^2 \end{array}\right. \tag{24}
$$

The new set of bare parameters is therefore

$$\{g, g', M', M'_H, \beta_t\} \quad \text{instead of} \{g, g', \mu, \lambda, v\}. \tag{25}$$

Remember that β_t (like v and β_h) is not an independent parameter.
Note that, contrary to β_h, the parameter β_t appears in the Higgs doublet K via $\zeta = H + v$, with $v = 2M'(1 + \beta_t)/g$ [Eq.(24)].
As a consequence, all three terms of the scalar Lagrangian $\mathcal{L}_S$ [Eq.(1)] depend on it. In particular, the interaction part of the scalar Lagrangian becomes

$$
\begin{aligned}
\mathcal{L}_S^I &= -\mu^2 K^\dagger K - (\lambda/2)(K^\dagger K)^2 \qquad (26)\\
&= (1+\beta_t)^2 \Big(1 - \beta_t\,(2+\beta_t)\Big)\frac{M_H'^2 M'^2}{2g^2}\\
&- \beta_t\,(\beta_t+1)\,(\beta_t+2)\,\frac{M_H'^2 M'}{g} H\\
&\quad -\frac{1}{2}M_H'^2 H^2 - \frac{1}{4}M_H'^2 \beta_t\,(\beta_t+2)\,\Big(3H^2 + \phi_0^2 + 2\phi_+\phi_-\Big)\\
&\quad -g\,(1+\beta_t)\,\frac{M_H'^2}{4M'} H\,\Big(H^2 + \phi_0^2 + 2\phi_+\phi_-\Big)\\
&- g^2\frac{M_H'^2}{32M'^2}\Big(H^2 + \phi_0^2 + 2\phi_+\phi_-\Big)^2, \qquad (27)
\end{aligned}
$$

while the term involving the covariant derivatives, $-(D_\mu K)^\dagger(D_\mu K)$, results in the same (lengthy) β_t-*independent* expression of the β_h scheme *plus* the following β_t-*dependent* terms

$$\begin{aligned}
\beta_t \quad \times \quad & \Big[igsM'\left(\phi^- W_\mu^+ - \phi^+ W_\mu^-\right)\left(A_\mu - \frac{s}{c}Z_\mu\right) \\
& -\frac{gM'}{2}H\left(2W_\mu^+W_\mu^- + \frac{Z_\mu Z_\mu}{c^2}\right) \\
- \quad & \frac{M'^2}{2}(\beta_t+2)\left(2W_\mu^+W_\mu^- + \frac{Z_\mu Z_\mu}{c^2}\right) + \frac{M'}{c}Z_\mu\partial_\mu\phi_0 \\
+ \quad & M'W_\mu^+\partial_\mu\phi_- + M'W_\mu^-\partial_\mu\phi_+\Big],
\end{aligned} \tag{28}$$

where, as usual, $W_\mu^\pm = (B_\mu^1 \mp iB_\mu^2)/\sqrt{2}$, $s = \sin\theta$, $c = \cos\theta$, and

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} B^3_\mu \\ B^0_\mu \end{pmatrix}. \tag{29}$$

Where else, in the SM Lagrangian, does the parameter β_t appear? Wherever v does — as it can be readily seen from Eq.(24). Let's quickly discuss the other sectors of the SM: Yang–Mills, fermionic, Faddeev–Popov (FP) and gauge-fixing. The pure Yang–Mills Lagrangian obviously contains no β_t terms.

The gauge-fixing part of the Lagrangian, $\mathcal{L}_{gf}$, cancels in the R_ξ gauges the gauge–scalar mixing terms Z–ϕ_0 and $W^\pm$–$\phi^\pm$ contained in the scalar Lagrangian $\mathcal{L}_S$. These terms are proportional to $gv/2$, i.e., in the β_t scheme, to $M'(1+\beta_t)$.

gauge-fixing

The gauge-fixing Lagrangian $\mathcal{L}_{gf}$ is matter of choice: we adopt the usual definition

$$\mathcal{L}_{gf} = -\mathcal{C}_+\mathcal{C}_- - \frac{1}{2}\mathcal{C}_Z^2 - \frac{1}{2}\mathcal{C}_A^2, \tag{30}$$

$$\mathcal{C}_A = -\frac{1}{\xi_A}\partial_\mu A_\mu, \quad \mathcal{C}_Z = -\frac{1}{\xi_Z}\partial_\mu Z^0_\mu + \xi_Z \frac{M'}{c}\phi_0, \quad \mathcal{C}_\pm = -\frac{1}{\xi_W}\partial_\mu W^\pm_\mu + \xi_W M'\phi_\pm \tag{31}$$

(that is, no β_t terms), thus canceling the $\mathcal{L}_S$ g-independent gauge–scalar mixing terms proportional to M', but not those proportional to $M'\beta_t$ [appearing at the end of Eq.(28)], which are of $\mathcal{O}(g^2)$. Clearly, this gauge fixing Lagrangian is *different* from the usual one of the β_h scheme because M and M' are not the same $[M = M'(1 + \beta_t)]$.

Alternatively, one could choose $M'(1 + \beta_t)$ instead of M' in eq. (31), thus canceling all $\mathcal{L}_S$ gauge–scalar mixing terms, both proportional to M' and $M'\beta_t$, but introducing then other new two-leg β_t vertices. In this latter case, the gauge fixing Lagrangian is indeed identical to the one of the β_h scheme. We will not follow this latter approach. Of course it's only matter of choice, but the explicit form of $\mathcal{L}_{gf}$ determines the FP ghost Lagrangian.

The parameter β_t shows up also in the FP ghost sector. The FP Lagrangian depends on the gauge variations of the chosen gauge-fixing functions $\mathcal{C}_A$, $\mathcal{C}_Z$ and $\mathcal{C}_\pm$. If, under gauge transformations, the functions $\mathcal{C}_i$ transform as

$$\mathcal{C}_i \to \mathcal{C}_i + \left(M_{ij} + gL_{ij}\right)\Lambda_j, \tag{32}$$

with $i = A, Z, \pm$, FP ghost Lagrangian is given by

$$\mathcal{L}_{FP} = \bar{X}_i \left(M_{ij} + gL_{ij}\right) X_j. \tag{33}$$

With the choice for $\mathcal{L}_{gf}$ given in eq. (30) [and the relation $gv/2 = M'(1+\beta_t)$] it's easy to check that the FP ghost Lagrangian contains the β_t terms

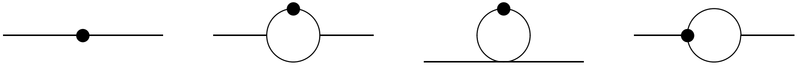
$$\mathcal{L}_{FP} = -\left(M'\right)^2 \beta_t \left(\xi_W \bar{X}^+ X^+ + \xi_W \bar{X}^- X^- + \xi_Z \bar{X}_Z X_Z / c^2\right) + \cdots, \tag{34}$$

where the dots indicate the usual β_t–independent terms. Had we chosen $\mathcal{L}_{gf}$ with $M'(1+\beta_t)$ instead of M' in eq. (31), additional β_t terms would now arise in the FP Lagrangian.

In the fermionic sector, the parameter β_t appears in the mass terms:

$$\frac{v}{\sqrt{2}}\left(-\alpha\bar{u}u + \beta\bar{d}d\right) = -\left(1+\beta_t\right)\left(m_u\bar{u}u + m_d\bar{d}d\right) \tag{35}$$

[$v = 2M'(1+\beta_t)/g$], α and β are the Yukawa couplings, and m_u, m_d are the masses of the fermions. The rest of the fermion Lagrangian does not contain β_t, as it doesn't depend on v. In the β_t scheme, contrary to the β_h one, we have (many) two- and three-leg β_t vertices containing also fields outside the scalar sector. Note that three-leg β_t vertices introduce a fourth irreducible topology for $\mathcal{O}(g^4)$ self-energy diagrams containing β_t vertices, namely:



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β_t up to one loop

Define

$$\beta_t = \beta_{t_0} + \beta_{t_1} g^2 + \beta_{t_2} g^4 + \cdots . \tag{36}$$

As we did for β_h, we will now set the parameter β_t such that the VEV of the Higgs field H remains zero to each order of perturbation theory.

At the lowest order, the only diagram contributing to $\langle H \rangle_0$ is the same one depicted in (Eq.(9)), originated by the term in $\mathcal{L}_S^I$ linear in H, $-\beta_t(\beta_t + 1)(\beta_t + 2)(M_H'^2 M'/g)H$. Therefore, at the lowest order we can simply set $\beta_t = 0$, i.e. $\beta_{t_0} = 0$.

Up to one loop, the diagrams T_0' and T_1' contributing to the Higgs VEV are analogous to T_0 and T_1 appearing in (Eq.(11)), so that β_{t_1} can be set in analogy with β_{h_1}:

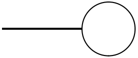
$$\beta_{t_1} = \frac{1}{(2\pi)^4 i} \left(\frac{T_1'}{2M' g M_H'^2} \right) . \tag{37}$$

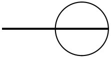
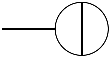
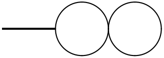
Note that T_1' and T_1 have the same functional form, but depend on different mass parameters; moreover, one gets $\beta_{t_1} = \beta_{h_1}/M_H^2 + \mathcal{O}(g^2)$.

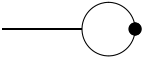
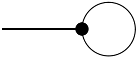
β_t up to two loops

The two-loop β_t fixing slightly differs from the β_h one. Up to terms of $\mathcal{O}(g^3)$, $\langle H \rangle_0$ gets contributions from the following diagrams:

T_0' :  (1) +

T_1' :  (1/2) +

T_2' :  (1/6) +  (1/4) +  (1/4) +

T_3' :  (1/2) +  (1/2),

plus *reducible* diagrams (analogous to those appearing in T_4–T_7 of section 2.4) which add up to zero because of our choice for β_{t_0} and β_{t_1}.

Note the new diagrams in T_3', with three-leg β_t vertices, not present in the β_h case (T_3). The parameter β_{t_2} can be set in the usual manner, requiring

$$\sum_{i=0}^{3} T_i' = 0, \qquad \Longrightarrow \qquad \beta_{t_2} = \frac{1}{(2\pi)^4 i}\left(\frac{T_2' + T_3'}{2M' g^3 M_H'^2}\right) - \frac{3}{2}\beta_{t_1}^2. \tag{38}$$

Note that $T_{1,2}'$ and $T_{1,2}$ have the same functional form (but depend on different mass parameters) while T_3' and T_3 are different also in form.

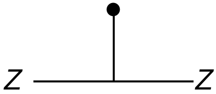
A comment on WST identities and mass renormalization

Consider the (doubly-contracted) WST identity relating the Z self-energy $\Pi_{\mu\nu,ZZ}(p)$, the ϕ_0 self-energy $\Pi_{\phi_0\phi_0}(p)$, and the Z–ϕ_0 transition $\Pi_{\mu,Z\phi_0}(p)$:

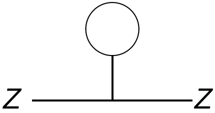
$$p_\mu p_\nu \Pi_{\mu\nu,ZZ}(p) \;+\; M_0^2 \Pi_{\phi_0\phi_0}(p) \;+\; 2ip_\mu M_0 \Pi_{\mu,Z\phi_0}(p) \;=\; 0. \qquad (39)$$

Both in β_h and β_t schemes, each of the three terms in Eq.(39) contains tadpoles diagrams, but they add up to zero, within each term.

For example, at the one-loop level, the first term in Eq.(39) contains the tadpoles diagrams



and



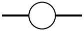
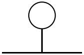
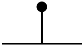
(40)

which cancel each other. In the β_h scheme at the one-loop level, only the second term of the identity (Eq.(39)) includes a diagram with a two-leg β_h vertex (Eq.(17)); in higher orders, two-leg β_h vertices will appear in all three terms. In the β_t scheme, all three terms of Eq.(39) contain the two-leg β_t vertices already at the one-loop level. Similar comments are valid for the WST identity involving the W self-energy.

Renormalization

Concerning renormalization, the constraint imposed on β_h (or β_t) in the previous sections is the renormalization condition to insure that $\langle 0|H|0\rangle = 0$, also in the presence of radiative corrections. In particular, the renormalized $\beta_{h,t}$ parameters are $\beta^{(R)}_{h,t} = \beta_{h,t} + \delta\beta_{h,t} = 0$. The equivalent of Eq.(6)) and Eq.(24) for the renormalized parameters are just the same equations with $\beta^{(R)}_h = \beta^{(R)}_t = 0$.

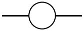
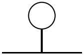
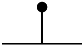
In the β_h scheme, the one-loop renormalization of the W and Z masses involves the diagrams

(a)  (b)  (c)  . (41)

(Diagrams (a) have two possible loop topologies.)

Both (a) and (b) are gauge-dependent, but their sum is gauge-independent on-shell. However, as we choose the β_h tadpole (c) to cancel (b), the mass counterterm contains only (a) and is therefore gauge-dependent.

On the contrary, in the β_t scheme, the one-loop renormalization of the W and Z masses involves the diagrams

(a)  (c)  (b)  (d)  . (42)

Once again, both (a) and (b) diagrams are gauge-dependent, their sum is gauge-independent on-shell, and the β_t tadpole (d) is chosen to cancel (b). But, the mass counterterm is now gauge-independent, as it contains both (a) and the two-leg β_t vertex diagram (c) (which is missing in the β_h case).

TWO-LOOP Renormalization in the Making

Giampiero PASSARINO

Torino

July 12, 2006

Outline of Part II

1. New coupling constant in the β_h scheme
2. New coupling constant in the β_t scheme
3. The Γ–β_t mixing
4. WSTI for two-loop gauge boson self-energies
5. WSTI at two loops: the role of reducible diagrams
6. The photon self-energy
7. The photon–Z mixing
8. The Z self-energy
9. The W self-energy
10. Dyson resummed propagators and their WSTI
11. The charged sector
12. The neutral sector
13. The LQ basis

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New coupling constant in the β_h scheme

The Z–γ transition in the SM does not vanish at zero squared momentum transfer. Although this fact does not pose any serious problem, not even for the renormalization of the electric charge, it is preferable to use an alternative strategy. Let's introduce the new $SU(2)$ coupling constant $\bar{g}$, the new mixing angle $\bar{\theta}$ and the new W mass $\bar{M}$ in the β_h scheme:

$$
\begin{aligned}
g &= \bar{g}\,(1+\Gamma) \qquad g' = -(\sin\bar{\theta}/\cos\bar{\theta})\,\bar{g} \\
v &= 2\bar{M}/\bar{g} \quad \lambda = \left(\bar{g}M_{_H}/2\bar{M}\right)^2 \quad \mu^2 = \beta_h - \tfrac{1}{2}M_{_H}^2
\end{aligned}
\tag{1}
$$

note: $g \sin\theta / \cos\theta = \bar{g} \sin\bar{\theta} / \cos\bar{\theta}$, where $\Gamma = \Gamma_1\, \bar{g}^2 + \Gamma_2\, \bar{g}^4 + \cdots$ is a new parameter yet to be specified. This change of parameters entails new $\bar{A}_\mu$ and $\bar{Z}_\mu$ fields related to B^3_μ and B^0_μ by

$$\begin{pmatrix} \bar{Z}^0_\mu \\ \bar{A}_\mu \end{pmatrix} = \begin{pmatrix} \cos\bar{\theta} & -\sin\bar{\theta} \\ \sin\bar{\theta} & \cos\bar{\theta} \end{pmatrix} \begin{pmatrix} B^3_\mu \\ B^0_\mu \end{pmatrix}. \tag{2}$$

The replacement $g \to \bar{g}(1+\Gamma)$ introduces in the SM Lagrangian several terms containing the new parameter Γ. Let us take a close look at these 'Γ terms' in each sector of the SM.

- The pure Yang–Mills Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{4} F^0_{\mu\nu} F^0_{\mu\nu}, \tag{3}$$

with $F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g \epsilon^{abc} B^b_\mu B^c_\nu$ and $F^0_{\mu\nu} = \partial_\mu B^0_\nu - \partial_\nu B^0_\mu$, contains the following new Γ terms when we replace g by $\bar{g}(1+\Gamma)$:

$$\begin{aligned}
\Delta\mathcal{L}_{YM} = & -i\bar{g}\Gamma\bar{c}\Big[\partial_\nu \bar{Z}^0_\mu \left(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu\right) - \bar{Z}^0_\nu \left(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu\right) \\
& + \bar{Z}^0_\mu \left(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu\right)\Big] - i\bar{g}\Gamma\bar{s}\Big[\partial_\nu \bar{A}_\mu \left(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu\right) \\
& - \bar{A}_\nu \left(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu\right) + \bar{A}_\mu \left(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu\right)\Big] \\
& + \bar{g}^2\Gamma(2+\Gamma)\Big[\frac{1}{2}\left(W^+_\mu W^-_\nu W^+_\mu W^-_\nu - W^+_\mu W^-_\mu W^+_\nu W^-_\nu\right) \\
& + \bar{c}^2 \left(\bar{Z}^0_\mu W^+_\mu \bar{Z}^0_\nu W^-_\nu - \bar{Z}^0_\mu \bar{Z}^0_\mu W^+_\nu W^-_\nu\right) + \bar{s}^2 \left(\bar{A}_\mu W^+_\mu \bar{A}_\nu W^-_\nu - \bar{A}_\mu \bar{A}_\mu W^+_\nu W^-_\nu\right) \\
& + \bar{s}\bar{c}\left(\bar{A}_\mu \bar{Z}^0_\nu (W^+_\mu W^-_\nu + W^+_\nu W^-_\mu) - 2\bar{A}_\mu \bar{Z}^0_\mu W^+_\nu W^-_\nu\right)\Big],
\end{aligned} \tag{4}$$

where $\bar{s} = \sin\bar{\theta}$ and $\bar{c} = \cos\bar{\theta}$. As these terms are of $\mathcal{O}(\bar{g}^3)$ or $\mathcal{O}(\bar{g}^4)$, they do not contribute to the calculation of self-energies at the one-loop level, but they do beyond it.

• The scalar Lagrangian $\mathcal{L}_S$ contains several new Γ terms when we employ the relation $g = \bar{g}(1+\Gamma)$ and the β_h scheme of eqs. (1). Actually, the last two equations in (1) are not needed here, as the interaction part of the scalar Lagrangian does not induce Γ terms. They can be arranged in the following three classes

$$\Delta\mathcal{L}_{S,h} = \Delta\mathcal{L}_{S,h}^{(n_f=2)} + \Delta\mathcal{L}_{S,h}^{(n_f=3)} + \Delta\mathcal{L}_{S,h}^{(n_f=4)}, \tag{5}$$

according to the number of fields (n_f) appearing in each interaction term (indicated by the superscript in parentheses. Note that this superscript does not indicate, in general, the order in $\bar{g}$). The explicit expressions, up to terms of $\mathcal{O}(\bar{g}^4)$, are

$$
\begin{aligned}
\Delta\mathcal{L}_{S,h}^{(n_f=2)} &= \bar{M}\Gamma\Big[-\frac{1}{2}\bar{M}\bar{s}^2\Gamma\bar{A}_\mu\bar{A}_\mu - \frac{1}{2}\bar{M}\left(2+\Gamma\bar{c}^2\right)\bar{Z}^0_\mu\bar{Z}^0_\mu \\
&\quad -\bar{M}\,\frac{\bar{s}}{\bar{c}}\left(1+\Gamma\bar{c}^2\right)\bar{A}_\mu\bar{Z}^0_\mu + \partial_\mu\phi_0\left(\bar{s}\bar{A}_\mu + \bar{c}\bar{Z}^0_\mu\right) \\
&\quad -\bar{M}\,(2+\Gamma)\,W^+_\mu W^-_\mu + W^-_\mu\partial_\mu\phi^+ + W^+_\mu\partial_\mu\phi^-\Big],
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
\Delta\mathcal{L}_{S,h}^{(n_f=3)} &= \bar{g}\Gamma\Big[-\bar{M}H\left(\bar{Z}^0_\mu\bar{Z}^0_\mu + \frac{\bar{s}}{\bar{c}}\bar{A}_\mu\bar{Z}^0_\mu + 2W^+_\mu W^-_\mu\right) \\
&\quad +\frac{1}{2}\left(\bar{s}\bar{A}_\mu + \bar{c}\bar{Z}^0_\mu\right)\left(H\partial_\mu\phi^0 - \phi^0\partial_\mu H + i\phi^+\partial_\mu\phi^- - i\phi^-\partial_\mu\phi^+\right) \\
&\quad +i\left(\phi^- W^+_\mu - \phi^+ W^-_\mu\right)\left(\bar{s}\bar{M}\bar{A}_\mu - (\bar{s}^2/\bar{c})\bar{M}\bar{Z}^0_\mu + \frac{1}{2}\partial_\mu\phi^0\right) \\
&\quad +\frac{1}{2}W^-_\mu\partial_\mu\phi^+\left(H + i\phi^0\right) + \frac{1}{2}W^+_\mu\partial_\mu\phi^-\left(H - i\phi^0\right) - \frac{1}{2}\partial_\mu H\left(\phi^+ W^-_\mu + \phi^- W^+_\mu\right)\Big],
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\Delta\mathcal{L}_{S,h}^{(n_f=4)} &= \frac{\bar{g}^2}{2}\Gamma\Big\{-\frac{1}{2}\left(H^2 + \phi_0^2\right)\left(\bar{Z}^0_\mu\bar{Z}^0_\mu + \frac{\bar{s}}{\bar{c}}\bar{A}_\mu\bar{Z}^0_\mu + 2W^+_\mu W^-_\mu\right) \\
&\quad +\phi^+\phi^-\left(-2\bar{s}^2\bar{A}_\mu\bar{A}_\mu + (1-2\bar{c}^2)\bar{Z}^0_\mu\bar{Z}^0_\mu + (\bar{s}/\bar{c} - 4\bar{s}\bar{c})\bar{A}_\mu\bar{Z}^0_\mu\right) \\
&\quad -2W^+_\mu W^-_\mu\phi^+\phi^- + \left(\bar{s}\bar{A}_\mu - (\bar{s}^2/\bar{c})\bar{Z}^0_\mu\right)\times \\
&\qquad \times\Big[\phi_0\left(\phi^+ W^-_\mu + \phi^- W^+_\mu\right) - iH\left(\phi^+ W^-_\mu - \phi^- W^+_\mu\right)\Big]\Big\}.
\end{aligned}
\tag{8}
$$

The interaction part of the scalar Lagrangian, $\mathcal{L}_S^I = -\mu^2 K^\dagger K - (\lambda/2)(K^\dagger K)^2$, does not induce Γ terms; these are only originated by the term involving the covariant derivatives, $-(D_\mu K)^\dagger (D_\mu K)$. On the other hand, as $M/g = \bar{M}/\bar{g}$, the β_h terms induced by $\mathcal{L}_S^I$ are expressed in terms of the ratio of the barred parameters $\bar{M}/\bar{g}$.

