# Parma International School of Theoretical Physics

## September 3 - September 8, 2012

## Parma, Italy

***Scattering Amplitudes in QCD, Supersymmetric Gauge Theories and Supergravity***

# One-loop calculations in QCD and N=4 supersymmetric gauge theories

Zoltan Kunszt, ETH, Zurich

## Reviews:

*R. Keith Ellis, Zoltan Kunszt, Kirilll Melnikov, Giulia Zanderighi,*
**One-loop calculations in quantum field theory: from Feynman diagrams to unitarity cuts.**
**Phys.Rep. (in print), arXiv:1105.4319** [hep-ph]

*Harald Ita*, **Susy Theories in QCD: Numerical Approaches, J.Phys.A A44 (2011) 454005**

## Books:

*R. Keith Ellis, James W. Stirling, Bryan Webber,* **QCD and Collider Physics,** Cambridge University Press, 2003

*Taizo Muta,* **Foundation of Quantum Chromodynamics,** World Scientific, 1997

*M. Creutz,* **Lattice**

## Papers:

*Z. Bern, L. Dixon, D. Dunbar and D. Kosower*,
**Fusing gauge theory tree amplitudes into loop amplitudes, Nucl.Phys. B435 (1995) 59-101**

# OUTLINE

Lecture 1: QCD basics

Lecture 2: One loop tensor integrals and their reduction

Lecture 3: Unitarity method and amplitudes

Lecture 4: Analytic and numerical computations

Lecture 5: Outlook

# QCD basics

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$$(\mathbf{D}_\mu)_{\mathbf{ij}} = \partial_\mu \delta_{\mathbf{ij}} - \mathbf{igT}^{\mathbf{a}}_{\mathbf{ij}}\mathbf{G}^{\mathbf{a}}_\mu, \quad \mathbf{i,j = 1,2,3}\,; \quad \mathbf{a = 1,\ldots,8}$$

$$(\mathbf{D}_\mu)_{\mathbf{ab}} = \partial_\mu \delta_{\mathbf{ab}} - \mathbf{igF}^{\mathbf{c}}_{\mathbf{ab}}\mathbf{G}^{\mathbf{c}}_\mu, \quad \mathbf{a,b,c = 1,\ldots,8}$$

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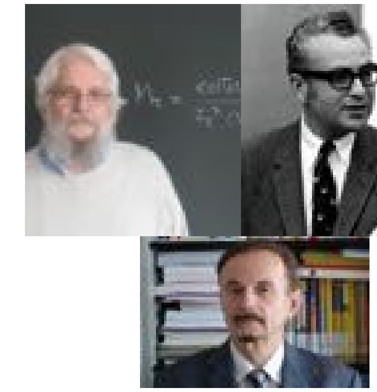

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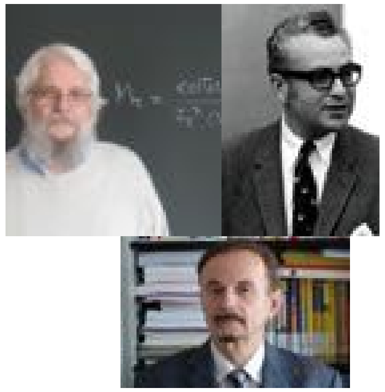

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$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G_a^{\mu\nu} + \sum_{i,j,f} \bar{q}_i^{(f)} (i\hat{D} - m)_{ij} q_j^{(f)} - \frac{1}{2\lambda} (\partial^\mu G^a_\mu)^2 + \partial_\mu c^{a\,\dagger} D^\mu_{abj} c^b \,,$$

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} - i g T^a_{ij} G^a_\mu, \quad i,j = 1,2,3\,; \quad a = 1,\ldots,8$$

$$(D_\mu)_{ab} = \partial_\mu \delta_{ab} - i g F^c_{ab} G^c_\mu, \quad a,b,c = 1,\ldots,8$$

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu \,,$$

Quarks come in six different flavors (u,d,s,c,b,t) and three colors (red, blue, green).

$$\left.\begin{aligned} &\hat{G}^\Omega_\mu(x) = \Omega(x)\hat{G}_\mu \Omega^{-1}(x) + \Omega(x)\partial_\mu \Omega(x)^{-1}\,, \quad \hat{G}_\mu = T^a G^a_\mu \\ &q(x)_i = \left(e^{\{i \sum_{a=1}^{8} T^a \xi^a(x)\}}\right)_{ij} q_j(x) = \Omega(x)_{ij} q_j(x) \\ &\delta G^a_\mu(x) = -\frac{1}{g}\partial_\mu \xi^a(x) - \sum_{b,c=1}^{8} f^{abc} \xi^b(x) G^c_\mu(x) \end{aligned}\right\} \text{SU(3) gauge transformations}$$

