Lecture 1: QCD as a Yang-Mills theory

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Outline

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Some facts about quarks

- All hadrons $(p, n, \pi, K, \ldots)$ are constituted of quarks
- Quarks are pointlike spin 1/2 particles
- Quark have fractional electric charges and appear in 6 different flavours

f	m (GeV/c^2)	Q (e)
u	$0.0015 - 0.003$	2/3
d	$0.003 - 0.007$	-1/3
c	1.25 ± 0.09	2/3
s	0.095 ± 0.025	-1/3
t	174.2 ± 3.3	2/3
b	4.2 ± 0.07	-1/3

- Quarks have an additional quantum number, the colour. Each quark can have three colours (R,G,B). Colour symmetry $SU(3)$ is an exact symmetry of the quark Lagrangian
- Gauging colour symmetry the quark interact via the exchange of other coloured objects, the gluons. Since $SU(3)$ is a non-abelian group, the theory underlying gluon exchange is a non-abelian gauge (a.k.a. Yang-Mills) theory, whose name is Quantum Chromo-Dynamics

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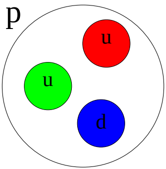

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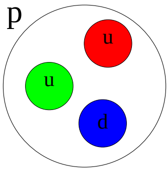

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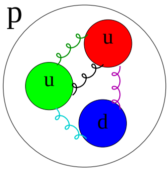

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The matter fields: the quarks

- Each quark has spin 1/2 and a colour i, the number of colours is N_c

$$\mathcal{L} = \bar{\psi}_i(x)\,(i\delta_{ij}\gamma^\mu\partial_\mu - m)\,\psi_j(x) = \bar{\psi}(x)\,(i\not{\partial} - m)\,\psi(x)$$

The matter field ψ_i is in the fundamental representation **3** of $SU(N_c)$

- $\mathcal{L}$ is invariant under the global $SU(N_c)$ transformation

$$\psi'(x) = U\psi(x) \qquad \bar{\psi}'(x) = \bar{\psi}(x)\,U^\dagger$$

- Fundamental representation of $SU(N_c)$ has $N_c^2 - 1$ generators t^a, $N_c \times N_c$ hermitian traceless matrices

$$U = e^{i\theta_a t^a} \;\Rightarrow\; \psi_i' \simeq \psi_i + \delta\psi_i \qquad \delta\psi_i = i\theta_a\, t^a_{ij}\,\psi_j$$

- The antiquarks $\bar{\psi}_i$ transform according to the conjugate representation $\bar{\mathbf{3}}$, whose generators are $\bar{t}^a$

$$\delta\bar{\psi}_i' = i\theta_a\,\bar{t}^a_{ij}\,\bar{\psi}_j \qquad \bar{t}^a_{ij} = -t^a_{ji} \;\Leftrightarrow\; \bar{t}^a = (-t^a)^T$$

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The gauge fields: the gluons

- The quark Lagrangian is not invariant under local $SU(N_c)$ (gauge)

$$\psi'(x) = U(x)\,\psi(x) \qquad \bar{\psi}'(x) = \bar{\psi}(x)\,U^\dagger(x)$$

- Invariance under gauge transformation $\Leftrightarrow$ covariant derivative D_μ

$$\partial_\mu \to D_\mu \quad \text{such that} \quad D'_\mu\, U(x)\,\psi(x) = U(x)\,D_\mu\,\psi(x)$$

- Vector field A_μ (gluon) in the adjoint representation $\mathbf{8}$ (connection)

$$D_\mu = \partial_\mu + igA_\mu \qquad A'_\mu = UA_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger = UA_\mu U^\dagger - \frac{i}{g}U(\partial_\mu U^\dagger)$$

- Dynamics for gluon field $A_\mu \Leftrightarrow$ field strength $F_{\mu\nu}$ (tensor form)

$$F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \;\Rightarrow\; F'_{\mu\nu} = UF_{\mu\nu}U^\dagger$$

Field strength not gauge-invariant $\Rightarrow$ Self interacting gluons

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The gluons in components

- There are $N_c^2 - 1$ gluons A_μ^a

$$A_\mu = A_\mu^a \, t^a \;\; \Rightarrow \;\; D_\mu = \partial_\mu + ig \, A_\mu^a \, t^a$$

- Using the definition of the structure constant f_{abc}

$$[t_a, t_b] = i f_{abc} t^c \;\; \Rightarrow \;\; f_{abc} = -f_{bac} = f_{cab}$$

we find the components of $F_{\mu\nu}$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c$$

- Generators of the adjoint representation

$$F'_{\mu\nu} = U F_{\mu\nu} U^\dagger \;\Rightarrow\; \delta F_{\mu\nu}^a = i\theta^b (T^b)_{ac} F_{\mu\nu}^c \qquad (T^a)_{bc} = -i f^{abc} = i f^{bac}$$