- We choose the gauge-fixing Lagrangian $\mathcal{L}_{gf}$ with the following gauge functions:

$$\mathcal{C}_A = -\frac{1}{\xi_A}\partial_\mu \bar{A}_\mu, \quad \mathcal{C}_Z = -\frac{1}{\xi_Z}\partial_\mu \bar{Z}^0_\mu + \xi_Z \frac{\bar{M}}{\bar{c}}\phi_0, \quad \mathcal{C}_\pm = -\frac{1}{\xi_W}\partial_\mu W^\pm_\mu + \xi_W \bar{M}\phi_\pm. \tag{9}$$

gauge fixing

This R_ξ gauge Γ-independent $\mathcal{L}_{gf}$ cancels the zeroth order (in $\bar{g}$) gauge–scalar mixing terms introduced by $\mathcal{L}_S$, but not those proportional to Γ. Had one chosen gauge-fixing functions eqs. (9) with unbarred quantities, all the gauge–scalar mixing terms of $\mathcal{L}_S$ would be canceled, including those proportional to Γ, but additional new Γ vertices would also be introduced.

• New Γ terms are also originated in the Faddeev–Popov ghost sector. Studying the gauge transformations of the gauge-fixing functions $\mathcal{C}_A$, $\mathcal{C}_Z$ and $\mathcal{C}_\pm$ defined in eqs. (9), the additional new Γ terms of the FP Lagrangian in the β_h scheme are:

$$\Delta\mathcal{L}_{FP,h} = \Delta\mathcal{L}_{FP,h}^{(n_f=2)} + \Delta\mathcal{L}_{FP,h}^{(n_f=3)}, \tag{10}$$

where the two-field terms are,

$$\Delta\mathcal{L}_{FP,h}^{(n_f=2)} = -\Gamma\bar{M}^2\left[\xi_Z\bar{X}_Z\left(X_Z + \frac{\bar{s}}{\bar{c}}X_A\right) + \xi_W\left(\bar{X}_+X_+ + \bar{X}_-X_-\right)\right], \tag{11}$$

and the three-field terms are

$$
\begin{aligned}
\Delta\mathcal{L}^{(n_f=3)}_{FP,h} \;=\;& \Gamma\bar{g}\Big\{ i\bar{c}W^{+}_{\mu}\Big((\partial_\mu\bar{X}_Z/\xi_Z)X_{-} - (\partial_\mu\bar{X}_{+}/\xi_W)X_Z\Big) \qquad (12)\\
+\;& i\bar{s}W^{+}_{\mu}\Big((\partial_\mu\bar{X}_A/\xi_A)X_{-} - (\partial_\mu\bar{X}_{+}/\xi_W)X_A\Big)\\
+\;& i\bar{c}W^{-}_{\mu}\Big((\partial_\mu\bar{X}_{-}/\xi_W)X_Z - (\partial_\mu\bar{X}_Z/\xi_Z)X_{+}\Big)\\
+\;& i\bar{s}W^{-}_{\mu}\Big((\partial_\mu\bar{X}_{-}/\xi_W)X_A - (\partial_\mu\bar{X}_A/\xi_A)X_{+}\Big)\\
+\;& i\bar{c}\bar{Z}^{0}_{\mu}\;\Big((\partial_\mu\bar{X}_{+}/\xi_W)X_{+} - (\partial_\mu\bar{X}_{-}/\xi_W)X_{-}\Big)\\
+\;& i\bar{s}\bar{A}_{\mu}\;\Big((\partial_\mu\bar{X}_{+}/\xi_W)X_{+} - (\partial_\mu\bar{X}_{-}/\xi_W)X_{-}\Big)\\
+\;& \frac{1}{2}\xi_W\bar{M}\Big[i\phi_0\left(\bar{X}_{+}X_{+} - \bar{X}_{-}X_{-}\right) - H\left(\bar{X}_{+}X_{+} + \bar{X}_{-}X_{-}\right)\Big]\\
+\;& \frac{1}{2\bar{c}}\xi_Z\bar{M}\bar{X}_Z\Big[iX_{-}\phi_{+} - iX_{+}\phi_{-} - \bar{s}HX_A - \bar{c}HX_Z\Big]\\
+\;& \frac{i}{2}\xi_W\bar{M}\Big[\bar{X}_{-}\phi_{-}\left(\bar{c}X_Z + \bar{s}X_A\right) - \bar{X}_{+}\phi_{+}\left(\bar{c}X_Z + \bar{s}X_A\right)\Big]\Big\}.
\end{aligned}
$$

FP ghost fields

The bars over the FP ghost fields indicate conjugation. Obviously, the new FP fields X_A and X_Z should also be denoted with the bar for the field rediagonalization, just like the new fields $\bar{A}_\mu$ and $\bar{Z}_\mu$. However, this notation would be too messy and we will leave this point understood.

Note that the FP ghost – gauge boson vertices are simply the usual ones with g replaced by $\bar{g}\Gamma$. This is not the case, in general, for the FP ghost – scalar terms.

- Finally, the fermionic sector. The fermion – gauge boson Lagrangian,

$$\mathcal{L}_{fG} = \frac{i}{2\sqrt{2}}\, g\left[W^{+}_{\mu}\bar{u}\gamma_{\mu}\,(1+\gamma_5)\,d \;+ W^{-}_{\mu}\bar{d}\gamma_{\mu}\,(1+\gamma_5)\,u\,\right] + \frac{i}{2c}\, g\, Z_{\mu}\,\bar{f}\gamma_{\mu}\Big(I_3 - 2Q_f s^2 + I_3\gamma_5\Big)f \;+ i\, gs\, Q_f A_{\mu}\bar{f}\gamma_{\mu}f, \quad (13)$$

(where $I_3 = \pm 1/2$ is the weak isospin third component of the fermion f, and Q_f its charge in units of $|e|$) becomes, under the replacement $g \to \bar{g}(1+\Gamma)$ and the θ, A_{μ} and Z_{μ} redefinitions,

Fermions

$$
\begin{aligned}
\mathcal{L}_{fG} \;=\;& \frac{i}{2\sqrt{2}}\,\bar{g}\,(1+\Gamma)\left[W^{+}_{\mu}\bar{u}\gamma_{\mu}\,(1+\gamma_5)\,d \;+ W^{-}_{\mu}\bar{d}\gamma_{\mu}\,(1+\gamma_5)\,u\,\right] \\
&+ \frac{i}{2\bar{c}}\,\bar{g}\,\bar{Z}^{0}_{\mu}\,\bar{f}\gamma_{\mu}\left(I_3 - 2Q_f\bar{s}^2 + I_3\gamma_5\right)f \;+\; i\,\bar{g}\bar{s}\,Q_f\bar{A}_{\mu}\bar{f}\gamma_{\mu}f \\
&+ \frac{i}{2}\,\bar{g}\,\Gamma\left(\bar{s}\bar{A}_{\mu} + \bar{c}\bar{Z}^{0}_{\mu}\right) I_3\,\bar{f}\gamma_{\mu}\,(1+\gamma_5)\,f\,. \qquad (14)
\end{aligned}
$$

The new neutral and charged current Γ vertices are immediately recognizable. The CKM matrix has been set to unity.

The fermion–scalar Lagrangian does not induce Γ terms. Indeed, the Yukawa couplings α and β in

$$\mathcal{L}_{fS} = -\alpha \bar{\psi}_L K u_R - \beta \bar{\psi}_L K^c d_R + \text{h.c.} \tag{15}$$

(where $K^c = i\tau_2 K^\star$ is the conjugate Higgs doublet) are set by $\alpha v/\sqrt{2} = m_u$ and $\beta v/\sqrt{2} = -m_d$. As $v = 2\bar{M}/\bar{g}$, it is $\alpha = \bar{g} m_u/\sqrt{2}\bar{M}$ and $\beta = -\bar{g} m_d/\sqrt{2}\bar{M}$, and no Γ appears in Eq.(15).

Yang–Mills

The Feynman rules for all these new Γ vertices are computed, up to terms of $\mathcal{O}(\bar{g}^4)$. Those corresponding to the pure Yang–Mills Lagrangian [Eq.(4)] are not listed, as they are identical to the usual Yang–Mills ones, except for the replacement $g \to \bar{g}\Gamma$ in the three-leg vertices, and $g^2 \to \bar{g}^2\Gamma(2+\Gamma)$ in the four-leg ones. In Appendix C, all bars over the various symbols (indicating rediagonalization) have been dropped, except over $\bar{g}$.

New coupling constant in the β_t scheme

The β_t scheme equations corresponding to Eq.(1) are the following

$$
\begin{aligned}
g &= \bar{g}\,(1+\Gamma) \qquad g' = -(\sin\bar{\theta}/\cos\bar{\theta})\,\bar{g} \\
v &= 2\bar{M}'(1+\beta_t)/\bar{g} \quad \lambda = \left(\bar{g}M'_H/2\bar{M}'\right)^2 \quad \mu^2 = -\frac{1}{2}(M'_H)^2.
\end{aligned}
\tag{16}
$$

(Note: $g\sin\theta/\cos\theta = \bar{g}\sin\bar{\theta}/\cos\bar{\theta}$.) The analysis of the Γ terms presented in the previous section for the β_h scheme can be repeated for the β_t scheme using Eq.(16) instead of Eq.(1). The new fields $\bar{A}_\mu$ and $\bar{Z}_\mu$ are related to B^3_μ and B^0_μ by Eq.(2). Thus, we obtain the following results:

• The replacement $g \to \bar{g}(1+\Gamma)$ in the pure Yang–Mills sector introduces new Γ vertices collected in $\Delta\mathcal{L}_{YM}$, which does not depend on the parameters of the $\beta_{h,t}$ schemes. $\Delta\mathcal{L}_{YM}$ has already been given in Eq.(4).

• The new Γ terms introduced in $\mathcal{L}_S$ by eqs. (16) can be arranged once again in the three classes

$$\Delta\mathcal{L}_{S,t} = \Delta\mathcal{L}_{S,t}^{(n_f=2)} + \Delta\mathcal{L}_{S,t}^{(n_f=3)} + \Delta\mathcal{L}_{S,t}^{(n_f=4)}, \tag{17}$$

according to the number of fields appearing in the Γ terms. The explicit expression for $\Delta\mathcal{L}_{S,t}^{(2)}$ is, up to terms of $\mathcal{O}(\bar{g}^4)$,

$$
\begin{aligned}
\Delta\mathcal{L}^{(n_f=2)}_{S,t} \quad = \quad & \bar{M}'\Gamma\Big[-\frac{1}{2}\bar{M}'\bar{s}^2\Gamma\bar{A}_\mu\bar{A}_\mu - \frac{1}{2}\bar{M}'\left(2+\Gamma\bar{c}^2+4\beta_t\right)\bar{Z}^0_\mu\bar{Z}^0_\mu \quad (\\
& -\bar{M}'\,\frac{\bar{s}}{\bar{c}}\left(1+\Gamma\bar{c}^2+2\beta_t\right)\bar{A}_\mu\bar{Z}^0_\mu + \partial_\mu\phi_0\left(\bar{s}\bar{A}_\mu+\bar{c}\bar{Z}^0_\mu\right)(1+\beta_t)\\
& -\bar{M}'\,(2+\Gamma+4\beta_t)\,W^+_\mu W^-_\mu + \left(W^-_\mu\partial_\mu\phi^+ + W^+_\mu\partial_\mu\phi^-\right)(1+
\end{aligned}
$$

with $\bar{s} = \sin\bar{\theta}$ and $\bar{c} = \cos\bar{\theta}$, while, up to the same $\mathcal{O}(\bar{g}^4)$,

more fields

$$\Delta\mathcal{L}_{S,t}^{(n_f=3,4)} = \Delta\mathcal{L}_{S,h}^{(n_f=3,4)}\left(\bar{M} \to \bar{M}'\right) \tag{19}$$

[$\Delta\mathcal{L}_{S,h}^{(n_f=3)}$ and $\Delta\mathcal{L}_{S,h}^{(n_f=4)}$ are given in eqs. (7) and (8)]. The subscripts t and h indicate the β_t and β_h schemes. Note the presence of β_t factors in the new Γ terms of Eq.(18). We will comment on this in sec. 23.

• Our recipe for gauge-fixing is the same as in the previous sections: we choose the R_ξ gauge $\mathcal{L}_{gf}$ to cancel the zeroth order (in $\bar{g}$) gauge–scalar mixing terms introduced by $\mathcal{L}_S$, but not those of higher orders (see discussions in 2). Here, this prescription is realized by $\mathcal{L}_{gf}$ with

$$\mathcal{C}_A = -\frac{1}{\xi_A}\partial_\mu \bar{A}_\mu, \quad \mathcal{C}_Z = -\frac{1}{\xi_Z}\partial_\mu \bar{Z}^0_\mu + \xi_Z \frac{\bar{M}'}{\bar{c}}\phi_0, \quad \mathcal{C}_\pm = -\frac{1}{\xi_W}\partial_\mu W^\pm_\mu + \xi_W \bar{M}'\phi_\pm, \tag{20}$$

clearly Γ-independent.

The new Γ terms of the FP ghost Lagrangian in the β_t scheme are:

$$\Delta\mathcal{L}_{FP,t} = \Delta\mathcal{L}^{(n_f=2)}_{FP,t} + \Delta\mathcal{L}^{(n_f=3)}_{FP,t} \ , \tag{21}$$

where the two-field terms are

$$\Delta\mathcal{L}^{(n_f=2)}_{FP,t} = -\left(1+\beta_t\right)\Gamma\bar{M}'^2\left[\xi_Z\bar{X}_Z\left(X_Z + \frac{\bar{s}}{\bar{c}}X_A\right) + \xi_W\left(\bar{X}_+X_+ + \bar{X}_-X_-\right)\right], \tag{22}$$

and the three-field terms are the same as in the β_h scheme, with $\bar{M}$ replaced by $\bar{M}'$: $\Delta\mathcal{L}^{(n_f=3)}_{FP,t} = \Delta\mathcal{L}^{(n_f=3)}_{FP,h}(\bar{M} \to \bar{M}')$ [Eq.(12)]. Like in the scalar sector, the Γ and β_t factors are entangled; see sec. 23 for a comment.

• We conclude this analysis with the fermionic sector. As in the Yang–Mills case, the fermion – gauge boson Lagrangian $\mathcal{L}_{fG}$ does not depend on the parameters of the β_h or β_t schemes. Its expression in terms of the new coupling constant $\bar{g}$ contains new Γ terms and is given in Eq.(14). The neutral sector rediagonalization induces no Γ terms in the fermion–scalar Lagrangian $\mathcal{L}_{fS}$ [Eq.(15)], which contains, however, the β_t vertices (the ratio M'/g is now replaced by the identical ratio $\bar{M}'/\bar{g}$).

The Feynman rules for all Γ vertices are listed in Appendix C, up to terms of $\mathcal{O}(\bar{g}^4)$. All primes and bars over A_μ, Z_μ, M, M_H and θ have been dropped (but not over $\bar{g}$). As we mentioned at the end of the previous section, the Γ vertices of the pure Yang–Mills sector need not be listed.

The Γ–β_t mixing

A comment on the presence of β_t factors in the new Γ vertices is now appropriate. Consider the scalar Lagrangian $\mathcal{L}_S$. As we already pointed out in sec. 2, the interaction part of $\mathcal{L}_S$, $\mathcal{L}_S^I = -\mu^2 K^\dagger K - (\lambda/2)(K^\dagger K)^2$, does not induce Γ terms. On the other hand, $\mathcal{L}_S^I$ gives rise to β_t terms: as $M'/g = \bar{M}'/\bar{g}$, these β_t terms are simply expressed in terms of $\bar{M}'/\bar{g}$ instead of M'/g.

The derivative part of the scalar Lagrangian, $-(D_\mu K)^\dagger (D_\mu K)$, induces both Γ and β_t vertices, plus mixed ones which we still call Γ vertices (see the β_t factors in the two-leg Γ terms of $\Delta\mathcal{L}_{S,t}^{(n_f=2)}$).

It works like this: first, we replace $g \to \bar{g}(1+\Gamma)$ and $g' \to -\bar{g}(\bar{s}/\bar{c})$ in $-(D_\mu K)^\dagger (D_\mu K)$, splitting the result in two classes of terms, both written in terms of $\bar{g}$, with or without Γ.
Then we substitute in both classes $v \to 2\bar{M}'(1+\beta_t)/\bar{g}$: the class containing Γ is, up to terms of $\mathcal{O}(\bar{g}^4)$, $\Delta\mathcal{L}_{S,\,t}$ [Eq.(17)], and includes also β_t factors, while the class free of Γ has the same β_t vertices as Eq.(**??**) with g, θ, M', A_μ and Z_μ replaced by $\bar{g}$, $\bar{\theta}$, $\bar{M}'$, $\bar{A}_\mu$ and $\bar{Z}^0_\mu$. The upshot is that you need both the results for the new Γ vertices derived in the previous section 16 (containing β_t), *and* the expressions for the β_t terms.

The Γ and β_t terms of the Faddeev–Popov sector are intertwined just as in the case of the scalar Lagrangian.

Summary of the special vertices

The upshot of these first sections of the paper lies in the Appendices. There you find the full set of Standard Model Γ [up to $\mathcal{O}(\bar{g}^4)$] and $\beta_{h,t}$ special vertices in the R_ξ gauges. All primes and bars over A_μ, Z_μ, M, $M_{_H}$ and θ have been dropped, but not over $\bar{g}$, the $SU(2)$ coupling constant of the rediagonalized neutral sector. Just pick your tadpole scheme, β_h or β_t, and compute your Feynman diagrams including the $\beta_{h,t}$ vertices of Appendix A or B, respectively.

If you prefer to work with the rediagonalized neutral sector, you should simply replace g by $\bar{g}$ in the $\beta_{h,t}$ vertices, and add to them the Γ ones of Appendix C. There, Γ vertices are listed for the β_t scheme (note that Γ and β_t terms are intertwined — see sec. 23); just set $\beta_t = 0$ if you are using the β_h scheme instead.

Finally, the following table graphically summarizes which of the SM sectors provide each type of special vertex. Note the overlap of Γ and β_t terms in the scalar and Faddeev–Popov sectors.

SECTOR	β_h	β_t	Γ
Scalar: $(D_\mu K)^\dagger (D_\mu K)$		•	•
Scalar: $\mu^2 K^\dagger K + (\lambda/2)(K^\dagger K)^2$	•	•	
Yang–Mills			•
Gauge-Fixing			
Faddeev–Popov		•	•
Fermion – gauge boson			•
Fermion – Higgs		•	

WSTI for two-loop gauge boson self-energies

WSTI

The purpose of this section is to discuss in detail the structure of the (doubly-contracted) Ward-Slavnov-Taylor identities (WSTI) for the two-loop gauge boson self-energies in the Standard Model, focusing in particular on the role played by the reducible diagrams. This analysis is performed in the 't Hooft–Feynman gauge.

Definitions and WST identities

Let Π_{ij} be the sum of all diagrams (both one-particle reducible and irreducible) with two external boson fields, i and j, to all orders in perturbation theory (as usual, the external Born propagators are not to be included in the expression for Π_{ij})

$$\Pi_{ij} = \sum_{n=1}^{\infty} \frac{g^{2n}}{(16\pi^2)^n}\, \Pi_{ij}^{(n)}\,. \tag{23}$$

In the subscripts of the quantities $\Pi_{ij}^{(n)}$ we will also explicitly indicate, when necessary, the appropriate Lorentz indices with Greek letters. At each order in the perturbative expansion it is convenient to make explicit the tensor structure of these functions by employing the following definitions:

$$\Pi^{(n)}_{\mu\nu,VV} = D^{(n)}_{VV}\,\delta_{\mu\nu} + P^{(n)}_{VV}\,p_\mu\,p_\nu \quad \Pi^{(n)}_{\mu,VS} = -ip_\mu\,M_S\,G^{(n)}_{VS} \quad \Pi^{(n)}_{SS} = R^{(n)}_{SS}\,, \tag{24}$$

where the subscripts V and S indicate vector and scalar fields, M_S is the mass of the Nambu–Goldstone scalar S, and p is the incoming momentum of the vector boson (note: $\Pi^{(n)}_{\mu,SV} = -\Pi^{(n)}_{\mu,VS}$). The quantities D_{ij}, P_{ij}, G_{ij}, and R_{ij} depend only on the squared four-momentum and are symmetric in i and j. Furthermore, D and R have the dimensions of a mass squared, while G and P are dimensionless.