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  Spectrum of the theory is given by colorless states (hadrons).
  The color degrees of freedom are permanently confined.
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  *1 000 000 $ Milleneum Problem*, Clay Mathematics Institute, Cambridge, MA
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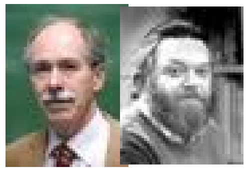
- **Non-trivial topological gauge configurations:**
  Instantons, $\mathbf{U(1)_A}$ anomaly, Theta term in Lagrangian: $-\Theta \frac{g^2}{64\pi^2}\epsilon^{\mu\nu\lambda\sigma}G^i_{\mu\nu}G^i_{\lambda\sigma}$

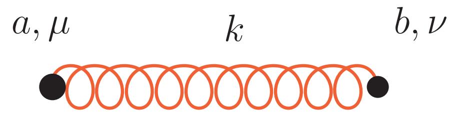
- ....



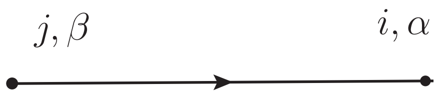
# Weak coupling  perturbation theory

Perturbative series of  renormalizable field theory  in terms of gluons an quarks with standard  Feynman rules and Feynman diagrams.

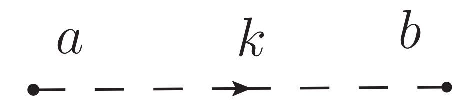
# Propagators:




$$\delta_{ab}\frac{-i}{k^2+i\epsilon}\left[g^{\mu\nu}+(\lambda-1)\frac{k^\mu k^\nu}{k^2+i\epsilon}\right]$$

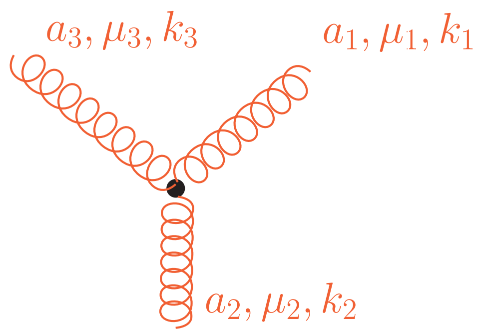



$$\delta_{ij}\left(\frac{i}{\hat{k}-m+i\epsilon}\right)_{\alpha\beta}$$

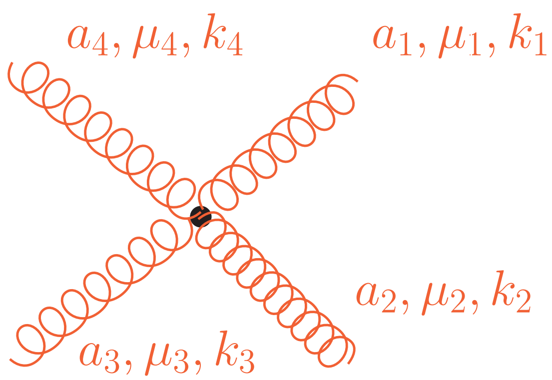



$$\delta_{ab}\frac{i}{k^2+i\epsilon}$$

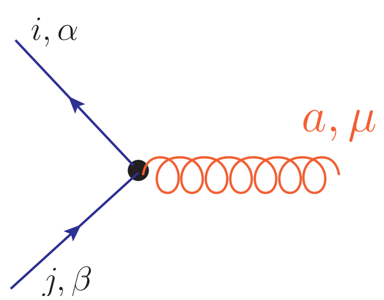
# Vertices:



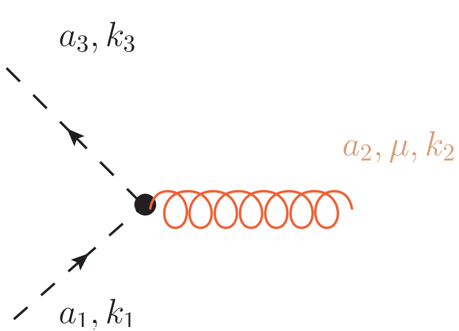

$$gf^{a_1a_2a_3}\left[(k_1-k_2)^{\mu_3}g^{\mu_1\mu_2}+(k_2-k_3)^{\mu_1}g^{\mu_2\mu_3}+(k_3-k_1)^{\mu_2}g^{\mu_3\mu_1}\right]$$




$$-ig^2f^{a_1a_3b}f^{a_2a_4b}\left(g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right)+(1\leftrightarrow 2)+(2\leftrightarrow 3)$$




$$ig(T^a)_{ij}(\gamma^\mu)_{\alpha\beta}$$




$$-gf^{a_1a_2a_3}k_3^\mu$$

# Weak coupling perturbation theory:

The diagrammatic rules  mathematically are not always well defined. In higher orders diagrams with closed loops can appear when the expressions given  by the Feynman diagrams have to be integrated over the internal loop momenta. Loop integrals  can become divergent both in the ultraviolet region ( at large values of the loop momenta) and in the the infrared region (soft and collinear regions of the loop momenta).