- Transformation rule for A_μ^a

$$A'_\mu = U A_\mu U^\dagger + \frac{i}{g}(\partial_\mu U) U^\dagger \Rightarrow \delta A_\mu^a = -\frac{1}{g} D_\mu^{ab} \theta_b \qquad D_\mu^{ab} = \partial_\mu \delta^{ab} + ig T_{ab}^c A_\mu^c$$

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Classical Yang-Mills Lagrangian

- Gauge invariant Yang-Mills Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}\,(i\not{D} - m)\,\psi$$

- Using $\mathrm{Tr}(t^a t^b) = T_R\delta^{ab}$ and setting $T_R = 1/2$ we have $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_F$

$$\mathcal{L}_G = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)$$
$$+\frac{g}{2}f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^\mu_b A^\nu_c \quad \text{(3-gluon coupling)}$$
$$-\frac{g^2}{4}f^{abe}f^{cde}A^\mu_a A^\nu_b A^c_\mu A^d_\nu \quad \text{(4-gluon coupling)}$$
$$\mathcal{L}_F = \bar{\psi}\,(i\not{\partial} - m)\,\psi - g\bar{\psi}\not{A}^a t^a\psi$$

- A term $A_\mu A^\mu$ is forbidden by gauge invariance $\Rightarrow$ the gluons are massless

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\end{aligned}$$

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Fixing the gauge

- Classical equation of motion for A_a^μ

$$K^{ab}_{\mu\nu}\,A^\nu_b = \delta^{ab}(-\Box g_{\mu\nu} + \partial_\mu\partial_\nu)A^\nu_b = J^a_\mu$$

is not solvable because the operator $K^{ab}_{\mu\nu}$ is not invertible $\Rightarrow$ fix the gauge

- At classical level the gauge fixing condition $G[A[\theta(x)]] = 0$ determines $A^a_\mu[\theta(x)]$ uniquely
- At quantum level, integrating away all configurations obtained from $A^a_\mu[\theta(x)]$ via a gauge transformation requires the addition of gauge-fixing and Fadeev-Popov ghost Lagrangian

$$\mathcal{L}_{GF} = -\frac{1}{2\alpha}(G[A])^2 \qquad \mathcal{L}_{FP} = -\bar{c}^a(x)\frac{\delta G[A^a[\theta(x)]]}{\delta\theta^b(y)}c^b(y)$$

The ghosts $c^a(x)$ are scalar fields with Fermi statistics

- Example: linear gauge-fixing condition

$$O^\mu A^a_\mu[\theta(x)] = O^\mu(A^a_\mu - \frac{1}{g}D^{ab}_\mu\theta_b) \;\Rightarrow\; \frac{\delta G[A^a[\theta(x)]]}{\delta\theta^b(y)} = O^\mu D^{ab}_\mu\delta(x-y)$$

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- Classical equation of motion for A_a^μ

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  is not solvable because the operator $K_{\mu\nu}^{ab}$ is not invertible $\Rightarrow$ fix the gauge

- At classical level the gauge fixing condition $G[A[\theta(x)]] = 0$ determines $A_\mu^a[\theta(x)]$ uniquely

- At quantum level, integrating away all configurations obtained from $A_\mu^a[\theta(x)]$ via a gauge transformation requires the addition of gauge-fixing and Fadeev-Popov ghost Lagrangian

$$\mathcal{L}_{GF} = -\frac{1}{2\alpha}(G[A])^2 \qquad \mathcal{L}_{FP} = -\bar{c}^a(x)\frac{\delta G[A^a[\theta(x)]]}{\delta\theta^b(y)} c^b(y)$$

  The ghosts $c^a(x)$ are scalar fields with Fermi statistics

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Covariant gauge

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$$\mathcal{L}_{GF} = -\frac{1}{2\alpha}(\partial_\mu A_a^\mu)^2 \quad \Rightarrow \quad \Delta_{\mu\nu}^{ab}(k) = \frac{i}{k^2} d_{\mu\nu}$$

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- Ghost Lagrangian:

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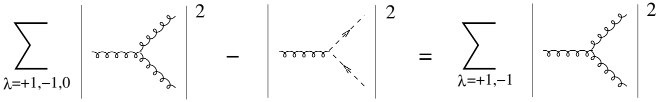

Axial gauge

- Gauge-fixing condition: $n_\mu A_a^\mu = 0$

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Ghosts do not couple to gluons ⇒ Only physical polarisations $k\cdot\epsilon = n\cdot\epsilon = 0$

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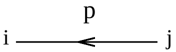
Outline

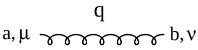
Feynman rules: propagators

Quark propagator: $\mathcal{L} = \bar{\psi}\,(i\not{\partial} - m)\,\psi$



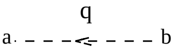
$$\frac{i}{\not{p} - m + i\epsilon}\delta_{ij}$$