The WST identities require that, at each perturbative order, the gauge-boson self-energies

satisfy the equations

$$
\begin{aligned}
p_\mu\, p_\nu\, \Pi^{(n)}_{\mu\nu,AA} &= 0 \\
p_\mu\, p_\nu\, \Pi^{(n)}_{\mu\nu,AZ} + i p_\mu\, M_0\, \Pi^{(n)}_{\mu,A\phi_o} &= 0 \\
p_\mu\, p_\nu\, \Pi^{(n)}_{\mu\nu,ZZ} + M_0^2\, \Pi^{(n)}_{\phi_o\phi_o} + 2\, i p_\mu\, M_0\, \Pi^{(n)}_{\mu,Z\phi_o} &= 0 \\
p_\mu\, p_\nu\, \Pi^{(n)}_{\mu\nu,WW} + M^2\, \Pi^{(n)}_{\phi\phi} + 2\, i p_\mu\, M\, \Pi^{(n)}_{\mu,W\phi} &= 0\,,
\end{aligned}
\tag{25}
$$

which imply the following relations among the form factors D, P, G, and R

$$
\begin{aligned}
D^{(n)}_{AA} + p^2\, P^{(n)}_{AA} &= 0 \qquad (26)\\
D^{(n)}_{AZ} + p^2\, P^{(n)}_{AZ} + M_0^2\, G^{(n)}_{A\phi_o} &= 0 \qquad (27)\\
p^2\, D^{(n)}_{ZZ} + p^4\, P^{(n)}_{ZZ} + M_0^2\, R^{(n)}_{\phi_o\phi_o} &= -2\, M_0^2\, p^2 G^{(n)}_{Z\phi_o} \qquad (28)\\
p^2\, D^{(n)}_{WW} + p^4\, P^{(n)}_{WW} + M^2\, R^{(n)}_{\phi\phi} &= -2\, M^2\, p^2 G^{(n)}_{W\phi}\,. \qquad (29)
\end{aligned}
$$

The subscripts A, Z, W, ϕ and ϕ_0 clearly indicate the SM fields. We have verified these WST Identities at the two-loop level (i.e. $n = 2$) with our code `GraphShot`.

WSTI at two loops: the role of reducible diagrams

At any given order in the coupling constant expansion, the SM gauge boson self-energies satisfy the WSTI (25). For $n \geq 2$, the quantities $\Pi^{(n)}_{ij}$ contain both one-particle irreducible (1PI) and reducible (1PR) contributions. At $\mathcal{O}(g^4)$, the SM $\Pi^{(n)}_{ij}$ functions contain the following *irreducible* topologies:

- eight two-loop topologies,
- three one-loop topologies with a β_{t_1} vertex,
- four one-loop topologies with a Γ_1 vertex,
- and one tree-level diagram with a two-leg $\mathcal{O}(g^4)$ β_t or Γ vertex .

Reducible $\mathcal{O}(g^4)$ graphs involve the product of two $\mathcal{O}(g^2)$ ones:

- two one-loop diagrams,
- one one-loop diagram and a tree-level diagram with a $\mathcal{O}(g^2)$ two-leg vertex insertion,
- or two tree-level diagrams, each with a $\mathcal{O}(g^2)$ two-leg vertex insertion.

There are also $\mathcal{O}(g^4)$ topologies containing tadpoles but, as we discussed in previous sections, their contributions add up to zero as a consequence of our choice for β_t.
In the following we analyze the structure of the $\mathcal{O}(g^4)$ WSTI for photon, Z, and W self-energies, as well as for the photon–Z mixing, emphasizing the role played by the reducible diagrams.

The photon self-energy

The contribution of the 1PR diagrams to the photon self-energy at $\mathcal{O}(g^4)$ is given, in the 't Hooft–Feynman gauge, by (with obvious notation)

$$\Pi^{(2)R}_{\mu\nu,AA} = \frac{1}{(2\pi)^4 i}\left[\frac{1}{p^2}\,\tilde{\Pi}^{(2)R}_{\mu\nu,AA} + \frac{1}{p^2+M_0^2}\,\hat{\Pi}^{(2)R}_{\mu\nu,AA}\right], \tag{30}$$

where

$$\tilde{\Pi}^{(2)R}_{\mu\nu,AA} = \Pi^{(1)}_{\mu\alpha,AA}\,\Pi^{(1)}_{\alpha\nu,AA} \qquad \hat{\Pi}^{(2)R}_{\mu\nu,AA} = \Pi^{(1)}_{\mu\alpha,AZ}\,\Pi^{(1)}_{\alpha\nu,ZA} + \Pi^{(1)}_{\mu,A\phi_o}\,\Pi^{(1)}_{\nu,\phi_o A}\,.$$

It is interesting to consider separately the reducible diagrams that involve an intermediate photon propagator ($\tilde{\Pi}^{(2)R}_{\mu\nu,AA}$) and those including an intermediate Z or ϕ_0 propagator ($\hat{\Pi}^{(2)R}_{\mu\nu,AA}$). By employing the definitions given in the previous subsection and eq. (26) with $n = 1$, one verifies that $\tilde{\Pi}^{2R}_{\mu\nu,AA}$ obeys the photon WSTI by itself,

Theorem

$$p_\mu\, p_\nu\, \tilde{\Pi}^{(2)R}_{\mu\nu,AA} = p^2\, \left[D^{(1)}_{AA} + p^2\, P^{(1)}_{AA} \right]^2 = 0\,. \tag{31}$$

This is not the case for $\hat{\Pi}^{(2)R}_{\mu\nu,AA}$, although most of its contributions cancel when contracted by $p_\mu p_\nu$ as a consequence of eq. (27) ($n = 1$),

$$p_\mu\, p_\nu \hat{\Pi}^{(2)R}_{\mu\nu,AA} = p^2\, M_0^2\, \left(p^2 + M_0^2\right) \left[G^{(1)}_{A\phi_0}\right]^2 . \qquad (32)$$

The only diagrams contributing to the A–ϕ_0 mixing up to $\mathcal{O}(g^2)$ are those with a W–ϕ or FP ghosts loop, and the tree-level diagram with a Γ insertion. Their contribution, in the 'tHooft–Feynman gauge, is

$$G^{(1)}_{A\phi_0} = (2\pi)^4 i\, sc\, \left[2B_0(p^2, M, M) + 16\pi^2\Gamma_1\right] . \qquad (33)$$

A direct calculation (e.g. with `GraphShot`) shows that this residual contribution of the reducible diagrams to the $\mathcal{O}(g^4)$ photon WSTI, eq. (32), is exactly canceled by the contribution of the $\mathcal{O}(g^4)$ irreducible diagrams, which include two-loop diagrams as well as one-loop graphs with a two-leg vertex insertion.

The photon–Z mixing

We now consider the second of eqs. (25) for $n = 2$. Reducible diagrams contribute to both A–Z and A–ϕ_0 transitions. Following the example of Eq.(30), we divide these contributions in two classes: the diagrams that include an intermediate photon propagator and those mediated by a Z or a ϕ_0, namely, for the photon–Z transition in the 't Hooft–Feynman gauge,

$$
\begin{aligned}
\Pi^{(2)R}_{\mu\nu,AZ} &= \frac{1}{(2\pi)^4 i}\left[\frac{1}{p^2}\,\tilde{\Pi}^{(2)R}_{\mu\nu,AZ} + \frac{1}{p^2+M_0^2}\,\hat{\Pi}^{(2)R}_{\mu\nu,AZ}\right] \\
\tilde{\Pi}^{(2)R}_{\mu\nu,AZ} &= \Pi^{(1)}_{\mu\alpha,AA}\,\Pi^{(1)}_{\alpha\nu,AZ} \\
\hat{\Pi}^{(2)R}_{\mu\nu,AZ} &= \Pi^{(1)}_{\mu\alpha,AZ}\,\Pi^{(1)}_{\alpha\nu,ZZ} + \Pi^{(1)}_{\mu,A\phi_o}\,\Pi^{(1)}_{\nu,\phi_o Z}\,,
\end{aligned}
\tag{34}
$$

and, for the photon–ϕ_0 transition in the same gauge,

$$
\begin{aligned}
\Pi^{(2)R}_{\mu,A\phi_0} &= \frac{1}{(2\pi)^4 i}\left[\frac{1}{p^2}\,\tilde{\Pi}^{(2)R}_{\mu,A\phi_0} + \frac{1}{p^2+M_0^2}\,\hat{\Pi}^{(2)R}_{\mu,A\phi_0}\right] \\
\tilde{\Pi}^{(2)R}_{\mu,A\phi_0} &= \Pi^{(1)}_{\mu\alpha,AA}\,\Pi^{(1)}_{\alpha,A\phi_0} \\
\hat{\Pi}^{(2)R}_{\mu,A\phi_0} &= \Pi^{(1)}_{\mu\alpha,AZ}\,\Pi^{(1)}_{\alpha,Z\phi_0} + \Pi^{(1)}_{\mu,A\phi_0}\,\Pi^{(1)}_{\phi_0\phi_0}\,.
\end{aligned}
\tag{35}
$$

The reducible diagrams with an intermediate photon propagator satisfy the WSTI by themselves. Indeed,

$$p_\mu\, p_\nu\, \tilde{\Pi}^{(2)R}_{\mu\nu,AZ} + iM_0 p_\mu \tilde{\Pi}^{(2)R}_{\mu,A\phi_o} = 0\,, \tag{36}$$

as it can be easily checked using eq. (26) with $n = 1$. On the contrary, the remaining reducible diagrams must be added to the irreducible $\mathcal{O}(g^4)$ contributions in order to satisfy the WSTI for the photon–Z mixing:

Theorem

$$p_\mu p_\nu \left[\frac{\hat{\Pi}^{(2)R}_{\mu\nu,AZ}}{(2\pi)^4 i (p^2 + M_0^2)} + \Pi^{(2)I}_{\mu\nu,AZ} \right] + iM_0 p_\mu \left[\frac{\hat{\Pi}^{(2)R}_{\mu,A\phi_0}}{(2\pi)^4 i (p^2 + M_0^2)} + \Pi^{(2)I}_{\mu,A\phi_0} \right] = 0. \tag{37}$$

The Z self-energy

Also in the case of the WSTI for the $\mathcal{O}(g^4)$ Z self-energy it is convenient to separate the reducible contributions mediated by a photon propagator from the rest of the reducible diagrams. In the 't Hooft–Feynman gauge it is

$$
\begin{aligned}
\Pi^{(2)R}_{\mu\nu,ZZ} &= \frac{1}{(2\pi)^4 i}\left[\frac{1}{p^2}\,\tilde{\Pi}^{(2)R}_{\mu\nu,ZZ} + \frac{1}{p^2+M_0^2}\,\hat{\Pi}^{(2)R}_{\mu\nu,ZZ}\right] \\
\tilde{\Pi}^{(2)R}_{\mu\nu,ZZ} &= \Pi^{(1)}_{\mu\alpha,ZA}\,\Pi^{(1)}_{\alpha\nu,AZ} \\
\hat{\Pi}^{(2)R}_{\mu\nu,ZZ} &= \Pi^{(1)}_{\mu\alpha,ZZ}\,\Pi^{(1)}_{\alpha\nu,ZZ} + \Pi^{(1)}_{\mu,Z\phi_0}\,\Pi^{(1)}_{\nu,\phi_0 Z}\,,
\end{aligned}
\qquad (38)
$$

$$
\begin{aligned}
\Pi^{(2)R}_{\mu,Z\phi_0} &= \frac{1}{(2\pi)^4 i}\left[\frac{1}{p^2}\,\tilde{\Pi}^{(2)R}_{\mu,Z\phi_0} + \frac{1}{p^2+M_0^2}\,\hat{\Pi}^{(2)R}_{\mu,Z\phi_0}\right] \\
\tilde{\Pi}^{(2)R}_{\mu,Z\phi_0} &= \Pi^{(1)}_{\mu\alpha,ZA}\,\Pi^{(1)}_{\alpha,A\phi_0} \\
\hat{\Pi}^{(2)R}_{\mu,Z\phi_0} &= \Pi^{(1)}_{\mu\alpha,ZZ}\,\Pi^{(1)}_{\alpha,Z\phi_0} + \Pi^{(1)}_{\mu,Z\phi_0}\,\Pi^{(1)}_{\phi_0\phi_0}\,,
\end{aligned}
\qquad (39)
$$

$$
\begin{aligned}
\Pi^{(2)R}_{\phi_0\phi_0} &= \frac{1}{(2\pi)^4 i}\left[\frac{1}{p^2}\,\tilde{\Pi}^{(2)R}_{\phi_0\phi_0} + \frac{1}{p^2+M_0^2}\,\hat{\Pi}^{(2)R}_{\phi_0\phi_0}\right] \\
\tilde{\Pi}^{(2)R}_{\phi_0\phi_0} &= \Pi^{(1)}_{\alpha,\phi_0 A}\,\Pi^{(1)}_{\alpha,A\phi_0} \\
\hat{\Pi}^{(2)R}_{\phi_0\phi_0} &= \Pi^{(1)}_{\alpha,\phi_0 Z}\,\Pi^{(1)}_{\alpha,Z\phi_0} + \Pi^{(1)}_{\phi_0\phi_0}\,\Pi^{(1)}_{\phi_0\phi_0},
\end{aligned}
\tag{40}
$$

and, once again, the reducible diagrams mediated by a photon propagator satisfy the WSTI by themselves, i.e.

$$
p_\mu\,p_\nu\,\tilde{\Pi}^{(2)R}_{\mu\nu,ZZ} + M_0^2\,\tilde{\Pi}^{(2)R}_{\phi_0\phi_0} + 2\,i\,p_\mu\,M_0\,\tilde{\Pi}^{(2)R}_{\mu,Z\phi_0} = 0\,, \tag{41}
$$

as it can be easily checked using the one-loop WSTI for the photon–Z mixing [eq. (27) with $n=1$].

The W self-energy

All the $\mathcal{O}(g^4)$ 1PR contributions to the WSTI for the W self-energy are mediated, in the 't Hooft–Feynman gauge, by a charged particle of mass M. A separate analysis of their contribution does not lead, in this case, to particularly significant simplifications of the structure of the WSTI. However, some cancellations among the reducible terms occur, allowing to obtain a relation that will be useful in the discussion of the Dyson resummation of the W propagator. The 1PR quantities that contribute to the $\mathcal{O}(g^4)$ WSTI for the W self-energy have the following form:

$$\Pi^{(2)R}_{\mu\nu,WW} = \frac{1}{(2\pi)^4 i\;(p^2+M^2)} \Big\{ \left(D^{(1)}_{WW}\right)^2 \delta_{\mu\nu} + p_\mu\, p_\nu \left[2\, D^{(1)}_{WW}\, P^{(1)}_{WW} + p^2\, \left(P^{(1)}_{WW}\right)^2 + M^2\, \left(G^{(1)}_{W\phi}\right)^2 \right] \Big\} \tag{42}$$

$$\Pi^{(2)R}_{\mu,W\phi} = \frac{-i\,p_\mu\,M}{(2\pi)^4 i\,(p^2+M^2)}\,G^{(1)}_{W\phi}\,\Big[D^{(1)}_{WW} + p^2\,P^{(1)}_{WW} + R^{(1)}_{\phi\phi}\Big]$$

$$\Pi^{(2)R}_{\phi\phi} = \frac{1}{(2\pi)^4 i\,(p^2+M^2)}\,\Big[p^2\,M^2\,\Big(G^{(1)}_{W\phi}\Big)^2 + \Big(R^{(1)}_{\phi\phi}\Big)^2\Big]\,. \qquad (43)$$

Contracting the free indices with the corresponding external momenta, summing the three contributions and employing eq. (29) with $n = 1$, we obtain

$$(2\pi)^4 i\Big[p_\mu\,p_\nu\,\Pi^{(2)R}_{\mu\nu,WW} + M^2\,\Pi^{(2)R}_{\phi\phi} + 2\,i\,p_\mu\,M\,\Pi^{(2)R}_{\mu,W\phi}\Big] = p^2\,M^2\,\Big(G^{(1)}_{W\phi}\Big)^2 - R^{(1)}_{\phi\phi}\,\Big[D^{(1)}_{WW} + p^2\,P^{(1)}_{WW}\Big]\,. \qquad (44)$$

Dyson resummed propagators and their WSTI

Dyson resummed propagators

We will now present the Dyson resummed propagators for the electroweak gauge bosons. We will then employ the results of sec. 27 to show explicitly, up to terms of $\mathcal{O}(g^4)$, that the resummed propagators satisfy the WST identities.
Following definition (23) for Π_{ij}, the function Π^{I}_{ij} represents the sum of all 1PI diagrams with two external boson fields, i and j, to all orders in perturbation theory (as usual, the external Born propagators are not to be included in the expression for Π^{I}_{ij}).

As we did in eqs. (24), we write explicitly its ,

Lorentz structure

$$\Pi^{I}_{\mu\nu,VV} = D^{I}_{VV}\,\delta_{\mu\nu} + P^{I}_{VV}\,p_\mu\,p_\nu \tag{45}$$

$$\Pi^{I}_{\mu,VS} = -ip_\mu\,M_S\,G^{I}_{VS} \qquad \Pi^{I}_{SS} = R^{I}_{SS}\,, \tag{46}$$

where V and S indicate SM vector and scalar fields, and p_μ is the incoming momentum of the vector boson [note: $\Pi^{I}_{\mu,SV} = -\Pi^{I}_{\mu,VS}$].

We also introduce the

transverse and longitudinal projectors

$$
t^{\mu\nu} = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\,, \qquad l^{\mu\nu} = \frac{p_\mu p_\nu}{p^2}\,,
$$
$$
t^{\mu\alpha}\,t^{\alpha\nu} = t^{\mu\nu}\,, \quad l^{\mu\alpha}\,l^{\alpha\nu} = l^{\mu\nu}\,, \quad t^{\mu\alpha}\,l^{\alpha\nu} = 0\,,
$$
$$
\Pi^l_{\mu\nu,VV} = D^l_{VV}\,t_{\mu\nu} + L^l_{VV}\,l_{\mu\nu}, \quad L^l_{VV} = D^l_{VV} + p^2\,P^l_{VV}\,. \tag{47}
$$

The full propagator for a field i which mixes with a field j via the function Π^{I}_{ij} is given by the perturbative series

$$\bar{\Delta}_{ii} = \Delta_{ii} + \Delta_{ii} \sum_{n=0}^{\infty} \prod_{l=1}^{n+1} \sum_{k_l} \Pi^{I}_{k_{l-1}k_l} \Delta_{k_l k_l} \tag{48}$$

$$= \Delta_{ii} + \Delta_{ii} \Pi^{I}_{ii} \Delta_{ii} + \Delta_{ii} \sum_{k_1=i,j} \Pi^{I}_{ik_1} \Delta_{k_1k_1} \Pi^{I}_{k_1 i} \Delta_{ii} + \cdots ,$$

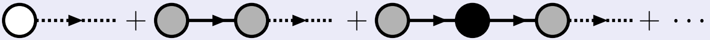
where $k_0 = k_{n+1} = i$, while for $l \neq n+1$, k_l can be i or j. Δ_{ii} is the Born propagator of the field i.

We rewrite Eq.(48) as

$$\bar{\Delta}_{ii} = \Delta_{ii}\,\left[1 - (\Pi\,\Delta)_{ii}\right]^{-1}, \tag{49}$$

and refer to $\bar{\Delta}_{ii}$ as the *resummed* propagator. The quantity $(\Pi\,\Delta)_{ii}$ is the sum of all the possible products of Born propagators and self-energies, starting with a 1PI self-energy Π^{I}_{ii}, or transition Π^{I}_{ij}, and ending with a propagator Δ_{ii}, such that each element of the sum cannot be obtained as a product of other elements in the sum.

A diagrammatic representation of $(\Pi\,\Delta)_{ii}$ is the following,

$(\Pi\,\Delta)_{ii} \;=$ 

where the Born propagator of the field i (j) is represented by a dotted (solid) line, the white blob is the i 1PI self-energy, and the dots at the end indicate a sum running over an infinite number of 1PI j self-energies (black blobs) inserted between two 1PI i–j transitions (gray blobs).

It is also useful to define, as an auxiliary quantity, the *partially resummed* propagator for the field i, $\hat{\Delta}_{ii}$, in which we resum only the proper 1PI self-energy insertions Π^{I}_{ii}, namely,

$$\hat{\Delta}_{ii} = \Delta_{ii}\,\left[1 - \Pi^{I}_{ii}\,\Delta_{ii}\right]^{-1}. \tag{50}$$

If the particle i were not mixing with j through loops or two-leg vertex insertions, $\hat{\Delta}_{ii}$ would coincide with the resummed propagator $\bar{\Delta}_{ii}$.