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Integrals which are divergent in four dimensions are carried out in different dimensions where the integrals are finite and the singularities of the integrals are exhibited as poles in $\epsilon$ with analytic continuation into d=4-2 $\epsilon$ dimensions.

$$\int \frac{\mathbf{d^{4-2\epsilon}dk}}{(\mathbf{2}\pi)^{\mathbf{4-2}\epsilon}} \frac{\mathbf{1}}{[-\mathbf{k^2}+\mathbf{C}-\mathbf{i}\epsilon]^{\mathbf{s}}} = \mathbf{i}(\mathbf{4}\pi)^{-\mathbf{2}+\epsilon}[\mathbf{C}-\mathbf{i}\epsilon]^{\mathbf{2-s}-\epsilon}\frac{\mathbf{\Gamma}(\mathbf{s}-\mathbf{2}+\epsilon)}{\mathbf{\Gamma}(\mathbf{s})}$$

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The difference of the bare Lagrangian and the renormalized Lagrangian gives the counter terms:

$$\mathcal{L}_{\mathbf{ct}} = -(\mathbf{Z_3}-\mathbf{1})\frac{\mathbf{1}}{\mathbf{4}}\left(\partial_\mu \mathbf{G}^{\mathbf{i}}_\nu - \partial_\nu \mathbf{G}^{\mathbf{i}}_\mu\right)^{\mathbf{2}} - (\mathbf{Z_1}-\mathbf{1})\frac{\mathbf{g}}{\mathbf{2}}\mu^\epsilon \mathbf{f^{ijk}}(\partial_\mu \mathbf{G}^{\mathbf{i}}_\nu - \partial_\nu \mathbf{G}^{\mathbf{i}}_\mu)\mathbf{G}^{\mathbf{j},\mu}\mathbf{G}^{\mathbf{k},\nu} + \cdots$$

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Additional diagrams with counter term vertices cancel the UV divergences coming from loop integrals order by order of perturbation theory iteratively.

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If we change the parameter $\mu$, a change is generated,in the renormalized parameters $\mathbf{g}(\mu)$.

$$\mu^{\mathbf{2}}\frac{\mathbf{d}\alpha_{\mathbf{s}}(\mu)}{\mathbf{d}\mu^{\mathbf{2}}} = \beta(\alpha_{\mathbf{S}}) \ , \quad \alpha_{\mathbf{S}} = \frac{\mathbf{g^2}(\mu)}{\mathbf{4}\pi} \ , \ \beta(\alpha_{\mathbf{S}}) = -\alpha_{\mathbf{S}}{}^{\mathbf{2}}\mathbf{b_0} - \alpha_{\mathbf{S}}{}^{\mathbf{3}}\mathbf{b_1} + \mathcal{O}(\alpha_{\mathbf{S}}{}^{\mathbf{4}})$$

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$$\mathbf{b_0} = \frac{\mathbf{11N_C - 4n_f T_R}}{\mathbf{12}\pi}$$

$$\mathbf{b_0} \geq \mathbf{0} \quad \text{for} \quad \mathbf{N_c = 3, T_R = \frac{1}{2}} \quad \text{provided} \quad \mathbf{n_f \leq 16.5}$$

$$\alpha_S(\mu, \mathbf{n_f}, \mathbf{\Lambda}) = \frac{\mathbf{1}}{\mathbf{b_0} \ln \frac{\mu^2}{\mathbf{\Lambda^2}}} + \mathcal{O}\left(\frac{\ln \mathbf{ln} \frac{\mu^2}{\mathbf{\Lambda^2}}}{\mathbf{ln} \frac{\mu^2}{\mathbf{\Lambda^2}}}\right)$$