Gluon propagator: $\mathcal{L} = -\frac{1}{4}\,(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) - \frac{1}{2\alpha}(G[A])^2$



$$\frac{i}{q^2 + i\epsilon}d_{\mu\nu}(q)\delta_{ab}$$

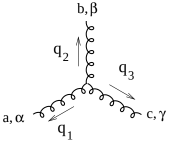
Ghost propagator (covariant gauge): $\mathcal{L} = \partial^\mu \bar{c}^a \partial_\mu c_a$



$$\frac{i}{q^2 + i\epsilon}\delta_{ab}$$

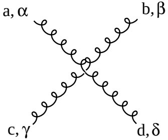
Feynman rules: interaction vertexes

3-gluon: $\mathcal{L} = \frac{g}{2} f^{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A_b^\mu A_c^\nu$



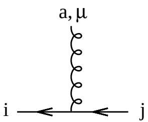
$$-ig\,(if_{abc})[g_{\alpha\beta}(q_1-q_2)_\gamma + g_{\beta\gamma}(q_2-q_3)_\alpha + g_{\gamma\alpha}(q_3-q_1)_\beta]$$

4-gluon: $\mathcal{L} = -\frac{g^2}{4} f^{abe} f^{cde} A_a^\mu A_b^\nu A_\mu^c A_\nu^d$



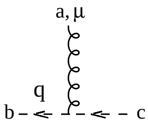
$$\begin{aligned} -ig^2 \Big[& f^{abe} f^{cde} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) \\ & + f^{ace} f^{bde} (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\beta\gamma}) \\ & + f^{ade} f^{cbe} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\beta} g_{\gamma\delta}) \Big] \end{aligned}$$

Quark-gluon: $\mathcal{L} = -g\bar{\psi} \not{A}^a\, t^a\, \psi$



$$-\,ig\, t^a_{ij}\, \gamma_\mu$$

Ghost-gluon: $\mathcal{L} = -g f_{abc} \partial^\mu \bar{c}^a A_\mu^b c^c$

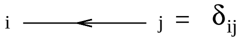


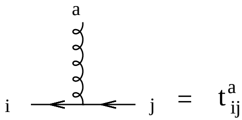
$$-\,ig\, q_\mu (-i f_{abc})$$

Outline

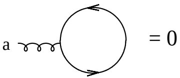
Fundamental relations

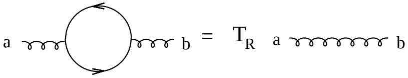
- Fundamental representation $\mathbf{3}$



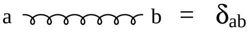


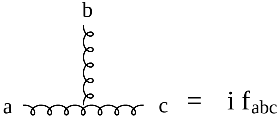
- Trace identities: $\mathrm{Tr}(t^a) = 0$ and $\mathrm{Tr}(t^a t^b) = T_R \delta^{ab}$





- Adjoint representation $\mathbf{8}$

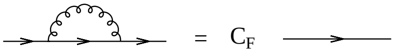




Casimir factors

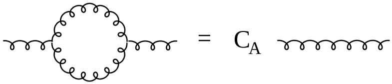
- Fundamental representation $\mathbf{3}$:

$$\sum_a t^a_{ik} t^a_{kj} = C_F \delta_{ij} \quad C_F = \frac{N_c^2 - 1}{2N_c}$$



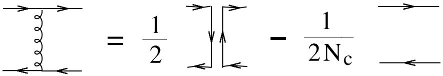
- Adjoint representation $\mathbf{8}$:

$$\sum_{cd} f^{acd} f^{bcd} = C_A \delta^{ab} \quad C_A = N_c$$

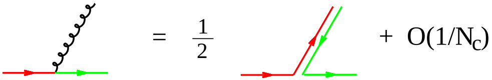


- Fierz identity:

$$(t^a)^i_k \, (t^a)^l_j = \frac{1}{2} \delta^i_j \, \delta^l_k - \frac{1}{2N_c} \delta^1_k \, \delta^l_j$$

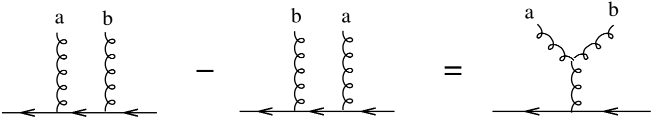


- Gluons as carriers of colour in the large-N_c limit



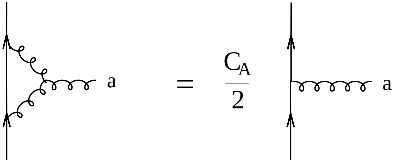
Useful identities

- Commutator and structure constants: $[t^a, t^b] = i f_{abc} t^c$



- 1-loop quark-gluon vertex: note pedestrian rule

$$i f_{abc} t^c t^b = -i f_{abc} t^b t^c = \frac{C_A}{2} t^a$$



$$t^b t^a t^b = \left(C_F - \frac{C_A}{2}\right) t^a = -\frac{1}{2N_c} t^a$$