$\hat{\Delta}_{ii}$ can be graphically depicted as

$\hat{\Delta}_{ii} =$  $+ \cdots .$

Partially resummed propagators allow for a compact expression for $(\Pi\,\Delta)_{ii}$,

$$(\Pi\,\Delta)_{ii} = \Pi^{I}_{ii}\Delta_{ii} + \Pi^{I}_{ij}\hat{\Delta}_{jj}\Pi^{I}_{ji}\Delta_{ii}, \tag{51}$$

so that the resummed propagator of the field i can be cast in the form

$$\bar{\Delta}_{ii} = \Delta_{ii}\,\left[1 - \left(\Pi^{I}_{ii} + \Pi^{I}_{ij}\hat{\Delta}_{jj}\Pi^{I}_{ji}\right)\Delta_{ii}\right]^{-1}. \tag{52}$$

We can also define a resummed propagator for the i–j transition. In this case there is no corresponding Born propagator, and the resummed one is given by the sum of all possible products of 1PI i and j self-energies, transitions, and Born propagators starting with Δ_{ii} and ending with Δ_{jj}. This sum can be simply expressed in the following compact form,

$$\bar{\Delta}_{ij} = \bar{\Delta}_{ii}\,\Pi^{I}_{ij}\,\hat{\Delta}_{jj}. \tag{53}$$

The charged sector

We now apply Eq.(50), Eq.(52), Eq.(53)) to W and charged Goldstone boson fields. The *partially* resummed propagator of the charged Goldstone scalar follows immediately from Eq.(50). The Born W and ϕ propagators in the 't Hooft–Feynman gauge are

$$\Delta^{\mu\nu}_{WW} = \frac{\delta_{\mu\nu}}{p^2 + M^2}, \quad \Delta_{\phi\phi} = \frac{1}{p^2 + M^2}, \tag{54}$$

where, for simplicity of notation, we have dropped the coefficients $(2\pi)^4 i$.

In the same gauge, the partially resummed ϕ and W propagators are

$$\hat{\Delta}_{\phi\phi} = \Delta_{\phi\phi}\left[1 - \Pi^{I}_{\phi\phi}\,\Delta_{\phi\phi}\right]^{-1} = \left[p^2 + M^2 - R^{I}_{\phi\phi}\right]^{-1} \tag{55}$$

$$\hat{\Delta}^{\mu\nu}_{WW} = \frac{1}{p^2 + M^2 - D^{I}_{WW}}\left(\delta_{\mu\nu} + \frac{p_\mu p_\nu P^{I}_{WW}}{p^2 + M^2 - D^{I}_{WW} - p^2 P^{I}_{WW}}\right). \tag{56}$$

Equation (56) assumes a more compact form when expressed in terms of the transverse and longitudinal projectors $t_{\mu\nu}$ and $l_{\mu\nu}$,

$$\hat{\Delta}^{\mu\nu}_{WW} = \frac{t^{\mu\nu}}{p^2 + M^2 - D^l_{WW}} + \frac{l^{\mu\nu}}{p^2 + M^2 - L^l_{WW}}. \tag{57}$$

The resummed W and ϕ propagators can be then derived from Eq.(52),

$$\bar{\Delta}_{\phi\phi} = \left[p^2 + M^2 - R^l_{\phi\phi} - \frac{p^2\, M^2\, (G^l_{W\phi})^2}{p^2 + M^2 - L^l_{WW}}\right]^{-1} \tag{58}$$

$$\bar{\Delta}^{\mu\nu}_{WW} = \frac{t^{\mu\nu}}{p^2 + M^2 - D^l_{WW}} + l^{\mu\nu}\left[p^2 + M^2 - L^l_{WW} - \frac{p^2 M^2 (G^l_{W\phi})^2}{p^2 + M^2 - R^l_{\phi\phi}}\right]^{-1}. \tag{59}$$

The resummed propagator for the W–ϕ transition is provided by Eq.(53),

$$\bar{\Delta}^{\mu}_{w\phi} = \frac{-ip_{\mu} M G^{I}_{\phi w}}{p^2 + M^2 - R^{I}_{\phi\phi}} \left[p^2 + M^2 - L^{I}_{ww} - \frac{p^2 M^2 (G^{I}_{w\phi})^2}{p^2 + M^2 - R^{I}_{\phi\phi}} \right]^{-1} . \qquad (60)$$

We will now show explicitly, up to terms of $\mathcal{O}(g^4)$, that the resummed propagators defined above satisfy the following WST identity:

Theorem

$$p_{\mu}\, p_{\nu}\, \bar{\Delta}^{\mu\nu}_{ww} \; + i\, p_{\mu}\, M\, \bar{\Delta}^{\mu}_{w\phi} - i\, p_{\nu}\, M\, \bar{\Delta}^{\nu}_{\phi w} + M^2\, \bar{\Delta}_{\phi\phi} = 1\,, \qquad (61)$$

which, in turn, is satisfied if

$$p^2 M^2 \left(G^I_{W\phi}\right)^2 + M^2 R^I_{\phi\phi} + p^2 L^I_{WW} - R^I_{\phi\phi} L^I_{WW} + 2p^2 M^2 G^I_{W\phi} = 0 \,. \quad (62)$$

This equation can be verified explicitly, up to terms of $\mathcal{O}(g^4)$, using the WSTI for the W self-energy: at $\mathcal{O}(g^2)$ Eq.(62) becomes simply

$$M^2 R^{(1)}_{\phi\phi} + p^2 L^{(1)}_{WW} + 2p^2 M^2 G^{(1)}_{W\phi} = 0 \,, \quad (63)$$

which coincides with eq. (29) for $n = 1$.

To prove Eq.(62) at $\mathcal{O}(g^4)$ we can combine the last of Eq.(25) with $n = 2$ and Eq.(44) to get [1]

$$p^2 M^2 \left(G^{(1)}_{W\phi}\right)^2 + M^2 R^{(2)\prime}_{\phi\phi} + p^2 L^{(2)\prime}_{WW} - R^{(1)}_{\phi\phi} L^{(1)}_{WW} + 2p^2 M^2 G^{(2)\prime}_{W\phi} = 0\,. \quad (64)$$

[1] For simplicity of notation, in this section we dropped the coefficients $(2\pi)^4 i$.

The neutral sector

neutral sector

The SM neutral sector involves the mixing of three boson fields, A_μ, Z_μ and ϕ_0. As the definitions for the resummed propagators presented at the beginning of sec. 44 refer to the mixing of only two boson fields, we will now discuss their generalization to the three-field case.

Consider three boson fields i, j and k mixing up through radiative corrections. For each of them we can define a *partially resummed* propagator $\hat{\Delta}_{ll}$ $(l = i, j,$ or $k)$ according to Eq.(50). For each pair of the three fields, say (j, k), we can also define the following *intermediate propagators*

$$
\begin{aligned}
\tilde{\Delta}_{jj}(j,k) &= \Delta_{jj}\left[1-\left(\Pi^{I}_{jj}+\Pi^{I}_{jk}\,\hat{\Delta}_{kk}\,\Pi^{I}_{kj}\right)\Delta_{jj}\right]^{-1} & (65)\\
\tilde{\Delta}_{jk}(j,k) &= \tilde{\Delta}_{jj}(j,k)\,\Pi^{I}_{jk}\,\hat{\Delta}_{kk}\,, & (66)
\end{aligned}
$$

where the parentheses on the l.h.s. indicate the chosen pair of fields. [$\tilde{\Delta}_{kk}(j,k)$ and $\tilde{\Delta}_{kj}(j,k)$ can be simply derived from the above definitions by exchanging $j \leftrightarrow k$.] The reader will immediately note that the r.h.s. of the above eqs. (65, 66) are almost identical to those of eqs. (52, 53), with the appropriate renaming of the fields. Equations (65, 66), introduced in the context of three-field mixing, define however only *intermediate* propagators (denoted by the tilde), while eqs. (52, 53), presented in the analysis of the two-field mixing case, define the complete resummed propagators (denoted by the bar).

Indeed, the definition of full resummed propagator in the three-field mixing scenario requires one further step: the resummed propagator for a field i mixing with the fields j and k via the functions Π'_{ij}, Π'_{ik} and Π'_{jk} can be cast in the following form

$$\bar{\Delta}_{ii} = \Delta_{ii}\left[1 - \left(\Pi'_{ii} + \sum_{l,m} \Pi'_{il}\,\tilde{\Delta}_{lm}(j,k)\,\Pi'_{mi}\right)\Delta_{ii}\right]^{-1}, \qquad (67)$$

where l and m can be j or k, while the resummed propagator for the transition between the fields i and k is

$$\bar{\Delta}_{ik} = \bar{\Delta}_{ii} \sum_{l=j,k} \Pi^{l}_{il} \tilde{\Delta}_{lk}(j,k)\,. \tag{68}$$

Armed with eqs. (65)–(68), we can now present the A_μ, Z_μ and A_μ–Z_μ propagators. First of all, the Born A_μ, Z_μ and ϕ_0 propagators in the 't Hooft–Feynman gauge are

$$\Delta^{\mu\nu}_{AA} = \frac{\delta_{\mu\nu}}{p^2}, \qquad \Delta^{\mu\nu}_{ZZ} = \frac{\delta_{\mu\nu}}{p^2 + M_0^2}, \qquad \Delta_{\phi_0\phi_0} = \frac{1}{p^2 + M_0^2}, \tag{69}$$

where, for simplicity of notation, we have dropped once again the coefficients $(2\pi)^4 i$. The *partially resummed* propagators (three) can be immediately computed via Eq.(50) and the *intermediate* ones (twelve) via eqs. (65) and (66). Finally, after some algebra, eqs. (67) and (68) provide us with the fully resummed propagators:
$\bar{\Delta}_{VV} = t_{\mu\nu}\bar{\Delta}^T_{VV} + l_{\mu\nu}\bar{\Delta}^L_{VV}$, with $V = A, Z$ and

$$\bar{\Delta}^T_{AA} = \left[p^2 - D^I_{AA} - \frac{(D^I_{AZ})^2}{p^2 + M_0^2 - D^I_{ZZ}}\right]^{-1} \quad (70)$$

$$\bar{\Delta}^T_{ZZ} = \left[p^2 + M_0^2 - D^I_{ZZ} - \frac{(D^I_{AZ})^2}{p^2 - D^I_{AA}}\right]^{-1} \quad (71)$$

$$\bar{\Delta}^T_{AZ} = D^I_{AZ}\left[\left(p^2 - D^I_{AA}\right)\left(p^2 + M_0^2 - D^I_{ZZ}\right) - (D^I_{AZ})^2\right]^{-1}. \quad (72)$$

The expressions of the longitudinal components of these propagators are more lengthy and we will only present them up to terms of $\mathcal{O}(g^4)$:

$$\bar{\Delta}^{L}_{AA} = \left[p^2 + \mathcal{O}(g^6)\right]^{-1} \tag{73}$$

$$\bar{\Delta}^{L}_{ZZ} = \left[p^2 + M_0^2 - L^{l}_{ZZ} - \frac{(L^{l}_{AZ})^2}{p^2} - \frac{p^2 M_0^2 (G^{l}_{Z\phi_0})^2}{p^2 + M_0^2} + \mathcal{O}(g^6)\right]^{-1} \tag{74}$$

$$\bar{\Delta}^{L}_{AZ} = \frac{L^{l}_{AZ}}{p^2\left(p^2 + M_0^2 - L^{l}_{ZZ}\right)} + \frac{M_0^2}{\left(p^2 + M_0^2\right)^2} G^{l}_{A\phi_0} G^{l}_{Z\phi_0} + \mathcal{O}(g^6). \tag{75}$$

Equation (73) achieves its compact form due to the use of the WSTI (26) and (27) with $n = 1, 2$. Also eq. (75) has been simplified using $L^{(1)}_{AA} = 0$ [i.e. eq. (26) with $n = 1$]. We point out that if we use the one-loop WSTI for the photon self-energy, eq. (26), the transverse part of the resummed A–Z propagator becomes, up to terms of $\mathcal{O}(g^4)$,

$$\bar{\Delta}^T_{AZ} = D^I_{AZ} \left[p^2 \left(1 + P^I_{AA}\right) \left(p^2 + M_0^2 - D^I_{ZZ}\right)\right]^{-1} + \mathcal{O}(g^6), \qquad (76)$$

thus showing a pole at $p^2 = 0$ if $D^I_{AZ}(p^2 = 0)$ were not vanishing because of the rediagonalization of the neutral sector.

In order to show explicitly, up to terms of $\mathcal{O}(g^4)$, that the above resummed propagators satisfy their WSTI, we also present the resummed propagators involving the neutral scalar ϕ_0:

$$\bar{\Delta}^{\mu}_{A\phi_0} = -ip_\mu \frac{M_0}{p^2}\left[\frac{G^{I}_{Z\phi_0} L^{I}_{AZ}}{\left(p^2+M_0^2\right)^2} + \frac{G^{I}_{A\phi_0}}{p^2+M_0^2-R^{I}_{\phi_0\phi_0}}\right] + \mathcal{O}(g^6) \tag{77}$$

$$\bar{\Delta}^{\mu}_{Z\phi_0} = \frac{-ip_\mu M_0}{p^2+M_0^2-L^{I}_{ZZ}}\left[\frac{G^{I}_{A\phi_0} L^{I}_{AZ}}{p^2\left(p^2+M_0^2\right)} + \frac{G^{I}_{Z\phi_0}}{p^2+M_0^2-R^{I}_{\phi_0\phi_0}}\right] + \mathcal{O}(g^6) \tag{78}$$

$$\bar{\Delta}_{\phi_0\phi_0} = \left[p^2+M_0^2-R^{I}_{\phi_0\phi_0} - M_0^2\left(G^{I}_{A\phi_0}\right)^2 - \frac{p^2M_0^2}{p^2+M_0^2}\left(G^{I}_{Z\phi_0}\right)^2\right]^{-1} + \mathcal{O}(g^6). \tag{79}$$

With these results, and with the WSTI (Eq.(26))–(Eq.(28)), (Eq.(37)) and (Eq.(41)), we can finally prove, up to $\mathcal{O}(g^4)$, the following WSTI for the resummed A, Z and A–Z propagators,

$$p_\mu\, p_\nu\, \bar{\Delta}^{\mu\nu}_{AA} \;=\; 1 \tag{80}$$

$$p_\mu\, p_\nu\, \bar{\Delta}^{\mu\nu}_{AZ} + i p_\mu M_0\, \bar{\Delta}^{\mu}_{A\phi_o} \;=\; 0 \tag{81}$$

$$p_\mu\, p_\nu\, \bar{\Delta}^{\mu\nu}_{ZZ} + M_0^2\, \bar{\Delta}_{\phi_o\phi_o} + 2 i p_\mu M_0\, \bar{\Delta}^{\mu}_{Z\phi_o} \;=\; 1\,. \tag{82}$$

The LQ basis

For the purpose of the renormalization, it is more convenient to extract from the quantities defined in the previous sections the factors involving the weak mixing angle θ. To achieve this goal, we employ the LQ basis, which relates the photon and Z fields to a new pair of fields, L and Q:

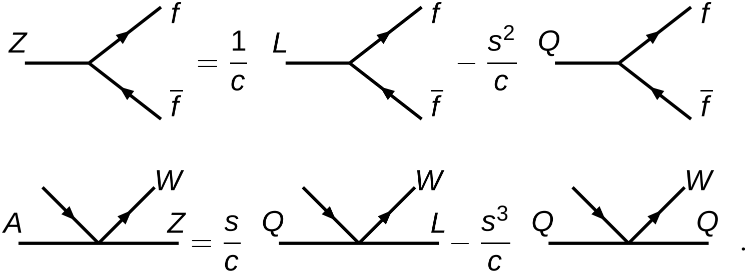
$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c & 0 \\ s & 1/s \end{pmatrix} \begin{pmatrix} L_\mu \\ Q_\mu \end{pmatrix}. \tag{83}$$

Consider the fermion currents j_A^μ and j_Z^μ coupling to the photon and to the Z. As the Lagrangian must be left unchanged under this transformation, namely $j_Z^\mu Z_\mu + j_A^\mu A_\mu = j_L^\mu L_\mu + j_Q^\mu Q_\mu$, the currents transform as

$$\begin{pmatrix} j_Z^\mu \\ j_A^\mu \end{pmatrix} = \begin{pmatrix} 1/c & -s^2/c \\ 0 & s \end{pmatrix} \begin{pmatrix} j_L^\mu \\ j_Q^\mu \end{pmatrix}. \tag{84}$$

If we rewrite the SM Lagrangian in terms of the fields L and Q, and perform the same transformation (83) on the FP ghosts fields [from (X_A,X_Z) to (X_L,X_Q)], then all the interaction terms of the SM Lagrangian are independent of θ. Note that this is true only if the relation $M/c = M_0$ is employed, wherever necessary, to remove the remaining dependence on θ. In this way the dependence on the weak mixing angle is moved to the kinetic terms of the L and Q fields which, clearly, are not mass eigenstates.

The relevant fact for our discussion is that the couplings of Z, photon, X_Z and X_A are related to those of the fields L and Q, X_L and X_Q by identities like the one described, in a diagrammatic way, in the following figure:


Z
f
f̄
= 1/c
L
f
f̄
− s²/c
Q
f
f̄
A
W
Z
= s/c
Q
W
L
− s³/c
Q
W
Q
.

As the couplings of the fields L, Q, X_L and X_Q do not depend on θ, all the dependence on this parameter is factored out in the coefficients in the r.h.s. of these identities.
Since θ appears only in the couplings of the fields A, Z, X_A and X_Z (once again, the relation $M/c = M_0$ must also be employed, wherever necessary), it is possible to single out this parameter in the two-loop self-energies of the vector bosons. Consider, for example, the transverse part of the photon two-loop self-energy $D_{AA}^{(2)}$ (which includes the contribution of both *irreducible* and *reducible* diagrams). All diagrams contributing to $D_{AA}^{(2)}$ can be classified in two classes: those including (i) one internal A, Z, X_A or X_Z field, and (ii) those not containing any of these fields. The complete dependence on θ can be factored out by expressing the external photon couplings and the internal A, Z X_A or X_Z couplings of the diagrams of class (i) in terms of the couplings of the fields L, Q, X_L and X_Q, namely

$$D^{(2)}_{AA} = s^2 \left[\frac{1}{c^2} f^{AA}_1 + f^{AA}_2 + s^2 f^{AA}_3 \right], \tag{85}$$

where the functions f^{AA}_i $(i = 1, 2, 3)$ are θ-independent. Similarly, we can factor out the θ dependence of the transverse part of the two-loop photon–Z mixing and Z self-energy,

$$D^{(2)}_{AZ} = \frac{s}{c} \left[\frac{1}{c^2} f^{AZ}_1 + f^{AZ}_2 + s^2 f^{AZ}_3 + s^4 f^{AZ}_4 \right], \tag{86}$$

$$D^{(2)}_{ZZ} = \frac{1}{c^2} \left[\frac{1}{c^2} f^{ZZ}_1 + f^{ZZ}_2 + s^2 f^{ZZ}_3 + s^4 f^{ZZ}_4 + s^6 f^{ZZ}_5 \right], \tag{87}$$

where, once again, the functions f_i^{AZ} and f_i^{ZZ} ($i = 1, \dots, 5$) do not depend on θ. Analogous relations hold for the longitudinal components of the two-loop self-energies.
We note that $D_{AZ}^{(2)}$ and $D_{ZZ}^{(2)}$ also contain a third class of diagrams containing more than one internal Z (or X_z) field (up to three, in $D_{ZZ}^{(2)}$). However, the diagrams of this class involve the trilinear vertex ZHZ (or $\bar{X}_z H X_z$), which does not induce any new θ dependence.

However, from the point of view of renormalization it is more convenient to distinguish between the θ dependence originating from external legs and the one introduced by external legs. We define, to all orders,

$$
\begin{aligned}
D_{AA} &= s^2\,\Pi_{QQ\,;\,\mathrm{ext}}\,p^2 = s^2 \sum_{n=1}^{\infty} \left(\frac{g^2}{16\,\pi^2}\right)^n \Pi^{(n)}_{QQ\,;\,\mathrm{ext}}\,p^2, \\
D_{AZ} &= \frac{s}{c}\,\Sigma_{AZ\,;\,\mathrm{ext}} = \frac{s}{c} \sum_{n=1}^{\infty} \left(\frac{g^2}{16\,\pi^2}\right)^n \Sigma^{(n)}_{AZ\,;\,\mathrm{ext}}, \\
D_{ZZ} &= \frac{1}{c^2}\,\Sigma_{ZZ\,;\,\mathrm{ext}} = \frac{1}{c^2} \sum_{n=1}^{\infty} \left(\frac{g^2}{16\,\pi^2}\right)^n \Sigma^{(n)}_{ZZ\,;\,\mathrm{ext}}, \\
\Sigma^{(n)}_{AZ\,;\,\mathrm{ext}} &= \Sigma^{(n)}_{3Q\,;\,\mathrm{ext}} - s^2\,\Pi^{(n)}_{QQ\,;\,\mathrm{ext}}\,p^2, \\
\Sigma^{(n)}_{ZZ\,;\,\mathrm{ext}} &= \Sigma^{(n)}_{33\,;\,\mathrm{ext}} - 2\,s^2\,\Sigma^{(n)}_{3Q\,;\,\mathrm{ext}} + s^4\,\Pi^{(n)}_{QQ\,;\,\mathrm{ext}}\,p^2.
\end{aligned}
\tag{88}
$$

Furthermore, our procedure is such that

$$\Sigma^{(n)}_{3Q\,;\,\mathrm{ext}} = \Pi^{(n)}_{3Q\,;\,\mathrm{ext}}\, p^2, \tag{89}$$

with $\Pi^{(n)}_{3Q\,;\,\mathrm{ext}}$ regular at $p^2 = 0$. At $\mathcal{O}\left(g^2\right)$ the *external* quantities are θ-independent while, at $\mathcal{O}\left(g^4\right)$ the relation with the coefficients of Eqs.(85)–(87) is

$$\begin{aligned}
\Pi^{(2)}_{QQ\,;\,\mathrm{ext}}\, p^2 &= \frac{1}{c^2}\, f^{AA}_1 + f^{AA}_2 + f^{AA}_3\, s^2, \\
\Sigma^{(2)}_{3Q\,;\,\mathrm{ext}} &= \frac{1}{c^2}\, (f^{AA}_1 + f^{AZ}_1) - f^{AA}_1 + f^{AZ}_2 + s^2\, (f^{AA}_2 + f^{AZ}_3) + s^4\, (f^{AA}_3 + f^{AZ}_4) \\
\Sigma^{(2)}_{33\,;\,\mathrm{ext}} &= \frac{1}{c^2}\, (f^{AA}_1 + 2\, f^{AZ}_1 + f^{ZZ}_1) - f^{AA}_1 - 2\, f^{AZ}_1 + f^{ZZ}_2 \\
&\quad + s^2\, (-f^{AA}_1 + 2\, f^{AZ}_2 + f^{ZZ}_3) + s^4\, (f^{AA}_2 + 2\, f^{AZ}_3 + f^{ZZ}_4) \\
&\quad + s^6\, (f^{AA}_3 + 2\, f^{AZ}_4 + f^{ZZ}_5),
\end{aligned} \tag{90}$$

and s, c in Eq.(90) should be evaluated at $\mathcal{O}\left(g^0\right)$.