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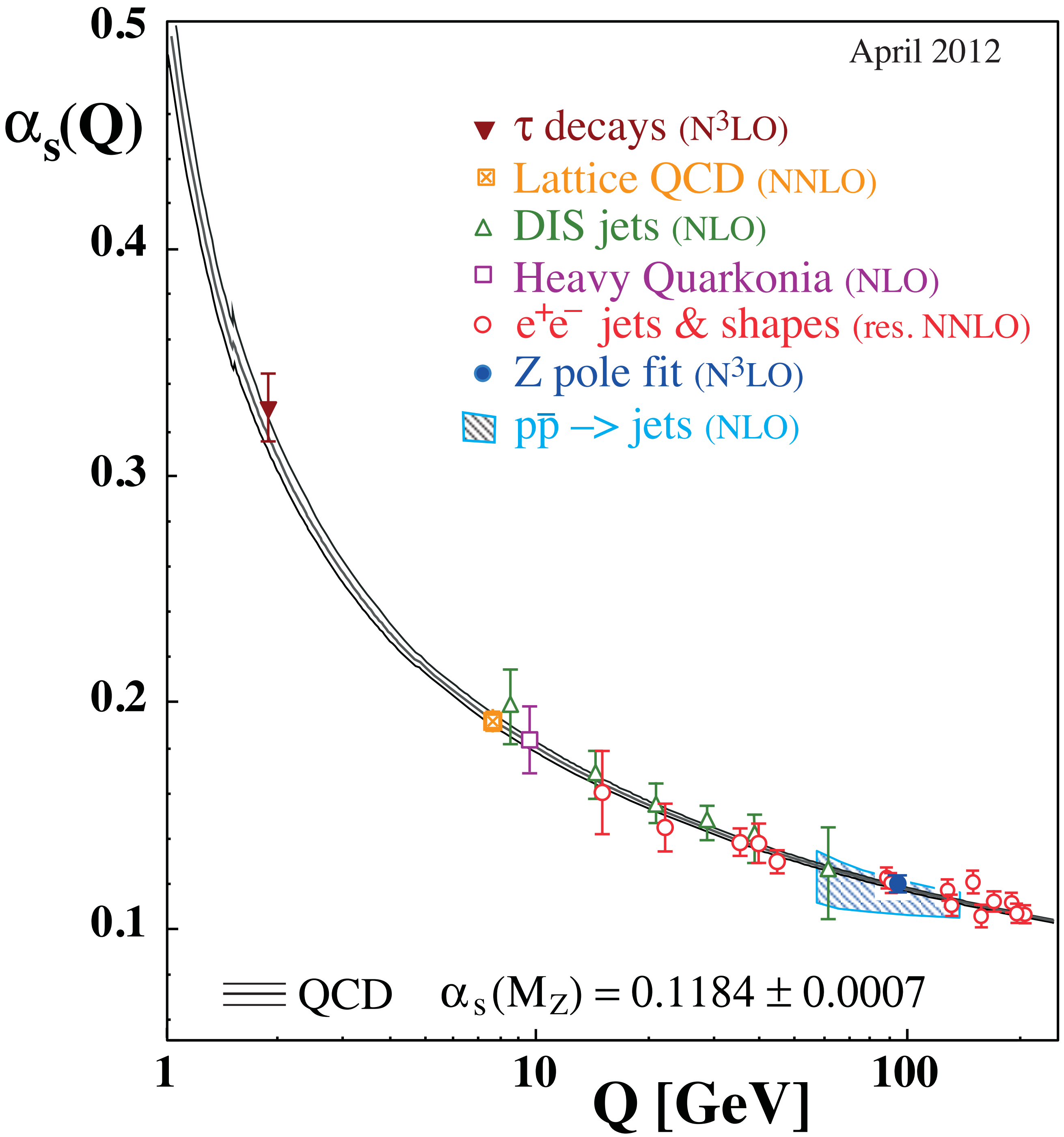
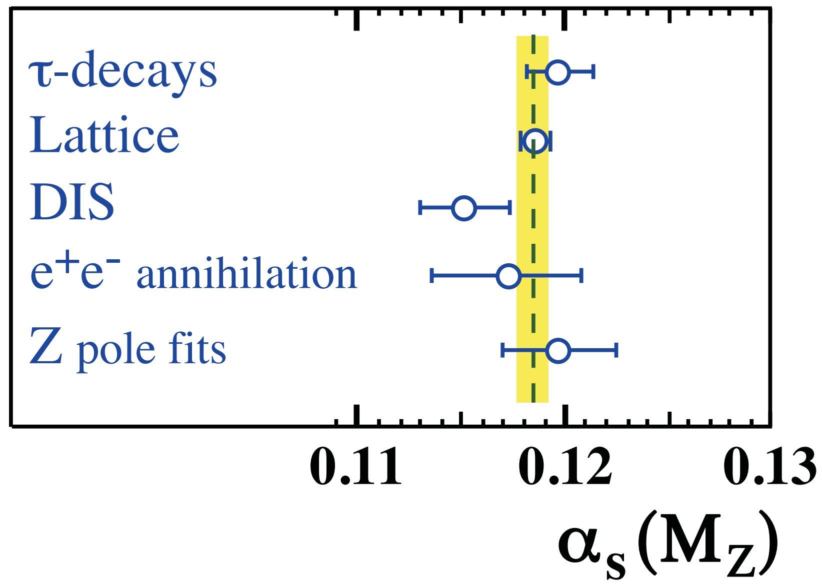
$$\alpha_\mathbf{S}(\mu, \mathbf{n_f}, \mathbf{\Lambda}) = \frac{\mathbf{1}}{\mathbf{b_0} \ln \frac{\mu^2}{\mathbf{\Lambda^2}}} + \mathcal{O}\left( \frac{\ln \mathbf{ln} \frac{\mu^2}{\mathbf{\Lambda^2}}}{\mathbf{ln} \frac{\mu^2}{\mathbf{\Lambda^2}}} \right)$$