Consider the process $\overline{f}f \to \overline{h}h$; taking into account Dyson re-summed propagators and neglecting, for the moment, vertices and boxes we write

$$\begin{aligned}\mathcal{M}(\overline{f}f \to \overline{h}h) = (2\pi)^4\, i \Big[& -e^2\, Q_f Q_h\, \gamma^\mu \otimes \gamma^\mu\, \bar{\Delta}^T_{AA} \\ & - \frac{eg}{2\,c}\, Q_f\, \gamma^\mu \otimes \gamma^\mu\, (v_h + a_h\,\gamma_5)\, \bar{\Delta}^T_{ZA} \\ & - \frac{eg}{2\,c}\, Q_h\, \gamma^\mu\, (v_f + a_f\,\gamma_5) \otimes \gamma^\mu\, \bar{\Delta}^T_{ZA} \\ & - \frac{g^2}{4\,c^2}\, \gamma^\mu\, (v_f + a_f\,\gamma_5) \otimes \gamma^\mu\, (v_h + a_h\,\gamma_5)\, \bar{\Delta}^T_{ZZ} \Big] \end{aligned} \tag{91}$$

where f and h are fermions with quantum numbers Q_I, I_{3i}, $i = f, h$;

furthermore we have introduced

$$v_f = I_{3f} - 2\,Q_f\,s^2, \qquad a_f = I_{3f}, \tag{92}$$

with $e^2 = g^2\,s^2$. Always neglecting terms proportional to fermion masses it is useful to introduce an effective weak-mixing angle as follows:

Definition

$$s^2_{\rm eff} = s^2\left[1 - \frac{\Pi_{AZ\,;\,\rm ext}}{1 - s^2\,\Pi_{AA\,;\,\rm ext}}\right], \qquad V_f = I_{3f} - 2\,Q_f\,s^2_{\rm eff}. \tag{93}$$

The amplitude of Eq.(91) can be cast into the following form:

$$\mathcal{M}(\bar{f}f \to \bar{h}h) = (2\pi)^4\, i \Big[-\gamma^\mu \otimes \gamma^\mu \frac{1}{1 - s^2\,\Pi_{AA\,;\,\mathrm{ext}}} \frac{e^2\, Q_f Q_h}{p^2} - \frac{g^2}{4\,c^2}\, \gamma^\mu\, (V_f + a_f\,\gamma_5) \otimes \gamma^\mu\, (V_h + a_h\,\gamma_5)\, \bar{\Delta}^T_{ZZ} \Big]. \quad (94)$$

The functions $\Pi_{AA\,;\,\mathrm{ext}}$, $\Pi_{AZ\,;\,\mathrm{ext}}$ and $\Sigma_{ZZ\,;\,\mathrm{ext}}$ start at $\mathcal{O}\left(g^2\right)$ in perturbation theory. Eq.(94) shows the nice effect of absorbing – to all orders – non-diagonal transitions into a redefinition of s^2 and forms the basis for introducing renormalization equations in the neutral sector, e.g. the one associated with the fine-structure constant α. Questions related to gauge-parameter independence of Dyson re-summation, e.g. in Eq.(93), will not be addressed here.

TWO-LOOP Renormalization in the Making

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Outline of Part III

1. The QED case
2. The standard model case
3. Fermion mass fitting equations
4. W mass fitting equation

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The QED case

To understand renormalization at the two-loop level we consider first the case of pure QED where we have

$$\Pi_{QED}(s,m) = \frac{e^2}{16\,\pi^2}\,\Pi^{(1)}(s,m) + \frac{e^4}{256\,\pi^4}\,\Pi^{(2)}(s,m), \tag{1}$$

where $p^2 = -s$ and where we have indicated a dependence of the result on the (bare) electron mass. Suppose that we compute the two-loop contribution (3 diagrams) in the limit $m = 0$. The result is

$$\Pi^{(2)}(s,0) = -\frac{4}{\epsilon} + \mathcal{O}(1)\,, \tag{2}$$

where $n = 4 - \epsilon$. This is a well-known result which shows the cancellation of the double ultraviolet pole as well as of any non-local residue. The latter is related to the fact that the four one-loop diagrams with one-loop counterterms cancel due to a Ward identity. Let us repeat the calculation with a non-zero electron mass;

after *scalarization* of the result we consider the ultraviolet divergent parts of the various diagrams. Collecting all the terms we obtain

$$\Pi^{(2)}(s,m) = -\frac{1}{\epsilon}\left[4\left(1+24\,\frac{m^2}{s}\right)+192\,\frac{m^4}{s^2}\,\frac{1}{\beta(m)}\,\ln\frac{\beta(m)+1}{\beta(m)-1}\right]+\mathcal{O}(1)\,. \tag{3}$$

Note that the *m* dependent part is not only finite but also zero in the limit $s \to 0$; indeed, in the limit $s \to 0$ and with $\mu^2 = m^2/s - i\,\delta$ we have

$$\beta = 2\,i\,\mu - \frac{i}{2\,\mu} + \mathcal{O}\left(\mu^{-2}\right), \qquad \frac{1}{\beta}\,\ln\frac{\beta+1}{\beta-1} = -\frac{1}{2\,\mu^2}, \tag{4}$$

so that

$$\Pi^{(2)}(0,m) = -\frac{4}{\epsilon} + \Pi^{(2)}_{\rm fin}(0,m). \tag{5}$$

Eq.(5) is the main ingredient to build our renormalization equation and contains only bare parameters, in the true spririt of the fitting equations that express a measurable input, α in this case, as a function of bare parameters, e and m in this case, and of ultraviolet singularites.
To make a prediction, the running of α in this case, is a different issue: the scattering of two charged particles is proportional to

$$
\begin{aligned}
\frac{e^2}{1 - f(s)} &= e^2 \left[1 + f(s) + f^2(s) + \cdots\right], \\
f(s) &= \frac{e^2}{16\,\pi^2}\,\Pi^{(1)}(s) + \frac{e^4}{(16\,\pi^2)^2}\,\Pi^{(2)}(s) + \mathcal{O}\left(e^6\right).
\end{aligned} \tag{6}
$$

Renormalization

Renormalization amounts to substituting

$$e^2 = 4\,\pi\,\alpha - \alpha^2\,\Pi^{(1)}(0) + \frac{\alpha^3}{4\,\pi}\left\{\left[\Pi^{(1)}(0)\right]^2 - \Pi^{(2)}(0)\right\} + \mathcal{O}\left(\alpha^4\right), \quad (7)$$

with the following result

$$\frac{e^2}{1 - f(s)} = 4\,\pi\,\alpha\left\{1 + \frac{\alpha}{4\,\pi}\,\Pi^{(1)}_R(s) + \left(\frac{\alpha}{4\,\pi}\right)^2\,\left[\Pi^{(1)}_R(s)\,\Pi^{(1)}_R(s)\right.\right.$$
$$\left.\left. + \Pi^{(2)}_R(s) + \mathcal{O}\left(\alpha^3\right)\right\},$$
$$\Pi^{(n)}_R(s) = \Pi^{(n)}(s) - \Pi^{(n)}(0). \quad (8)$$

If our result has to be ultraviolet finite then the poles in $\Pi^{(n)}(s)$ should not depend on the scale s. This is obviously true for the one-loop result but what is the origin of the scale-dependent extra term in Eq.(3)? One should take into account that

$$\Pi^{(1)}(s,m) = -\frac{8}{3}\frac{1}{\epsilon} + \frac{4}{3}\left[\ln\frac{m^2}{M^2} + \left(1 + 2\,\frac{m^2}{s}\,\beta(m)\,\ln\frac{\beta(m)+1}{\beta(m)-1}\right]\right. - \frac{20}{9} + \frac{4}{3}\Delta_{UV} - \frac{16}{3}\frac{m^2}{s}, \tag{9}$$

and that m is the bare electron mass. To proceed step-by-step we introduce a renormalized electron mass which is given by

$$m = m_R\left[1 + \frac{e^2}{16\,\pi^2}\left(-\frac{6}{\epsilon} + \text{finite part}\right)\right]. \tag{10}$$

If we write $m^2 = m_R^2\,(1+\delta)$ then

$$\beta(m) = \beta(m_R) - 2\,\frac{m_R^2}{\beta(m_R)\,s}\,\delta + \mathcal{O}\left(\delta^2\right),$$

$$\ln\frac{\beta(m)+1}{\beta(m)-1} = \ln\frac{\beta(m_R)+1}{\beta(m_R)-1} - \frac{\delta}{\beta(m_R)} + \mathcal{O}\left(\delta^2\right). \qquad (11)$$

Inserting this expansion into our results we obtain

$$\begin{aligned}\Pi_{QED}(s,m_R) = \frac{e^2}{\pi^2}\Big[&-\frac{1}{6\,\epsilon} + \frac{1}{12}\ln\frac{m_R^2}{M^2} \\ &+ \frac{1}{3}\left(\frac{1}{4} - \frac{1}{2}\frac{m_R^2}{s} - 2\,\frac{m_R^4}{s^2}\right)\frac{1}{\beta(m_R)}\ln\frac{\beta(m_R)+1}{\beta(m_R)-1} - \\ &- \frac{5}{36} + \frac{1}{12}\Delta_{UV} - \frac{1}{3}\frac{m_R^2}{s}\Big] \\ &+ \frac{e^4}{\pi^4}\left[-\frac{1}{64\,\epsilon} + \frac{1}{256}\Pi^{(2)}_{\rm fin}(s,m_R)\right],\end{aligned} \qquad (12)$$

showing cancellation of the ultraviolet poles in $\Pi^{(n)}_R(s, m_R)$ with $n = 1, 2$. Of course Eq.(10) is not yet a true renormalization equation since the latter should contain the physical electron mass m_e and not the intermediate parameter m_R but the relation between the two is ultraviolet finite. All of this is telling us that a renormalization equation has the structure

$$p_{\mathrm{phys}} = f\left(\frac{1}{\epsilon}, p_{\mathrm{bare}}\right), \tag{13}$$

where the residue of the ultraviolet poles must be local. A prediction,

$$O\left(\frac{1}{\epsilon}, p_{\mathrm{bare}}\right) \equiv O(p_{\mathrm{phys}}), \tag{14}$$

gives a finite quantity that can be computed in terms of some input parameter set.

The SM case

In the full standard model the one-loop result is

$$\Pi^{(1)} = \Pi^{(1)}_{\rm bos} + \sum_l \Pi^{(1)}_l + \Pi^{(1)}_{tb} + \Pi^{(1)}_{udcs}. \tag{15}$$

We introduce

$$x_W = \frac{M_W^2}{s}, \quad x_l = \frac{m_l^2}{M_W^2}, \quad \text{etc},$$
$$\Delta_{UV} = \gamma + \ln\pi + \ln\frac{M_W^2}{\mu^2}, \qquad L_\beta(x) = \ln\frac{\beta(x)+1}{\beta(x)-1}, \tag{16}$$

In the limit $s \to 0$ we have

$$\Pi^{(1)}_{\rm bos}(0) = -\,3\,\left(-\,\frac{2}{\epsilon} + \Delta_{UV}\right),$$

$$\Pi^{(1)}_{l}(0) = \frac{4}{3}\,\left(-\,\frac{2}{\epsilon} + \Delta_{UV}\right) + \frac{4}{9} + \frac{4}{3}\,\ln x_l,$$

$$\Pi^{(1)}_{tb}(0) = \frac{20}{9}\,\left(-\,\frac{2}{\epsilon} + \Delta_{UV}\right) + \frac{20}{27} + \frac{16}{9}\,\ln x_t + \frac{4}{9}\,\ln x_b. \qquad (17)$$

First we consider fermion mass renormalization, obtaining

$$m_f^2 = m_{f\,R}^2\,\left(1 + 2\,\frac{g^2}{16\,\pi^2}\,\frac{\delta Z^f_m}{\epsilon}\right), \qquad (18)$$

with renormalization constants given by

fermion mass renormalization

lepton

$$\delta Z_m^l = -\frac{3}{2}\frac{1}{c^4}x_H^{-1} - 3\frac{1}{c^2} + 3 + \frac{3}{4}x_L + 2\frac{x_L^2}{x_H} + 6\frac{x_B^2}{x_H} + 6\frac{x_T^2}{x_H} - \frac{3}{4}x_H - 3x_H^{-1}, \quad (19)$$

b quark

$$\delta Z_m^b = -\frac{3}{2}\frac{1}{c^4}x_H^{-1} + \frac{1}{3}\frac{1}{c^2} - \frac{1}{3} + \frac{3}{4}x_B - \frac{3}{4}x_T + 2\frac{x_L^2}{x_H} + 6\frac{x_B^2}{x_H} + 6\frac{x_T^2}{x_H} - \frac{3}{4}x_H - 3\,x_H^{-1}, \tag{20}$$

t quark

$$\delta Z^t_m = -\frac{3}{2}\frac{1}{c^4}x_H^{-1} - \frac{2}{3}\frac{1}{c^2} + \frac{2}{3} - \frac{3}{4}x_B + \frac{3}{4}x_T + 2\frac{x_L^2}{x_H} + 6\frac{x_B^2}{x_H} + 6\frac{x_T^2}{x_H} - \frac{3}{4}x_H - 3x_H^{-1}. \tag{21}$$

Consider the fermionic part of $\Pi^{(1)}$ relative to one fermion generation (ν_l, l, t and b) and perform fermion mass renormalization; we obtain

$$\Pi^{(1)}_{\text{fer}} \to \Pi^{(1)}_{\text{ferm}} + \frac{g^2}{\pi^2\,\epsilon}\,\Delta\Pi^{(1)}_{\text{ferm}}, \tag{22}$$

where

$$\begin{aligned}\Pi^{(1)}_{\text{fer}} = {}& \frac{32}{9}\left(-\frac{2}{\epsilon} + \Delta_{UV}\right) + \frac{4}{3}\left(\ln x_L + \frac{1}{3}\ln x_B + \frac{4}{3}\ln x_T\right)\\ & - \frac{160}{27} - \frac{16}{3}\,x_W\,(x_L + \frac{1}{3}x_B + \frac{4}{3}x_T) + \frac{4}{3}(1 - 2\,x_W x_L - 8\,x_W^2 x_L^2)\\ & + \frac{4}{3}\,\beta^{-1}(x_W x_L)\,L_\beta(x_W x_L) + \frac{4}{9}\,\beta^{-1}(x_W x_B)\,L_\beta(x_W x_B)\\ & + \frac{16}{9}\,\beta^{-1}(x_W x_T)\,L_\beta(x_W x_T),\end{aligned} \tag{23}$$

$$
\begin{aligned}
\Delta\Pi^{(1)}_{\mathrm{ferm}} = {} & \frac{3}{2}c^{-4}x_W x_L x_H^{-1} + \frac{1}{2}c^{-4}x_W x_B x_H^{-1} + 2c^{-4}x_W x_T x_H^{-1} + 3c^{-2}x_W x_L \\
& - \frac{1}{9}c^{-2}x_W x_B + \frac{8}{9}c^{-2}x_W x_T - 6x_W x_L x_B^2 x_H^{-1} - 6x_W x_L x_T^2 x_H^{-1} + \cdots \\
& + 2x_W^2 x_T^2 x_H - \frac{16}{9}x_W^2 x_T^2 - 2x_W^2 x_T^3 - 16x_W^2 x_T^4 x_H^{-1}\Big).
\end{aligned}
\tag{24}
$$

When we add the two-loop result we obtain

$$\frac{g^2}{16\,\pi^2}\,\Pi^{(1)}_{\mathrm{fer}} + \frac{g^4}{(16\,\pi^2)^2}\,\Pi^{(2)} = \text{one loop} + \frac{g^4}{\pi^4}\,\Big[R^{(2)}\,\epsilon^{-2} + R^{(1)}\,\epsilon^{-1} + \Pi_{\mathrm{fin}}\Big]. \tag{25}$$

The two residues are given by

$$\begin{aligned} R^{(2)} &= -\frac{11}{256}, \\ R^{(1)} &= \frac{11}{256}\,\Delta_{UV} + \frac{407}{27648} + \frac{9}{64}c^{-4}x_W x_H^{-1} - \frac{9}{128}c^{-2}x_W - \frac{131}{6912}c^{-2} \\ &+ \frac{3}{64}x_W x_L - \frac{3}{16}x_W x_L^2 x_H^{-1} + \frac{9}{64}x_W x_B - \frac{9}{16}x_W x_B^2 x_H^{-1} \end{aligned} \tag{26}$$

$$
\begin{aligned}
&+ \frac{9}{64} x_W x_T - \frac{9}{16} x_W x_T^2 x_H^{-1} + \frac{9}{32} x_W x_H^{-1} + \frac{9}{128} x_W x_H \\
&+ \frac{1}{32} x_{\boldsymbol{w}} + \frac{3}{512} x_L + \frac{7}{1536} x_B + \frac{13}{1536} x_T \\
&+ \beta^{-1}(x_W) L_\beta(x_W) \Big(-\frac{11}{768} + \frac{3}{64} c^{-4} x_W x_H^{-1} + \frac{9}{32} c^{-4} x_W^2 x_H^{-1} \\
&- \frac{1}{32} c^{-2} x_W - \frac{9}{64} c^{-2} x_W^2 + \frac{3}{128} x_W x_L - \frac{1}{16} x_W x_L^2 x_H^{-1} + \frac{9}{128} x_W x_B \\
&- \frac{3}{16} x_W x_B^2 x_H^{-1} + \frac{9}{128} x_W x_T - \frac{3}{16} x_W x_T^2 x_H^{-1} + \frac{3}{32} x_W x_H^{-1} \\
&+ \frac{3}{128} x_W x_H - \frac{13}{384} x_W + \frac{3}{32} x_W^2 x_L - \frac{3}{8} x_W^2 x_L^2 x_H^{-1} \\
&+ \frac{9}{32} x_W^2 x_B - \frac{9}{8} x_W^2 x_B^2 x_H^{-1} + \frac{9}{32} x_W^2 x_T - \frac{9}{8} x_W^2 x_T^2 x_H^{-1} \\
&+ \frac{9}{16} x_W^2 x_H^{-1} + \frac{9}{64} x_W^2 x_H + \frac{1}{16} x_W^2 \Big).
\end{aligned}
\tag{27}
$$

Theorem

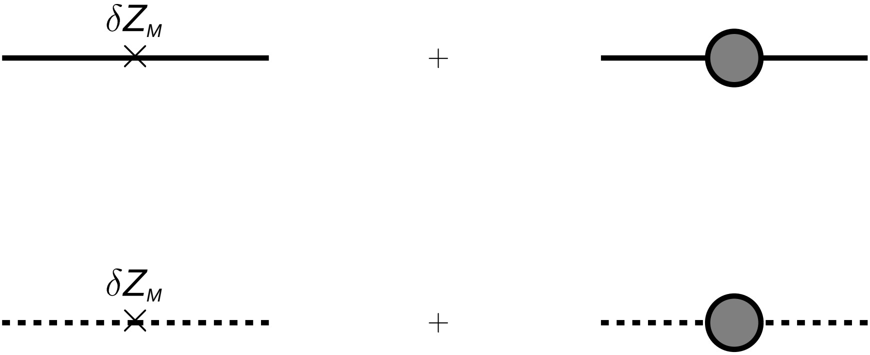
Therefore mass renormalization has removed

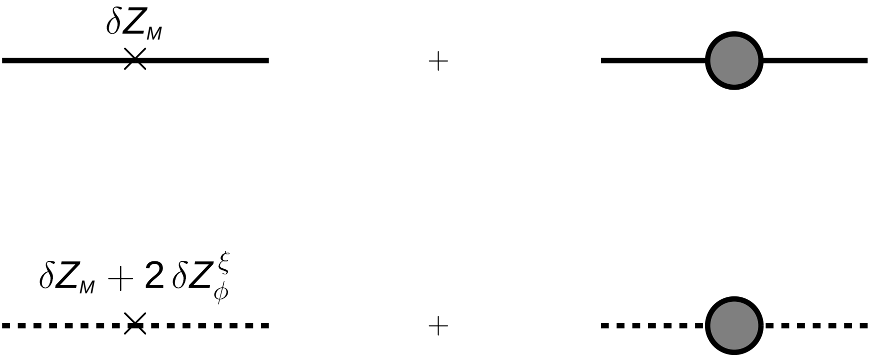
> *all logarithms in the residue of the simple ultraviolet pole for the fermionic part*
>
> *while a non-local residue remains in the bosonic part.*

Unfortunately a simple procedure of W mass renormalization is not enough to get rid of logarithmic residues in the bosonic component and the reason is that in a bosonic loop we may have three different fields, the W, the ϕ and the charged ghosts
and only one mass is available.