Experiment: $\alpha_\mathbf{s}(\mathbf{m_Z}, \mathbf{5}, \mathbf{\Lambda}^{(\mathbf{5})}_{\overline{\mathbf{MS}}}) = \mathbf{0.1183} \pm \mathbf{0.0010}$ , $\mathbf{\Lambda}^{(\mathbf{5})}_{\overline{\mathbf{MS}}} \approx \mathbf{300}\,\mathrm{MeV}$

$\mathbf{\Lambda}^{(\mathbf{5})}_{\overline{\mathbf{MS}}}$ : hidden fundamental scale of QCD

# Measured values of $\alpha_s(M_Z)$



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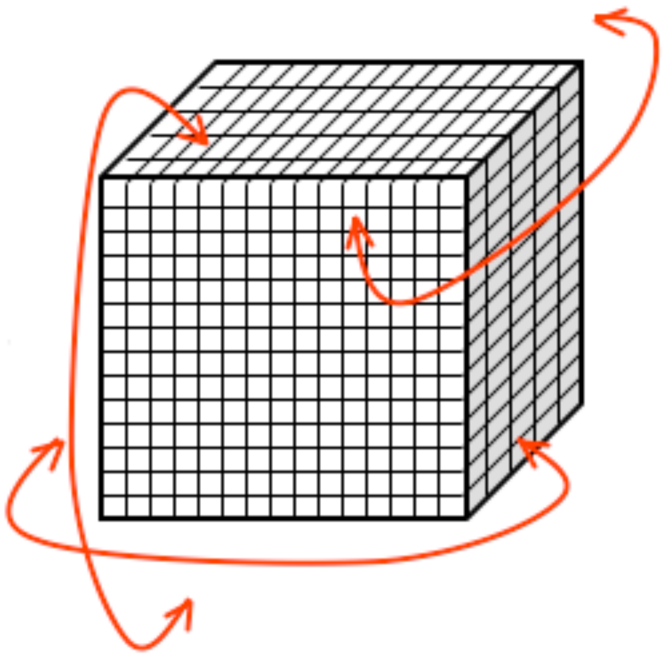

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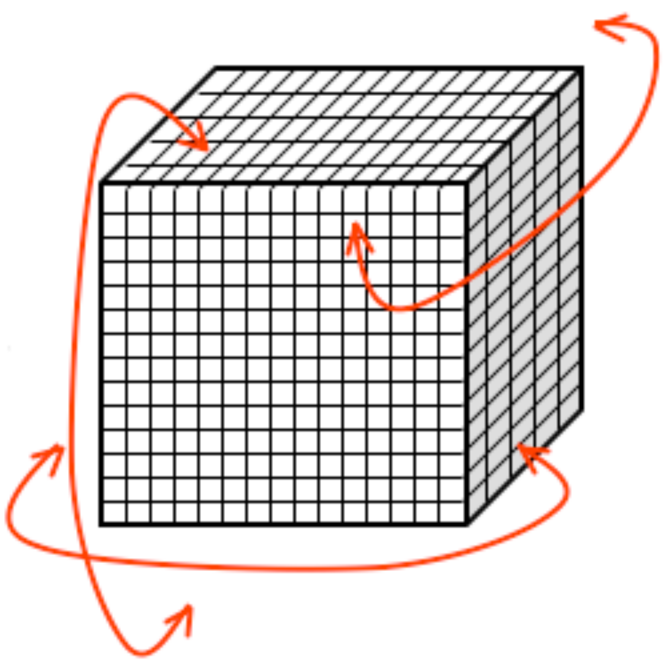

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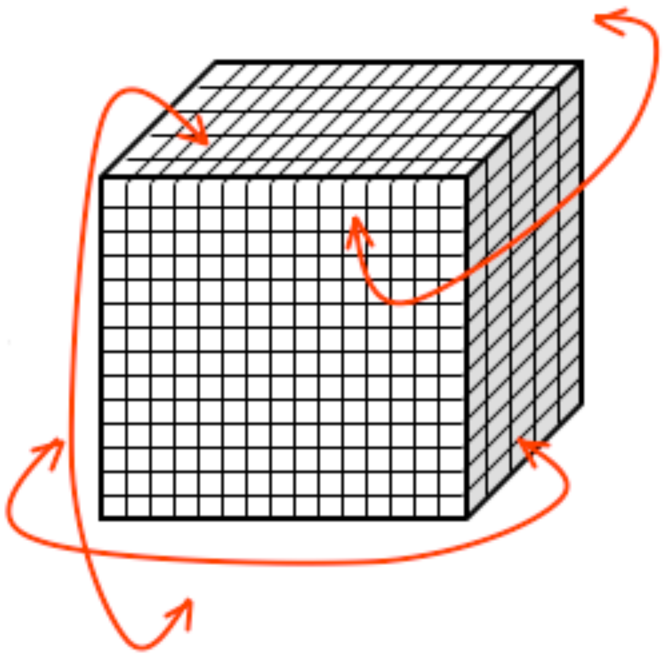