Example

The situation is illustrated in Fig. 1 where the cross denotes insertion of a counterterm δZ_M; the latter is fixed to remove the ultraviolet pole in the W self-energy and one easily verifies that the total in the second and third line of Fig. 1 (ϕ and X self-energies, respectively) is not ultraviolet finite.


δZ_M
$+$
δZ_M
$+$


δZ_M
$+$
$\delta Z_M + 2\,\delta Z_{\phi}^{\xi}$
$+$

The procedure has to be changed if we want to make the result in the bosonic sector as similar as possible to the one in the fermionic sector. With this goal in mind we introduce the following counterterms

$$W_\mu = Z_W^{1/2}\, W_\mu^R, \qquad \phi = Z_\phi^{1/2}\, \phi^R, \qquad M_W = Z_M^{1/2}\, M_W^R. \tag{28}$$

Our solution is to work in a $R_{\xi\xi}$-gauge where the gauge-fixing term (limited to the charged sector) is

$$\mathcal{C} = -\frac{1}{\xi_W}\,\partial_\mu W_\mu + \xi_\phi\, M_W\, \phi. \tag{29}$$

We also introduce additional counter-terms for the gauge parameters,

$$\xi_W = Z_W^{\xi}\, \xi_W^R, \qquad \xi_\phi = Z_\phi^{\xi}\, \xi_\phi^R. \tag{30}$$

Our scheme is further specified by imposing the condition

$$\xi_W^R = \xi_\phi^R = 1. \tag{31}$$

Dropping from now on the index R for renormalized fields and parameters we define the counter-Lagrangian to be

$$\mathcal{L}_{\text{ct}} = \frac{g^2}{16\,\pi^2}\left[\mathcal{L}_{\text{ct}}^{WW} + \mathcal{L}_{\text{ct}}^{\phi W} + \mathcal{L}_{\text{ct}}^{\phi\phi}\right], \qquad \mathcal{L}_{\text{ct}}^{ij} = \Phi_i^R\,\mathcal{O}_{ij}\,\Phi_i^R, \tag{32}$$

Φ_i being a vector or scalar field. We define δZ factors in the *MS*-scheme as

$$Z = 1 + \frac{g^2}{16\,\pi^2}\,\delta Z\,\frac{1}{\epsilon}, \tag{33}$$

and obtain

$$\epsilon\,\mathcal{O}^{WW}_{\mu\nu} = -\Big[\delta Z_W\,(p^2 + M^2_W) + \delta Z_M\,M^2_W\Big]\,\delta_{\mu\nu} + 2\,\delta Z^{\xi}_W\,p_\mu\,p_\nu,$$

$$\epsilon\,\mathcal{O}^{\phi\phi} = -\Big[\delta Z_\phi\,(p^2 + M^2_W) + M^2_W\,(\delta Z_M + 2\,\delta Z^{\xi}_\phi)\Big],$$

$$\epsilon\,\mathcal{O}^{W\phi}_{\mu} = (\delta Z^{\xi}_W - \delta Z^{\xi}_\phi)\,i\,M_W\,p_\mu. \tag{34}$$

These counter-terms are used to remove all poles from the transitions in the charged sector. After including the tadpole contribution and using Eq.(31) we find

$$\begin{aligned}
\delta Z^{\xi}_{W} &= \frac{11}{6}, \\
\delta Z^{\xi}_{\phi} &= -\frac{2}{3} + \frac{3}{2}c^{-4}x_H^{-1} - \frac{5}{4}c^{-2} + x_L - 2x_L^2 x_H^{-1} \\
&+ 3x_B - 6x_B^2 x_H^{-1} + 3x_T - 6x_T^2 x_H^{-1} + 3/4x_H + 3x_H^{-1}, \\
\delta Z_W &= \frac{11}{3} \\
\delta Z_{\phi} &= 2 + c^{-2} - x_L - 3x_B - 3x_T, \\
\delta Z_M &= -\frac{2}{3} - 3c^{-4}x_H^{-1} + \frac{3}{2}c^{-2} - x_L + 4x_L^2 x_H^{-1} - 3x_B \\
&+ 12x_B^2 x_H^{-1} - 3x_T + 12x_T^2 x_H^{-1} - \frac{3}{2}x_H - 6x_H^{-1}. \qquad (35)
\end{aligned}$$

Theorem

An important result follows, namely both

$$-Z_W^{1/2}\,(\xi_W Z_W^{\xi})^{-1}, \qquad +Z_M^{1/2}\,Z_\phi^{\xi}\,Z_\phi^{1/2}\,M\,\xi_\phi, \tag{36}$$

are ultraviolet finite so that the gauge-fixing term remains unrenormalized.

To continue our derivation we consider the ghost Lagrangian and the associated counter-terms,

$$\mathcal{L}_{\rm g} = Z_X\, \bar{X}^{\pm} \Big[\frac{1}{Z_W^{\xi}\, \xi_W}\, \partial^2 - Z_{\phi}^{\xi}\, Z_M\, \xi_{\phi}\, M_W^2 \Big]\, X^{\pm}. \tag{37}$$

To this Lagrangian corresponds an operator

$$\epsilon\, \mathcal{O}^{gg} = - \Big[(\delta Z_X - \delta Z_W^{\xi})\, (p^2 + M_W^2) + (\delta Z_M + \delta Z_W^{\xi} + \delta Z_{\phi}^{\xi})\, M_W^2 \Big]. \tag{38}$$

A simple calculation shows that, with the choice

$$\delta Z_x = \frac{23}{6}, \tag{39}$$

also the ghost Lagrangian is ultraviolet finite. The correct combination of mass counterterms is illustrated in Fig. 2. Note that in the $\overline{MS}$ scheme we define

$$Z = 1 + \frac{g^2}{16\,\pi^2}\,\delta Z\left[-\frac{2}{\epsilon} + \Delta_{UV}\right], \quad \delta Z_{\overline{MS}} = -\frac{1}{2}\,\delta Z_{MS}. \tag{40}$$

Note that the two-loop part of Π remains unchanged since modifications are of $\mathcal{O}\left(g^6\right)$ while for $\Pi^{(1)}_{\rm bos}$ we have to repeat the calculation, working in the new gauge.

The bare propagators for charged fields in the $R_{\xi\xi}$ gauge are

$$
\begin{aligned}
\bar{\Delta}^{ww}_{\mu\nu} &= \frac{1}{p^2+M^2}\left[\delta_{\mu\alpha}+\frac{\xi_w^2-1}{p^2+\xi_w^2\,M^2}\,p_\mu\,p_\alpha\right] \\
&\times\left[\delta_{\alpha\nu}+(1-\frac{\xi_\phi}{\xi_w})^2\,\frac{\xi_w^2\,M^2}{(p^2+\xi_w\xi_\phi\,M^2)^2}\,p_\alpha\,p_\nu\right], \\
\bar{\Delta}^{w\phi}_{\mu} &= i\,M\,p_\mu\,\frac{\xi_w\,(\xi_\phi-\xi_w)}{(p^2+\xi_w\xi_\phi\,M^2)^2}, \qquad \bar{\Delta}^{\phi\phi} = \frac{p^2+\xi_w^2\,M^2}{(p^2+\xi_w\xi_\phi\,M^2)^2}, \\
\bar{\Delta}^{gg} &= \frac{\xi_w}{p^2+\xi_w\xi_\phi\,M^2},
\end{aligned}
\tag{41}
$$

where the last propagator refers to the ghost - ghost transition.

One example will be enough to describe the procedure. Consider the following integral, corresponding to a ϕ loop in the *AA* self-energy:

$$I_{\mu\nu} = \int d^n q \, \frac{(q^2 + \xi_W^2 M_W^2)\,((q+p)^2 + \xi_W^2 M_W^2)}{(q^2 + \xi_W \xi_\phi M_W^2)^2\,((q+p)^2 + \xi_W \xi_\phi M_W^2)^2} \times (2\,q_\mu + p_\mu)\,(2\,q_\nu + p_\nu). \tag{42}$$

We expand the propagators,

$$\begin{aligned}
(q^2 + \xi_W^2 M_W^2)^{-k} &= (q^2 + M_W^2)^{-k} \\
&\quad - 2\,k\,\frac{g^2}{16\,\pi^2\,\epsilon}\, dZ_W^\xi\, M_W^2\,(q^2 + M_W^2)^{-k-1} + \cdots\,, \\
(q^2 + \xi_W \xi_\phi M_W^2)^{-k} &= (q^2 + M_W^2)^{-k} \\
&\quad - k\,\frac{g^2}{16\,\pi^2\,\epsilon}\,(dZ_W^\xi + dZ_\phi^\xi)\, M_W^2\,(q^2 + M_W^2)^{-k-1} + \cdot
\end{aligned} \tag{43}$$

and obtain

$$I_{\mu\nu} = I_0\,\delta_{\mu\nu} + I_1\,p_\mu p_\nu, \qquad (44)$$

with form factors

$$\begin{aligned} I_0 &= I_0(\xi = 1) + i\,\pi^2 g^2\,\Delta I_0\,dZ_\phi^\xi, \\ \Delta I_0 &= \frac{1}{8}\,\frac{n-2}{n-1}\,A_0(1, M_w^2) - \frac{n-1}{2}\,M_w^2\,B_0(1, 1, p^2, M_w, M_w) \\ &+ \frac{1}{4}\,\frac{1}{n-1}\,M_w^2\,(p^2 + M_w^2)\,B_0(1, 2, p^2, M_w, M_w), \end{aligned} \qquad (45)$$

where M_w is the bare W mass. Collecting all diagrams, renormalizing the W mass and inserting the solution for the renormalization constants we find the expression for the bosonic, one-loop, AA self-energy:

$$\Pi^{(1)}_{\mathrm{bos}} \to \frac{6}{\epsilon} + 6 - 3\Delta_{UV} + 8x_W + \cdots \qquad (46)$$

Including both components and taking into account the additional contribution arising from renormalization we finally get residues for the ultraviolet poles which show the expected properties:

$$
\begin{aligned}
R^{(2)} &= -\frac{55}{768}, \\
R^{(1)} &= \frac{11}{192}\Delta_{UV} + \frac{1199}{27648} - \frac{131}{6912}c^{-2} + \frac{3}{512}x_L + \frac{13}{1536}x_T \\
&+ \frac{7}{1536}x_B.
\end{aligned} \tag{47}
$$

Eq.(47) shows complete cancellation of poles with a logarithmic residue; furthermore the two residues in Eq.(47) are scale independent and cancel in the difference $\Pi(p^2) - \Pi(0)$.

Transitions

A final comment concerns the Z-photon transition which is not zero, at $p^2 = 0$, in any gauge where $\xi \neq 1$ even after the Γ_1 re-diagonalization procedure.

However, in our case, the non-zero result shows up only due to a different renormalization of the two bare gauge parameters and it is, therefore, of $\mathcal{O}\left(g^4\right)$; it can be absorbed into Γ_2 which does not modify our result for Π since there are no Γ_2-dependent terms in the AA transition (only Γ_1^2 appears).

renormalization procedure

One should observe that our procedure is completely equivalent to consider one-loop diagrams with the insertion of one-loop counterterms and one may wonder why we have not included $\delta Z_W, \delta Z_\phi, \delta Z_x$ and also a δZ_e, arising from charge renormalization and a δZ_A from the renormalization of the photon field.

about counterterms

The argument goes as follows: first we consider the relevant vertices with counterterms:

$$
\begin{aligned}
AWW &= Z_W\, Z_A^{1/2}\, Z_e \;\otimes\; \text{Born},\\
A\phi\phi &= Z_\phi\, Z_A^{1/2}\, Z_e \;\otimes\; \text{Born},\\
AW\phi &= (Z_W\, Z_\phi\, Z_A\, Z_M)^{1/2}\, Z_e \;\otimes\; \text{Born},\\
A\overline{X}^{\pm}X^{\pm} &= Z_X\, Z_A^{1/2}\, Z_e \;\otimes\; \text{Born}.
\end{aligned}
\tag{48}
$$

Next, we consider the ultraviolet divergent part of the corresponding one-loop diagrams and obtain:

$$V_{UV} = \frac{g^2}{16\,\pi^2}\,\frac{\delta V}{\epsilon}, \tag{49}$$

where

$$\begin{aligned}
\delta V^{AWW}_{\alpha\beta\gamma} &= -\frac{11}{3}\,\delta_{\alpha\beta}\,(p_2 + 2\,p_1)_\gamma + \frac{11}{3}\,\delta_{\alpha\gamma}\,(p_1 + 2\,p_2)_\beta \\
&+ \frac{11}{3}\,\delta_{\beta\gamma}\,(p_1 - p_2)_\alpha \\
\delta V^{A\phi\phi}_{\alpha} &= \Big(2 + c^{-2} - x_{L} - 3\,x_{T} - 3\,x_{B}\Big)\,(p_1 - p_2)_\alpha, \\
\delta V^{AXX}_{\alpha} &= 2\,p_{1\alpha}, \\
\delta V^{AW\phi}_{\alpha\gamma} &= i\,\delta_{\alpha\gamma}\,M_{W}\,\Big(\frac{3}{2}c^{-4}\frac{1}{x_H} - \frac{5}{4}c^{-2} - 2\frac{x_L^2}{x_H} - 6\frac{x_T^2}{x_H} - 6\frac{x_B^2}{x_H} + \frac{3}{x_H} + \frac{3}{4}x_H \\
&+ x_{L} + 3x_{T} + 3x_{B} - \frac{5}{2}\Big).
\end{aligned} \tag{50}$$

With these results we can prove that

$$\delta Z_e + \frac{1}{2}\,\delta Z_A = 0, \tag{51}$$

i.e. that, like in QED, charge renormalization is only due to vacuum polarization. Note that the Γ_1 prescription is crucial for proving the Ward identity of Eq.(51). Consider now the one-loop photon self-energy in our gauge; for instance, the diagrams with a ghost loop have vertices proportional to Z_x (thanks to Eq.(51)) and ghost propagators given by

$$\Delta^{gg} = \frac{1}{Z_x}\,\frac{\xi_W}{p^2 + \xi_W\,\xi_\phi\,mw^2}. \tag{52}$$

Clearly, δZ_x gives no contribution. The same holds for all other diagrams and for the remaining counterterms, δZ_ϕ and δZ_W. In conclusion, in computing Π we can forget about one-loop diagrams with field and charge counterterms and only worry about mass renormalization which we do, in some unconventional way, by expanding the explicit expression for $\Pi^{(1)}(s)$.

Inclusion of Δ_{UV}

In the previous section we have performed renormalization in the MS scheme and here we proceed by extending the same procedure to the $\overline{MS}$ scheme. The counterterms in the two schemes are connected by the simple relation $\delta Z_{\overline{MS}} = -\frac{1}{2}\,\delta Z_{MS}$ and what we may show that not only the double and single ultraviolet poles of $\Pi(s)$ have scale independent, local, residues but also the terms proportional to powers of Δ_{UV} have the same property.

Fermion mass fitting equations

For the complete answer we need fitting equations that relate the bare masses to the physical ones since the renormalized mass is only an intermediate parameter which is bound to disappear in the expresion for any physical observable. For a generic $u - d$ doublet we obtain

$$
\begin{aligned}
m_f &= m_f^{\mathrm{phys}} + \frac{g^2}{16\,\pi^2}\,\Sigma_f\Big|_{m=m^{\mathrm{phys}}}, \\
m^2_{f\,\mathrm{ren}} &= m^2_{f\,\mathrm{phys}}\left\{1 + \frac{g^2}{8\,\pi^2}\left[\frac{\Sigma_f}{m_f^2}\Big|_{m=m^{\mathrm{phys}}} - \delta Z_m^f\right]\right\}
\end{aligned}
\tag{53}
$$

W mass fitting equations

The relation between renormalized and physical W mass is

$$M^2_{W\,\mathrm{ren}} = M^2_{W\,\mathrm{phys}} \left\{ 1 + \frac{g^2}{16\,\pi^2} \left[\frac{\mathrm{Re}\,\Sigma_{WW}(-M^2_{W\,\mathrm{phys}})}{M^2_{W\,\mathrm{phys}}} - \delta Z_M \right] \right\}, \tag{54}$$

where the quantity within square brackets is ultraviolet finite by construction and where

$$\Sigma_{WW} = \sum_{\mathrm{gen}} \Sigma^f_{WW} + \Sigma^b_{WW} - 2\,(\beta_{t1} + \Gamma_1). \tag{55}$$

Part II

Introduction to the Fermi Coupling Constant

Definitions

Writing a renormalization equation that involves G_F should not be confused with making a prediction with the muon life-time.

In the following section we present few examples that are relevant in evaluating Δg (see Eq.(58)) up to two-loops and therefore in contructing one of our renormalization equations.

– The Lagrangian of the Fermi theory which is relevant for our pourposes can be written as:

$$\mathcal{L}_F = \mathcal{L}_{QED} + \frac{G_F}{\sqrt{2}} \overline{\psi}_{\nu_m u} \gamma^\mu \gamma_+ \psi_\mu \overline{\psi}_e \gamma^\mu \gamma_+ \psi_{\nu_e}, \tag{56}$$

where $\gamma_+ = 1 + \gamma_5$.

To leading order in G_F and to all orders in α the muon lifetime takes the form

$$\frac{1}{\tau_\mu} = \Gamma_0\,(1 + \Delta q), \qquad \Gamma_0 = \frac{G_F^2\, m_\mu^5}{192\,\pi^3}. \tag{57}$$

The standard model weak corrections to τ_μ are conventionally parametrized by the relation

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8\,M^2}\,(1 + \Delta g). \tag{58}$$

Our goal will be to derive an explicit expression for Δg so that one can use Eq.(58) as a relation where on the left hand side there is a quantity whose value is obtained by experiment and where on the right hand side we have bare quantities.

Ther quantity Δg may be written as the sum of various contributions, which are

$$\Delta g = \Delta g^{WF} + \Delta g^{V} + \Delta g^{B} + \Delta g^{S}. \qquad (59)$$

The various terms arise from wave-function renormalization factors, weak vertices, boxes and the W self-energy. Self-energy corrections always play a special role and will be dicussed separately, although they are crucial in establishing gauge parameter independence.

Strategy of the calculation

In the standard model and in the $\xi = 1$ gauge the lowest order amplitude is

$$\begin{aligned}\mathcal{M}_{SM\,;\,0} &= (2\pi)^4\, i\, \frac{g^2}{8}\, \frac{1}{Q^2 + M^2}\, \overline{u}(p_{\nu_\mu})\, \gamma^\alpha\, \gamma_+\, u(p_\mu)\, \overline{u}(p_e)\, \gamma^\alpha\, \gamma_+\, v(p_{\nu_e}) \\ &\approx \frac{G_F}{\sqrt{2}}\, \overline{u}(p_{\nu_\mu})\, \gamma^\alpha\, \gamma_+\, u(p_\mu)\, \overline{u}(p_e)\, \gamma^\alpha\, \gamma_+\, v(p_{\nu_e}) \equiv \mathcal{M}_F, \qquad (60)\end{aligned}$$

where we have introduced $Q = p_\mu - p_e$.

Note that at one loop we have

$$\frac{1}{\tau_\mu} = \frac{m_\mu^5}{192\,\pi^3}\,\frac{g^4}{32\,M^2}\,\left(1 + 2\,\Delta g^{(1)} + \Delta q^{(1)}\right), \tag{61}$$

and we have to separate the pure e.m. corrections evaluated in the Fermi theory to obtain $\Delta g^{(1)}$. To obtain the amplitude which generates the one-loop weak correction we consider first

$$\mathcal{M}_{W\,;\,1} = \mathcal{M}_{SM\,;\,1} - \mathcal{M}_{\mathrm{sub}\,;\,1}, \tag{62}$$

where $\mathcal{M}_{\mathrm{sub}\,;\,1}$ is obtained by
grouping the one-loop SM corrections with one photon line connected to external fermions and one W line,
by shrinking the W line to a point and by replacing the corresponding W propagator with $1/M^2$.

At the one-loop level and after the substitution $g^2/(8\,M^2) \to G_F/\sqrt{2}$ we obtain

$$\mathcal{M}_{\mathrm{sub}\,;\,1} \equiv \mathcal{M}_{F\,;\,1}, \tag{63}$$

where the latter generates $\Gamma_0\,\Delta q^{(1)}$. In the subtracted amplitude the soft terms have disappeared and we generate $\Delta g^{(1)}$ with the help of

$$\mathcal{M}^{\mathrm{leading}}_{W\,;\,1} = \lim_{p_i, m_i \to 0} \mathcal{M}_{\mathrm{sub}\,;\,1}, \tag{64}$$

i.e. we only retain the lading part, with vanishing lepton masses and external momenta, which amounts to neglect corrections of $\mathcal{O}\left(\alpha\,m^2/M^2\right)$. One-loop diagrams with no photons only have an hard component and do not need a subtraction.

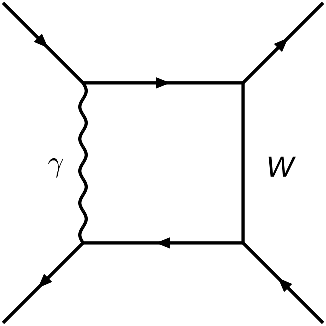


Figure: Infrared divergent one-loop box.