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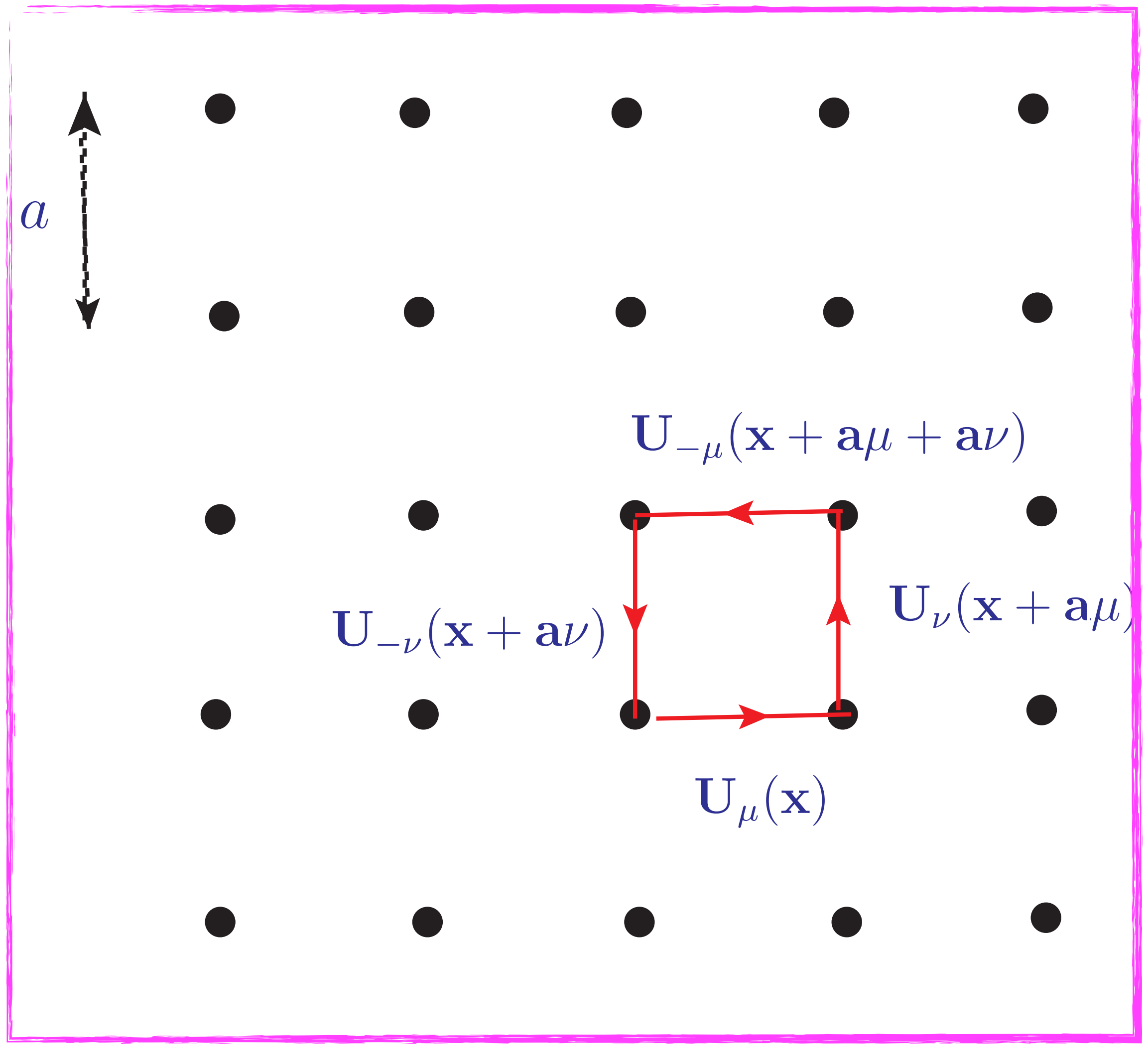
$$\mathcal{Z} = \int \mathcal{D}\mathbf{A}_\mu \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathbf{e}^{-(\bar{\psi}\mathbf{M}\psi+\mathbf{S_G})}$$