This amplitude contains two structures,

$$M_0 = \overline{u}\,\gamma^{\alpha}\,\gamma_+\,u\,\overline{u}\,\gamma^{\alpha}\,\gamma_+\,v, \qquad M_1 = \overline{u}\,\gamma^{\alpha}\,\gamma^{\mu}\,\gamma^{\beta}\gamma_+\,u\,\overline{u}\,\gamma^{\beta}\,\gamma^{\mu}\,\gamma^{\alpha}\,\gamma_+\,v. \tag{65}$$

However, M_1 is simply related to the current $\otimes$ current structure as it will be illustrated by considering the case of the one-loop box with W, γ exchange. We neglect for the moment all coupling constants and write

$$\mathcal{M}^{\rm sub}_{{\rm box}_{\gamma W}} = -\int d^n q\, \frac{q_\lambda\, q_\sigma}{(q^2+M^2)\,(q^2)^2}\, J^{\alpha\lambda\beta}\, J^{\beta\sigma\alpha},$$
$$J^{\alpha\lambda\beta} = \overline{u}(p_{\nu_\mu})\,\gamma^{\alpha}\,\gamma_+\,\gamma^{\lambda}\,\gamma^{\beta}\,u(p_\mu), \quad J^{\beta\sigma\alpha} = \overline{u}(p_e)\,\gamma^{\beta}\,\gamma^{\sigma}\,\gamma^{\alpha}\,\gamma_+\,v(p_{\nu_e}). \tag{66}$$

After integration we obtain

$$\mathcal{M}^{\rm sub}_{\rm box_{\gamma W}} = -\, i\, \pi^2\, B_0(2,1\,;\,0,0,M)\, J^{\alpha\lambda\beta}\, J^{\beta\lambda\alpha}. \qquad (67)$$

It can be shown that

$$J^{\alpha\lambda\beta}\, J^{\beta\lambda\alpha} = B^{(1)}\, M_0, \qquad (68)$$

where $B^{(1)}$ is obtained with the help of a projection operator,

$$\sum_{\text{spin}} \mathcal{P} \left(J^{\alpha\lambda\beta} J^{\beta\lambda\alpha} - B^{(1)} M_0 \right) = 0,$$
$$\mathcal{P} = \overline{v}(p_{\nu_e}) \gamma^\rho \gamma_+ u(p_{\nu_\mu}) \overline{u}(p_\mu) \gamma^\rho \gamma_+ u(p_e). \tag{69}$$

After a straightforward algebraic manipulation one obtains (in the limit $Q^2 \to 0$)

$$B^{(1)} = (n-2)^2, \tag{70}$$

which, after multiplication by $B_0(2,1\,;\,0,0,M)$ and in the limit $n \to 4$ reproduces the correct result, proportional to $B_0(2,1\,;\,0,0,M) - 1/2$.

Alternatively we start from the expression for the γ, W box without nullifying the soft scales,

$$\mathcal{M}_{\mathrm{box}_{\gamma W}} = \int d^q \frac{1}{d_0 d_1 d_2 d_3} \overline{u}(p_{\nu_\mu})\, \gamma^\alpha \gamma_+ \left[-i \, (\not{q} + \not{p}_\mu) + m_\mu \right] \gamma^\beta u(p_\mu)$$
$$\times\, \overline{u}(p_e)\, \gamma^\beta \left[-i \, (\not{q} + \not{p}_e) + m_e \right] \gamma^\alpha \gamma_+ v(p_{\nu_e}), \qquad (71)$$

where we introduce

$$d_0 = q^2, \quad d_1 = (q+p_\mu)^2+m_\mu^2, \quad d_2 = (q+P)^2+M^2, \quad d_3 = (q+p_e)^2+m_e^2, \tag{72}$$

$$\left(p_\mu - p_{\nu_\mu}\right)^2 = P^2, \qquad (p_\mu - p_e)^2 = Q^2. \tag{73}$$

A standard decomposition gives

$$\frac{1}{d_0 d_1 d_2 d_3} = \frac{1}{P^2+M^2}\left[\frac{1}{d_0 d_1 d_3} - \frac{1}{d_1 d_2 d_3} - 2\,\frac{q \cdot P}{d_0 d_1 d_2 d_3}\right]. \tag{74}$$

- The first term in the decomposition (in the limit $|P^2| \ll M^2$) is the QED vertex in the local Fermi theory that can be computed with standard techniques;
- The last two terms inside the square bracket of Eq.(74) are finite in the soft limit so that the extra contribution from the infrared SM box can be evaluated for $m_\mu, m_e = 0$ and $Q^2, P^2 = 0$.

In this limit only the term with three propagators survives and gives the well-known result.

With this technique (extracting instead of subtracting) we circumvent the puzzling procedure of Eq.(64) where the subtracted term is zero in dimensional regularization. However, the two procedures are totally equivalent.

If we neglect, for the moment, issues related to gauge parameter independence it is convenient to define a G constant that is totally process independent,

$$\Delta g = \delta_G + \Delta g^S, \qquad G = G_F \left(1 - \frac{g^2}{8\,M^2}\,\delta_G\right), \quad \delta_G = \sum_{n=1} \left(\frac{g^2}{16\,\pi^2}\right)^n \delta_G^{(n)}. \tag{75}$$

Alternatively, but always neglecting issues related to gauge parameter independence, we could resum δ_G by defyning $G_R = G_F/(1 + \delta_G)$.

In one case we obtain

$$G = \frac{g^2}{8\,M^2}\,\left[1 - \frac{g^2}{16\,\pi^2\,M^2}\,\Sigma_{WW}(0)\right]^{-1},$$

$$\Sigma_{WW}(0) = \Sigma^{(1)}_{WW}(0) + \frac{g^2}{16\,\pi^2}\,\Sigma^{(2)}_{WW}(0), \qquad (76)$$

where Σ_{WW} is the W self-energy, whereas with resummation we get

$$G_R = \frac{g^2}{8\,M^2}\left[1 - \frac{g^2}{16\,\pi^2\,M^2}\,\overline{\Sigma}_{WW}(0)\right]^{-1},$$

$$\overline{\Sigma}_{WW}(0) = \Sigma^{(1)}_{WW}(0) + \frac{g^2}{16\,\pi^2}\left[\Sigma^{(2)}_{WW}(0) - \Sigma^{(2)}_{WW}(0)\,\delta^{(1)}_{G}\right]. \qquad (77)$$

TWO-LOOP Renormalization in the Making

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Torino

July 12, 2006

Outline of Part IV

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1. Unstable particles
2. Solution of the renormalization equations
3. Some Input Parameter St
4. General structure of self-energies
5. Loop diagrams with dressed propagators
6. Unitarity, gauge parameter independence and WST identities
7. Unitarity
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9. Gauge parameter dependence
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Complex poles

To write additional renormalization equations we need *experimental* masses. For the W and Z bosons the IPS is defined in terms of pseudo-observables (PO); at first, OS quantities are derived by fitting the experimental lineshapes with

$$\Sigma_{VV}(s) = \frac{N}{(s - M_{OS}^2)^2 + s^2\,\Gamma_{OS}^2 / M_{OS}^2}, \qquad V = W\,, Z\,, \tag{1}$$

where N is an irrelevant (for our purposes) normalization constant. Secondly we define pseudo-observables (PO)

$$M_P = M_{OS}\,\cos\psi, \qquad \Gamma_P = \Gamma_{OS}\,\sin\psi, \qquad \psi = \arctan\frac{\Gamma_{OS}}{M_{OS}}, \tag{2}$$

which are inserted in the IPS.

Beyond one-loop

At one-loop level we can use directly the OS masses which are related to the zero of the real part of the inverse propagator. Beyond one-loop this would show a clash with gauge invariance since only the complex poles

$$s_V = \mu_V^2 - i\,\gamma_V\,\mu_V \qquad (3)$$

do not depend, to all orders, on gauge parameters. As a consequence, renormalization equations change their structure.

There is also a change of perspective with respect to old one-loop calculations.

- There one considers the cdb OS masses as input parameters independent of complex poles and *derive* the latter in terms of the former;
- Here the situation changes, renormalization equations are written for real, renormalized, parameters and solved in terms of (among other things) experimental complex poles.

When we constuct a propagator from an IPS that contains its complex pole, say s_V, we are left with a consistency relation between theoretical and experimental values of γ_V. If instead, we derive s_W from an IPS that contains s_Z, this is a prediction for the full W complex pole.

Furthermore, consistently with an order-by-order renormalization procedure, renormalized masses in loops and in vertices will be replaced with their real solutions of the renormalized equations, truncated to the requested order.

Alternatively, one could use Dyson resummed (*dressed*) propagators,

$$\bar{\Delta}_V = \frac{\Delta_V}{1 - i\,\Delta_V\,\Sigma_{VV}}, \tag{4}$$

also in loops, say two-loop resummed propagators in tree diagrams, one loop resummed in one-loop diagrams, tree in two-loop diagrams.

renormalization equations

Renormalization with complex poles

has more in it than the content of Eq.(3) and is not confined to prescribe a fixed width for unstable particles; it allows, al least in principle, for an elegant treatment of radiative corrections via effective, complex, couplings.
The corresponding formulation, however, cannot be extended naively beyond the fermion loop approximation; this is due, once again, to gauge parameter independence. We formulate the next renormalization equation in close resemblance with the language of effective couplings and will perform the proper expansions at the end.

We define residual functions according to

$$\Sigma_B(s) = \Sigma_{3Q}(s) + F_B(s), \qquad B = W, Z, \text{ and } H, \tag{5}$$

and discuss solutions of the renormalization equations for different IPS. As a consequence of introducing higher order corrections the coupling constant g will run according to

$$\frac{1}{g^2(s)} = \frac{1}{g^2} - \frac{1}{16\,\pi^2}\,\Pi^{(1)}_{3Q}(s) - \frac{g^2}{(16\,\pi^2)^2}\,\Pi^{(2)}_{3Q}(s). \tag{6}$$

The running of $e^2 = g^2 s^2$ is controlled by

$$e^2(s) = \frac{4\,\pi\,\alpha}{1 - \frac{\alpha}{4\,\pi}\,\Pi_R(s)}, \tag{7}$$

while the running of the weak-mixing angle is defined according to

$$s^2(s) = \frac{e^2(s)}{g^2(s)}. \tag{8}$$

Eqs.(6)–(8) still contain bare parameters and in the following sections we will show how to replace bare quantities in terms of some IPS.

Input Parameter St

We use α, G_F and μ_W and predict, among other things, γ_W which, in turn, can be compared with the measured OS Γ_W. We begin with two equations

$$G\left[M^2 - \frac{g^2}{16\,\pi^2}\,F_W(0)\right] = \frac{g^2}{8}$$

$$\mu_W^2 = M^2 - \frac{g^2}{16\,\pi^2}\,\mathrm{Re}\left[\Sigma_{3Q}(s_W) + F_W(s_W)\right], \qquad (9)$$

where, to second order, we have

$$F_W = F_W^{(1)} + \frac{g^2}{16\,\pi^2}\,F_W^{(2)}, \qquad \Sigma_{3Q} = \Sigma_{3Q}^{(1)} + \frac{g^2}{16\,\pi^2}\,\Sigma_{3Q}^{(2)}. \qquad (10)$$

The (finite) mass counterterm of Eq.(9) is to be contrasted with the conventional mass renormalization where $\mathrm{Re}\,\Sigma_{WW}(M_W^2)$ is used.

We look for a solution with the following form:

$$g^2 = 8\,G\,\mu_W^2 \left[1 + \sum_{n=1} C_g(n) \left(\frac{G}{\pi^2}\right)^n\right],$$
$$M^2 = \mu_W^2 \left[1 + \sum_{n=1} C_M(n) \left(\frac{G}{\pi^2}\right)^n\right]. \qquad (11)$$

The solution is

$$C_g(1) = \frac{1}{2}\left[\mathrm{Re}\,\Sigma_{WW}^{(1)}(s_W) - F_W^{(1)}(0)\right], \qquad C_M(1) = \frac{1}{2}\,\mathrm{Re}\,\Sigma_{WW}^{(1)}(s_W),$$
$$C_g(2) = C_g^2(1) + \frac{1}{4}\,\mu_W^2\left[\mathrm{Re}\,\Sigma_{WW}^{(2)}(s_W) - F_W^{(2)}(0)\right],$$
$$C_M(2) = C_M^2(1) + \frac{1}{4}\,\mathrm{Re}\left[\mu_W^2\,\Sigma_{WW}^{(2)}(s_W) - F_W^{(1)}(0)\,\Sigma_{WW}^{(1)}(s_W)\right]. \qquad (12)$$

In particular we obtain

$$\frac{M^2}{g^2} = \frac{1}{8\,G}\Bigl[1 + \frac{G}{2\,\pi^2}\,F_W^{(1)}(0) + \frac{G^2}{4\,\pi^4}\,\mu_W^2\,F_W^{(2)}(0)\Bigr]. \tag{13}$$

For this input parameter set renormalization of g is obtained after inserting Eq.(12) into Eq.(6),

$$\frac{1}{g^2(s)} = \frac{1}{8\,G\,\mu_W^2} - \frac{1}{16\,\pi^2\,\mu_W^2}\,\delta\,g^{(1)} - \frac{G}{32\,\pi^4}\,\delta\,g^{(2)},$$
$$\delta\,g^{(n)} = \mu_W^2\,\Pi_{3Q}^{(n)}(s) + \mathrm{Re}\,\Sigma_{WW}^{(n)}(s_W) - F_W^{(n)}(0). \tag{14}$$

The renormalization equation for s^2 is

$$g^2\, s^2 = 4\,\pi\,\alpha \left[1 - \frac{g^2\, s^2}{16\,\pi^2}\,\Pi_{QQ}(0)\right]. \tag{15}$$

with a solution given by

$$s^2 = \frac{1}{2}\,A\left[1 + \sum_{n=1} C_s(n)\left(\frac{G}{\pi^2}\right)^n\right], \qquad A = \frac{\pi\,\alpha}{G\,\mu_W^2},$$
$$C_s(1) = -\frac{1}{2}\,\delta s^{(1)}, \qquad C_s(2) = -\frac{1}{4}\left[\delta s^{(2)} - \mu_W^2\,A\,\Pi^{(n)}_{QQ}(0)\,\delta s^{(1)}\right],$$
$$\delta s^{(n)} = \mathrm{Re}\,\Sigma^{(n)}_{WW}(s_W) - F^{(n)}_W(0) + \mu_W^2\,A\,\Pi^{(n)}_{QQ\,;\,\mathrm{ext}}(0). \tag{16}$$

In $\delta s^{(2)}$ we have a residual dependence on s^2 which must be set to its lowest order value,

$$\bar{s}^2 = \frac{1}{2}\,A. \tag{17}$$

For the W propagator we factorize a g^2, insert the solution and write its inverse as

$$\left[g^2\,\Delta_W(s)\right]^{-1} = \frac{s}{g^2(s)} - \frac{1}{8\,G} + \frac{1}{16\,\pi^2}\left[F_W^{(1)}(s) - F_W^{(1)}(0)\right] + \frac{G\,\mu_W^2}{32\,\pi^4}\left[F_W^{(2)}(s) - F_W^{(2)}(0)\right]. \tag{18}$$

Using Eq.(14) the same expression can be rewritten as

$$\left[g^2\,\Delta_W(s)\right]^{-1} = \frac{s}{g^2(s)} - \frac{\mu_W^2}{g^2(s_W)} + \frac{i}{16\,\pi^2}\,R_W^{(1)}(s_W) + \frac{i\,G\,\mu_W^2}{32\,\pi^4}\,R_W^{(2)}(s_W), \tag{19}$$

where the remainders are:

$$R_W^{(n)}(s_W) = \mathrm{Im}\,\Sigma_{WW}^{(n)}(s_W) - \mu_W\,\gamma_W\,\Pi_{3Q\,;\,\mathrm{ext}}^{(n)}(s_W). \tag{20}$$

The complex zero of this expression is the theoretical prediction for the complex pole of the W boson. The real part has been fixed to μ_W^2; the solution for the imaginary part is

$$
\begin{aligned}
\gamma_W^{\rm th} &= \frac{G\,\mu_W}{2\,\pi^2}\,\Bigl(\gamma_1 + \frac{G}{2\,\pi^2}\,\gamma_2\Bigr)\,,\\
\gamma_1 &= \mathrm{Im}\,\Sigma^{(1)}_{WW}(\mu_W^2),\\
\gamma_2 &= \mathrm{Im}\,\Sigma^{(1)}_{WW}(\mu_W^2)\,\Bigl[\mathrm{Re}\,F^{(1)}_W(\mu_W^2) - F^{(1)}_W(0)\Bigr] + \mu_W^2\,\Bigl[\mathrm{Im}\,F^{(2)}_W(\mu_W^2)\\
&\quad - \mathrm{Im}\,F^{(1)}_W(\mu_W^2)\,\mathrm{Re}\,\Sigma^{(1)}_{WW\,;\,p}(\mu_W^2)\Bigr], \qquad (21)
\end{aligned}
$$

where the suffix p denotes derivation.

We have one consistency condition obtained by comparing the derived width of Eq.(21) with the experimental input γ_W. The goodness of the comparison is a precision test of the standard model.
Furthermore, the parameter controlling perturbative (non-resummed) expansion is $G_F\,\mu_W^2$ and we derive,

$$G = G_F \left\{ 1 - \delta_G^{(1)} \frac{G_F\,\mu_W^2}{2\,\pi^2} + \left[2\,(\delta_G^{(1)})^2 - \frac{2}{\mu_W^2}\,\delta_G^{(1)}\,C_g(1) - \delta_G^{(2)} \right] \left(\frac{G_F\,\mu_W^2}{2\,\pi^2} \right)^2 \right\}. \tag{22}$$

In other words, we can go from the G option to the G_F option by replacing in the previous results

$$\begin{aligned} F_W^{(1)}(0) \quad &\to \quad \overline{F}_W^{(1)} = F_W^{(1)}(0) + \mu_W^2\,\delta_G^{(1)}, \\ F_W^{(2)}(0) \quad &\to \quad \overline{F}_W^{(2)} = F_W^{(2)}(0) + \mu_W^2\,\delta_G^{(2)} + \delta_G^{(1)} \Big[\mu_W^2\,\delta_G^{(1)} + \mathrm{Re}\,F_W^{(1)}(s_W) \\ &\qquad + \mathrm{Re}\,\Sigma^{(1)}_{3Q\,;\,\mathrm{ext}}(s_W) - 2\,\overline{F}_W^{(1)} \Big], \end{aligned} \tag{23}$$

and $G \to G_F$.

All function appearing in the results depend also on internal masses, M etc. Therefore we always use, for and arbitrary f

$$f^{(1)}(s\,;\,M^2\,,\,\dots) = f^{(1)}(s\,;\,\mu_W^2\,,\,\dots) + \frac{G\,\mu_W^2}{2\,\pi^2}\,\mathrm{Re}\,\Sigma_{WW}^{(1)}(s_W\,;\,\mu_W^2\,,\,\dots) \times \frac{\partial}{\partial M^2}\,f^{(1)}(s\,;\,M^2\,,\,\dots)\Big|_{M^2=\mu_W^2}. \qquad (24)$$

A last subtlety in Eq.(18) is represented by the residual s^2 dependence of the W self-energy and of δ_G; we use

$$s^2 = \bar{s}^2\left[1 - \frac{G_F}{2\,\pi^2}\,\delta s^{(1)}\right] \quad \text{in} \quad F_W^{(1)},\ \delta_G^{(1)}$$

$$s^2 = \bar{s}^2 \quad \text{in} \quad F_W^{(2)},\ \delta_G^{(2)}. \qquad (25)$$

Self-energies

Consider a two-point function to all orders in perturbation theory,

$$\Sigma_{VV}(s,\xi) = \sum_{n=2}^{\infty} \Sigma^{(n)}_{VV}(s,\xi)\, g^{2n}. \tag{26}$$

All one-loop self-energies corresponding to physical particles are gauge-parameter independent when put on their, bare or renormalized, mass-shell and coincide with the corresponding $\xi = 1$ expression, i.e.

$$\Sigma^{(1)}_{VV}(s,\xi) = \Sigma^{(1)}_{VV\,;\,I}(s) + (s - M_V^2)\, \Phi_{VV}(s,\xi). \tag{27}$$

Theorem

from arguments based on Nielsen identities we know that

$$\frac{\partial}{\partial \xi}\, \Sigma_{VV}(s_P, \xi) = 0, \tag{28}$$

where

$$s_P - M_V^2 + \Sigma_{VV}(s_P) = 0. \tag{29}$$

We write

$$\Sigma^{(n)}_{VV}(s, \xi) = \Sigma^{(n)}_{VV\,;\,I}(s) + \Sigma^{(n)}_{VV\,;\,\xi}(s, \xi), \tag{30}$$

use

$$
\begin{aligned}
M_V^2 &= s_P + g^2\,\Sigma^{(1)}_{VV}(s_P) \\
&+ \boldsymbol{g}^4\left[\Sigma^{(1)}_{VV\,;\,I}(s_P)\,\Sigma^{(1)}_{VV\,;\,\xi}(s_P,\xi) - \Sigma^{(2)}_{VV\,;\,I}(s_P) - \Sigma^{(2)}_{VV\,;\,\xi}(s_P,\xi)\right] \\
&+ \mathcal{O}\left(g^6\right), \qquad (31)
\end{aligned}
$$

to derive, as a consequence of Eq.(28),

$$
\Sigma^{(n)}_{VV\,;\,\xi}(s_P,\xi) = \Sigma^{(n-1)}_{VV\,;\,I}(s_P)\,\Phi_{VV}(s_P,\xi), \qquad (32)
$$

etc. As a consequence we obtain

$$
\Sigma_{VV}(s_P) = \sum_{n=2}^{\infty} \Sigma^{(n)}_{VV\,;\,I}(s_P)\,g^{2n}. \qquad (33)
$$

Dressed propagators

Suppose that we have a simple model with an interaction Lagrangian

$$L = \frac{g}{2}\,\Phi(x)\,\phi^2(x). \tag{34}$$

The mass M of the Φ-field and m of the ϕ-field be such that the Φ-field be unstable. Let Δ_i be the lowest order propagators and $\overline{\Delta}_i$ the one-loop dressed propagators, i.e.

$$\overline{\Delta}_\Phi = \frac{\Delta_\Phi}{1 - \Delta_\Phi\,\Sigma_{\Phi\Phi}}, \qquad \overline{\Delta}_\phi = \frac{\Delta_\phi}{1 - \Delta_\phi\,\Sigma_{\phi\phi}}, \tag{35}$$

etc. In fixed order perturbation theory, the ϕ self-energy is given in Fig. 1.