# Lattice QCD defines the theory with infrared and ultraviolet cut-off

$x_\mu = n_\mu a \quad \text{with} \quad n_\mu \in \{0, \ldots, N_\mu - 1\}$




The parallel transport $\mathbf{U}_\mu(\mathbf{x})$ is the discretized version of the path ordered product of continuum gauge fields $\hat{\mathbf{G}}_\mu$

$$\mathbf{U}_\mu(\mathbf{x}) = \mathcal{P}\mathbf{e}^{\mathbf{ig}\int_{\mathbf{x}}^{\mathbf{x}+\hat{\mu}} \mathbf{dx}'_\mu \hat{\mathbf{G}}_\mu(\mathbf{x}')}$$

$$\mathbf{U}_\mu(\mathbf{x}) \rightarrow \mathbf{\Omega}(\mathbf{x})\mathbf{U}_\mu(\mathbf{x})\mathbf{\Omega}^\dagger(\mathbf{x} + \hat{\mu})$$

The trace over a closed loop of parallel transports is gauge invariant. The simplest of these loops is the plaquette

$$\mathbf{U}_{\mu\nu}(\mathbf{x}) = \mathbf{U}_\mu(\mathbf{x})\mathbf{U}_\nu(\mathbf{x} + \hat{\mu})\mathbf{U}^\dagger_\mu(\mathbf{x} + \hat{\nu})\mathbf{U}^\dagger_\nu(\mathbf{x})$$

$$\mathbf{S_G} = \beta \sum_{\mathbf{x},\mu>\nu} \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{6}}\mathrm{Tr}(\mathbf{U}^\dagger_{\mu\nu}(\mathbf{x}) + \mathbf{U}_{\mu\nu}(\mathbf{x}))\right) \quad , \quad \beta = \frac{\mathbf{6}}{\mathbf{g^2}}$$

The path integral

$$\langle \mathbf{O} \rangle = \frac{\mathbf{1}}{\mathcal{Z}} \int \mathcal{D}\mathbf{U}_\mu \int \mathcal{D}\mathbf{\Psi}\mathcal{D}\bar{\mathbf{\Psi}}\mathbf{O}\mathbf{e}^{-(\bar{\mathbf{\Psi}}\mathbf{M}(\mathbf{U})\mathbf{\Psi} + \mathbf{S_G}(\mathbf{U}))}$$

is carried out by numerical Monte Carlo integration

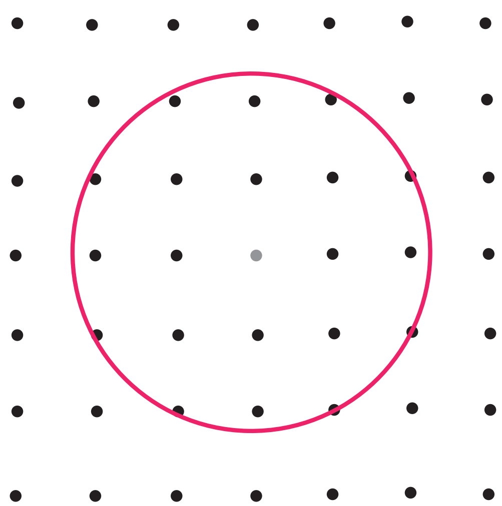
# Lattice QCD defines the theory with infrared and ultraviolet cut-off

## Taking the continuum limit.

Let us imagine that measuring the correlation function of proton creation operators on the lattice with Monte Carlo simulation, we extract the value of the proton mass.The cut-off scale on the lattice is the lattice distance “a” therefore all physical quantities are obtained in its physical units.The value of the proton mass in such a lattice measurement depends on the coupling constant and on the lattice distance as

$$\mathbf{m_P} = \frac{\mathbf{1}}{\mathbf{a}}\mathbf{f}(\mathbf{g}(\mathbf{a}))\,, \qquad \lim_{\mathbf{a}\to\mathbf{0}} \mathbf{m_P} = \frac{\mathbf{1}}{\mathbf{a}}\mathbf{f}(\mathbf{g}(\mathbf{a})) \longrightarrow \mathrm{const} \quad , \qquad \mathbf{a}\frac{\mathbf{d}}{\mathbf{da}}\mathbf{m_P} = \mathbf{0}$$

$$\mathbf{a} \ll \mathbf{R_P} \ll \mathbf{L} = \mathbf{N\,a}$$



$$\mathbf{f}(\mathbf{g}) = \mathbf{a}\frac{\partial \mathbf{g}}{\partial \mathbf{a}}\frac{\partial}{\partial \mathbf{g}}\mathbf{f}(\mathbf{g}) \qquad \mathbf{a}\frac{\partial \mathbf{g}}{\partial \mathbf{a}} = \frac{\mathbf{b}}{\mathbf{4\pi}}\mathbf{g^3}\left(\mathbf{1} + \mathbf{b}'\frac{\mathbf{g^2}}{\mathbf{4\pi}}\right)$$