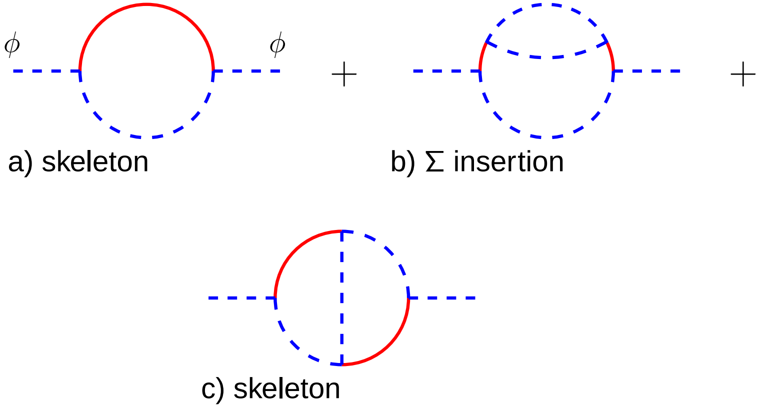

Figure: The ϕ self-energy with skeleton expansion, diagrams a) and c), and insertion of a sub-loop $\Sigma_{\Phi\Phi}$, diagram b).

ϕ imaginary part

Note that the imaginary part of $\Sigma_{\phi\phi}$ is non-zero only for

$$-p^2 > 9\,m^2, \qquad \text{(the three-particle cut of diagram b) in Fig. 1)}, \quad \text{if} \quad m \ll M. \tag{36}$$

When we use dressed propagators only diagrams a) and c) are retained in Fig. 1 (for two-loop accuracy) but in a) we use $\overline{\Delta}_\Phi$ with one-loop accuracy:

$$\Sigma^{(a)}_{\phi\phi} = \int \frac{d^n q_2}{\Bigl(q_2^2 + M^2 - \frac{g^2}{16\,\pi^2}\,\Sigma_{\Phi\Phi}(q_2^2)\Bigr)\,\Bigl((q_2+p)^2 + m^2\Bigr)},$$

$$\Sigma_{\Phi\Phi}(q_2^2) = B_0(q_2^2\,;\,m\,,\,m), \tag{37}$$

where we assume $p^2 < 0$.

Since the complex Φ pole is defined by

$$M^2 - s_M - \frac{g^2}{16\,\pi^2}\,\Sigma_{\Phi\Phi}(-s_M) = 0, \tag{38}$$

we write the inverse (dressed) propagator as

$$\left[1 - \frac{g^2}{16\,\pi^2}\,\frac{\Sigma_{\Phi\Phi}(q_2^2) - \Sigma_{\Phi\Phi}(-s_M)}{q_2^2 + s_M}\right]\left(q_2^2 + s_M\right), \tag{39}$$

expand in g as if we were in a gauge theory with problems of gauge parameter dependence and obtain

$$\Sigma^{(a)}_{\phi\phi} = g^2 \int \frac{d^n q}{(q^2 + s_M)\,\left((q+p)^2 + m^2\right)} \times \left[1 + \frac{g^2}{16\,\pi^2}\,\frac{\Sigma_{\Phi\Phi}(q^2) - \Sigma_{\Phi\Phi}(-s_M)}{q^2 + s_M}\right] \tag{40}$$

$$= \frac{i}{2} g^2 \pi^2 B_0 \left(1, 1\,;\, p^2\,;\, s_M\,,\, m^2\right) + i\,\frac{g^4}{16}\, S^E \left(p^2\,;\, m^2, m^2, s_M, m^2, s_M\right)$$
$$+ i\,\frac{g^4}{16} B_0 \left(2, 1\,;\, p^2\,;\, s_M\,,\, m^2\right) \left[\Delta_{UV} - \ln \frac{m^2}{\mu^2} + 2 - \beta \ln \frac{\beta + 1}{\beta - 1}\right], \qquad (41)$$

where

$$\beta^2 = 1 - 4\,\frac{m^2}{s_M}. \qquad (42)$$

More on dressed propagators

Note that there is an interply between using dressed propagators for all internal lines of a diagram and combinatorial factors and number of diagrams with and without dressed propagators.

Note that the poles in the q^0 complex plane remain in the same quadrants as in the Feynman prescription and Wick rotation can be carried out, as usual.

Evaluation of diagrams with complex masses does not pose a serious problem; in the analytical approach one should, hovever, pay the due attention to splitting of logarithms.

Consider a B_0 function,

$$B_0(p^2\,;\,M_1\,,\,M_2) = \Delta_{UV} - \int_0^1 dx\,\frac{\chi(x)}{\mu^2},$$
$$\chi(x) = -p^2\,x^2 + (p^2 + M_2^2 - M_1^2)\,x + M_1^2, \qquad (43)$$

where one usually writes

$$\ln\frac{\chi(x)}{\mu^2} = \ln(-\frac{p^2}{\mu^2} - i\,\delta) + \ln(x - x_-) + \ln(x - x_+). \qquad (44)$$

Since $\mathrm{Im}\,\chi(x)$ does not change sign with in $[0,1]$ the correct recipe for $M^2 = m^2 - i\,m\,\gamma$ is

$$\ln\frac{\chi(x)}{\mu^2} = \ln|\,p^2\,| + \ln(x - x_-) + \theta(-p^2)\,\Big[\ln(x - x_+) + \eta(-x_-\,,\,-x_+)\Big]$$
$$+\,\theta(p^2)\,\Big[\ln(x_+ - x) + \eta(-x_-\,,\,x_+)\Big]. \qquad (45)$$

In the numerical treatent, instead, no splitting is performed and no special care is needed.

A t -channel propagator deserves some additional comment: one should not confuse the position of the pole which is always at $\mu^2 - i\,\mu\,\gamma$ with the fact that a dressed propagator function is real in the t -channel.

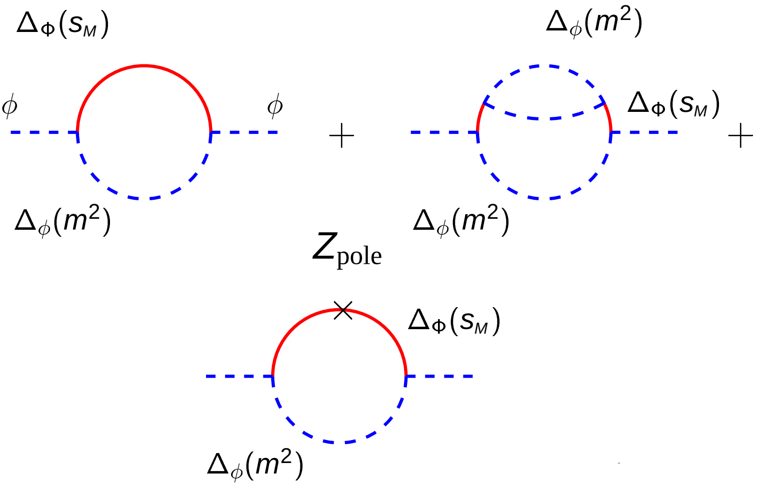

Figure: Diagram b) of Fig. 1 with one-loop dressed Φ propagators is equivalent, up to $\mathcal{O}\left(g^4\right)$, to the sum of three diagrams with lowest order

Theorem

Therefore, using one-loop diagrams with one-loop dressed Φ propagators is equivalent, to $\mathcal{O}\left(g^4\right)$, to use the sum of the three diagrams of Fig. 2 where Φ propagators are at lowest order but with complex mass s_M and where the vertex $Z_{\rm pole}$ is defined by

$$Z_{\rm pole} = \frac{g^2}{16\,\pi^2}\,B_0\left(-s_M\,;\,m\,,\,m\right). \qquad (46)$$

Unitarity and gauge invariance

When dealing with the calculation of physical processes, with one and two loops, that include unstable particles, one should construct a scheme that

- **a)** respects the unitarity of the $S-$matrix;
- **b)** gives results that are gauge-parameter independent;
- **c)** satisfies the whole set of WST identities.

Resummation will be part of any scheme, a fact that indroduces additional subtleties if $a - c)$ are to be respected. Consider in more details the definition of dressed propagator: we consider a skeleton expansion of the self-energy Σ with progators that are resummed up to $\mathcal{O}(n)$ and define

Recursion relation

$$\Delta^{(n+1)}(p^2) = \Delta^{(0)}(p^2)\left[\Delta^{(0)}(p^2) - \Sigma^{(n+1)}\left(p^2\,,\,\Delta^{(n)}(p^2)\right)\right]^{-1}, \qquad (47)$$

where

$$\Delta^{(0)}(p^2) = \frac{1}{p^2 + m^2}. \qquad (48)$$

If it exists, we define a dressed propagator as

$$\overline{\Delta}(p^2) = \lim_{n\to\infty} \Sigma^{(n)}(p^2),$$
$$\overline{\Delta}(p^2) = \Delta^{(0)}(p^2)\left[\Delta^{(0)}(p^2) - \Sigma\left(p^2, \overline{\Delta}(p^2)\right)\right], \tag{49}$$

which is not equivalent to a *rainbow* approximation and coincides with the Schwinger - Dyson solution for the propagator.

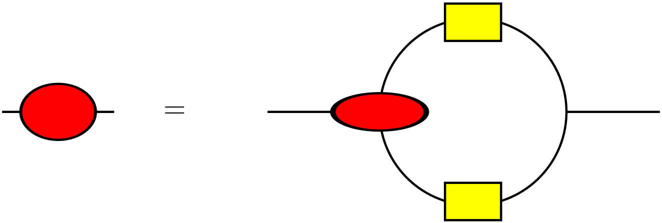

Figure: Schwinger - Dyson equation for the self-energy

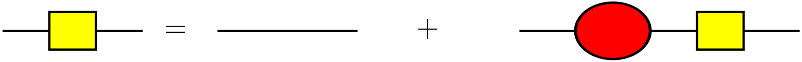

Figure: Dressed propagator

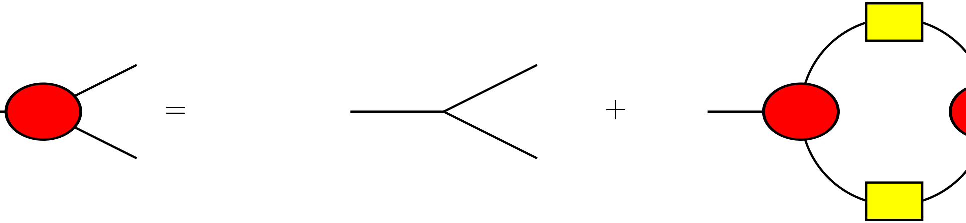

Figure: Dressed vertex

Cutting rules

– Cutting rules

We assume that Eq.(49) has a solution that obeys Källen - Lehmann representation,

$$\begin{aligned}\mathrm{Re}\,\overline{\Delta}(p^2) &= \mathrm{Im}\,\Sigma(p^2)\left[\Big(p^2+m^2-\mathrm{Re}\,\Sigma(p^2)\Big)^2+\Big(\mathrm{Im}\,\Sigma(p^2)\Big)^2\right]^{-1}\\ &= \pi\,\rho(-p^2).\end{aligned} \tag{50}$$

A dressed propagator, being the result of an infinite number of iterations,

$$\mathrm{Re}\,\overline{\Delta}(p^2) = \int_0^\infty ds\,\frac{\rho(s)}{p^2+s-i\,\delta}, \tag{51}$$

is a formal object which is difficult to handle for all practical pourposes.

Unitarity

Theorem

Unitarity follows if

- *we add all possible ways in which a diagram with given topology can be cut in two;*
- *the shaded line separates S from $S^\dagger$. F*

For a stable particle the cut line, proportional to $\overline{\Delta}^{+}$, contains a pole term

$$\overline{\Delta}^{+} = 2\, i\, \pi\, \theta(p_0)\, \delta(p^2 + m^2), \tag{52}$$

whereas there is no such contribution for an unstable particle. We express $\mathrm{Im}\,\Sigma$ in terms of cut self-energy diagrams and repeat the procedure ad libidum and prove that cut unstable lines are left with no contribution, i.e. unstable particles contribute to the unitarity of the $S-$matrix via their stable decay products.

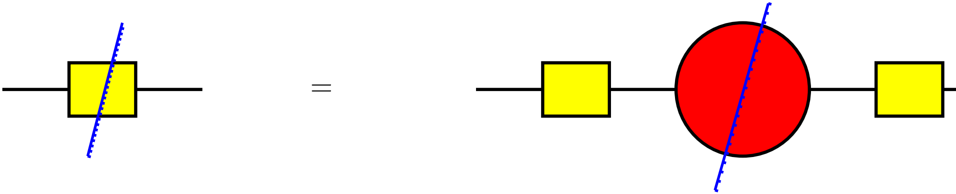

Figure: Cutting equation for dressed propagator.

Unitarity

The consistent use of dressed propagators gives a general scheme where unitarity is satisfied which is essentially a statement on the imaginary parts of the diagrams.
Approximated, or truncated, schemes (e.g. resummation of one-loop self energies, or *rainbow* approximation without further resummation of the vertex functions) usually lead to gauge dependent results.

WST identities

WST identities

We assume that WST identities hold at any fixed order in perturbation theory for diagrams that contain bare propagators and vertices; they again form dressed propagators and vertices when summed.
We expect that an arbitrary truncation that preferentially resums specific topologies will lead to violations of WST identities. Of course such violations are absent if exact calculations were possible.

Approximations

Gauge parameter dependence

A truncated approximation, e.g. simple resummation of two-point functions, necessarily leads to gauge dependent results. A convenient tool is to analyze the gauge invariance of the effective action where one can show that on-shell gauge dependence always occurs at higher order than the order of truncation.

Introducing complex poles

Complex pole

A property of the $S-$matrix is the complex pole

$$\overline{\Delta}^{-1}(p^2 = -s_P) = 0, \tag{53}$$

which is gauge parameter independent as shown by a study of Nielsen identities. An approximate solution of the unitarity constraint is as follows:

$$2\,\mathrm{Im}\,T_{ii} = \sum_n \left|T_{ni}\right|^2, \qquad \sum_n \left|T_{ni}\right|^2 = \left|D(p^2)\right|^2 \sum_n \int dPS_n \left|M_{1\to n}\right|^2, \tag{54}$$

where, $S = 1 + i\,T$ and where $D(p^2)$ is the unknown form of the propagator.

Making the approximation,

$$\sum_n \int dPS_n \left| M^{1\to n} \right|^2 \equiv m\,\Gamma_{\mathrm{tot}}, \tag{55}$$

we derive

$$\mathrm{Im}\, D(p^2) = m\,\Gamma_{\mathrm{tot}}. \tag{56}$$

A simple but, once again, approximate solution is

$$D(p^2) = \left(p^2 + m^2 - i\,m\Gamma_{\mathrm{tot}}\right)^{-1}, \tag{57}$$

which is valid far from the mass shell and where the invariant mass at which the decay is evaluated is identified with m^2.

We can improve upon this solution by writing instead

$$D(p^2) = \left(p^2 - s_P\right)^{-1}, \tag{58}$$

which is equivalent to resum only the self-energy (up to some fixed order), and to use $m^2 = s_P + \Sigma(s_P)$

$$\begin{aligned} D(p^2) &= -\left[s - s_P - \Sigma(s) + \Sigma(s_P)\right]^{-1} \\ &= -\left(p^2 - s_P\right)^{-1} + \text{h.o.}, \end{aligned} \tag{59}$$

where higher order terms are neglected. Another way to see that Eq.(58) is an improvement of Eq.(57) is to observe that

$$p^2 + m^2 + i\,\frac{\Gamma_{\rm tot}}{m}\,p^2 = \left(1 + i\,\frac{\Gamma_{\rm tot}}{m}\right)\,(p^2 + s_P) + \text{h.o.} \approx p^2 + s_P. \tag{60}$$

A propagator with the correct analytical structure, $p^2 - s_P$, will be represented with a thick dot. The approximation of Eq.(58) allows us to write the cutting equation of Fig. 7.

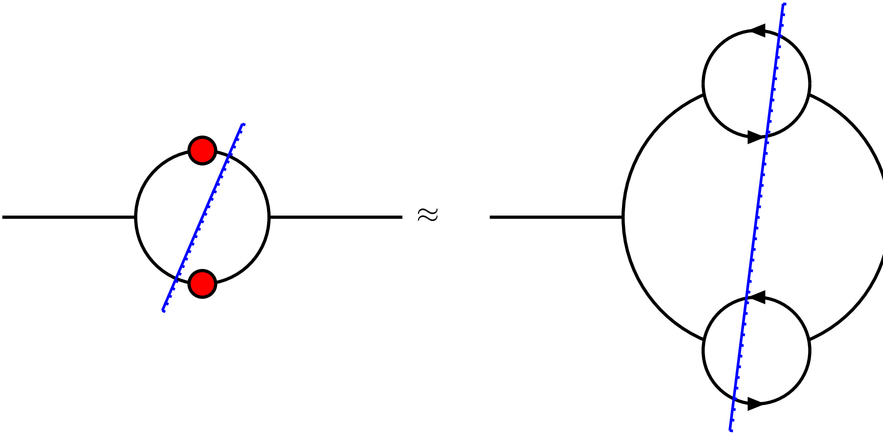

Figure: Cutting equation for a contribution to the Z self-energy using W propagators of Eq.(58).

truncated propagators

One can see that using truncated propagators with complex poles (at the one-loop level of accuracy) is still respecting unitarity of the $S-$matrix within the approximation of Eq.(55) if the complex pole is computed from fermions only; however, this scheme violates gauge invariance since vertices are not included.

There is a solution to this problem, namely replacing everywhere the (real) masses with the complex poles, couplings included; this is known in the literature as complex mass scheme.

CM scheme

The complex mass scheme

Since WST identities are algebraic relations satisfied separately by the real and the imaginary part one starts from WST identities with real masses, satisfied at any given order, replaces everywhere $m^2 \to s_P$ without violating the invariance.

In turns, this scheme violates unitarity, i.e. we cannot identify the two sides of any cut diagram with T and $T^\dagger$ respectively.

To summarize, the analytical structure of the $S-$matrix is correctly reproduced when we use propagator factors $p^2 - s_P$ but unitarity of S requires more, a dressed propagator

$p^2 + s_P$	$p^2 + s_P - \Sigma(p^2) + \Sigma(-s_P)$
analyticity	unitarity

Another drawback of the scheme is that all propagators for unstable particles will have the same functional form both in the time-like and in the space-like region while, for a dressed propagator the presence of a pole on the second Riemann sheet does not change the real character of the function if we are in a $t-$channel.

In some sense the scheme becomes more appealing when we go beyond one loop. WST identities are satisfied with bare (i.e. non-dressed) propagators and vertices up to two-loops; we may assume that they are verified order by order to all orders,

$$W^{(1)}(\{\Gamma\}) = W^{(2)}(\{\Gamma\}) = \cdots = 0, \tag{61}$$

where $\{\Gamma\}$ is a set of (off-shell) Green function and cdr $W = 0$ is the WST identity.

Next we write the same set of WST identities but using a skeleton expansion with one-loop dressed propagators. Calling the scheme *complex mass scheme* is somehow misleading; to the requested order we replace everywhere m^2 with $s_P + \Sigma(s_P)$ which is real by construction. If only one-loop is needed then $m^2 \to s_P$ everywhere (therefore justifying the name *complex mass*) and

$$W^{(1)}(\{\Gamma\})\Big|_{m^2=s_P} = 0, \tag{62}$$

is trivially true. Also,

$$W^{(2)}(\{\Gamma\})\Big|_{m^2=s_P} = 0. \tag{63}$$

At the two-loop level we have two-loop diagrams with no self-energy insertions where $m^2 = s_P$ and one-loop diagrams where $m^2 = s_P + \Sigma(s_P)$ and the factor

$$\frac{\Sigma(p^2) - \Sigma(s_P)}{p^2 + s_P}, \tag{64}$$

expanded to first order with $\Sigma = \Sigma^{(1)}$.

Furthermore, in vertices we use $m^2 = s_P$ in two-loop diagrams and $m^2 = s_P + \Sigma(s_P)$ in one-loop diagrams. Expanding the factor of Eq.(64) generates two-loop diagrams with insertion of one-loop self-energies plus one-loop diagrams with one more propagator and a vertex proportional to $\Sigma(s_P)$; furthermore one-loop diagrams with m^2 dependent vertices get multiplied by $\Sigma(s_P)$; it follows that

Theorem

$$
\begin{aligned}
W^{(1+2)}\left(\{\Gamma\}_{\text{skeleton}}\right)\Big|_{m^2 = s_P + \Sigma(s_P)} &= W^{(1+2)}\left(\{\Gamma\}\right)\Big|_{m^2 = s_P} \\
&+ \Sigma(s_P)\,\frac{d}{dm^2}\,W^{(1)}\left(\{\Gamma\}\right)\Big|_{m^2 = s_P} \\
&= 0, \qquad (65)
\end{aligned}
$$

as a consequence of Eqs.(62)–(63).