$$\mathbf{\Lambda_L} = \frac{\mathbf{1}}{\mathbf{a}}\exp\left(-\frac{\mathbf{2\pi}}{\mathbf{g_L^2}}\right)\left(\frac{\mathbf{4\pi} + \mathbf{b}'\mathbf{g_L^2}}{\mathbf{4\pi g_L^2}}\right)^{\frac{\mathbf{b}'}{\mathbf{2b}}}$$

$$\mathbf{m_P} = \mathbf{c\Lambda_L}$$

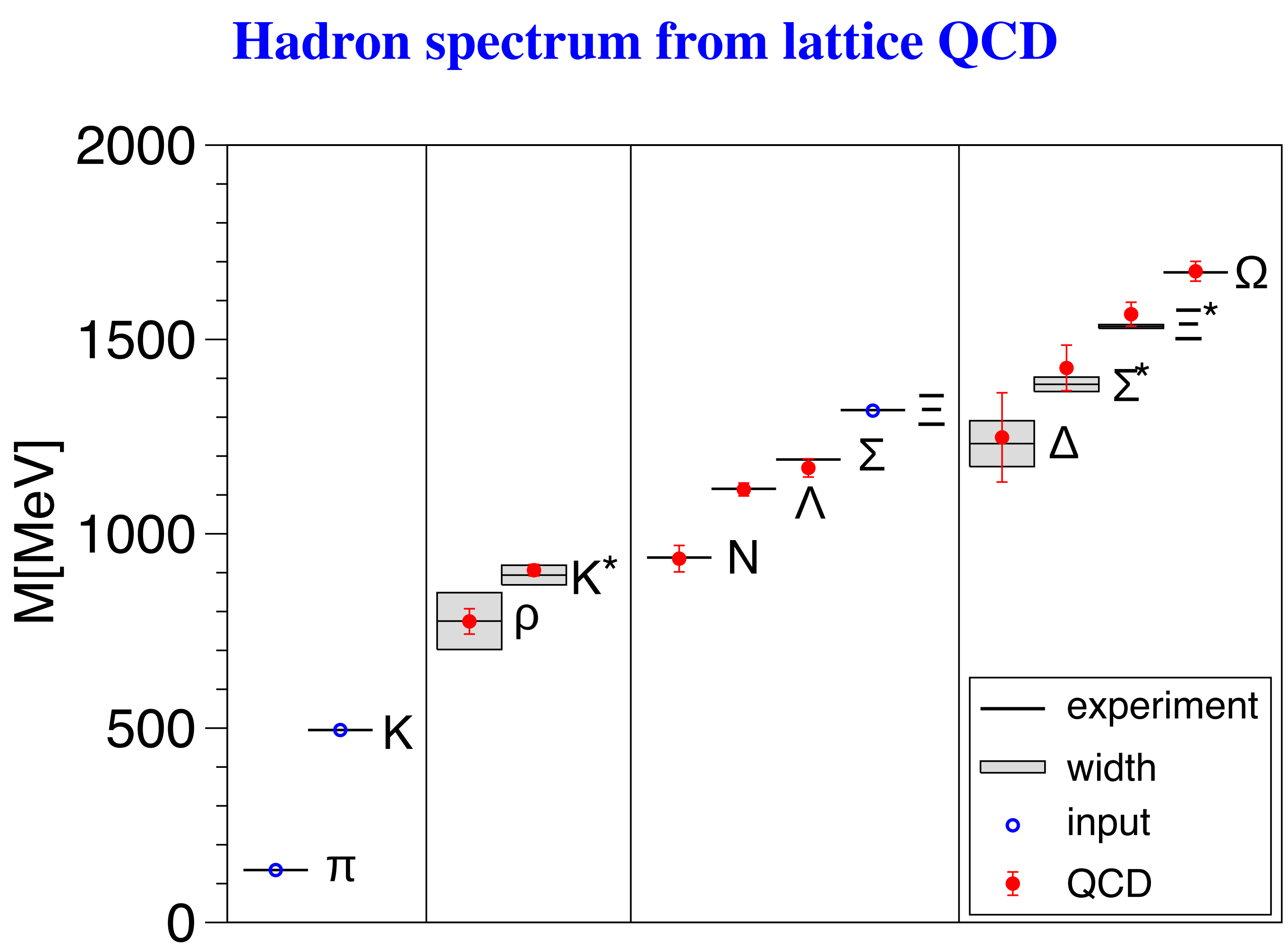
dimensional transmutation

$$\mathbf{\Lambda_L} \approx \mathbf{1.7}\mathrm{MeV} \qquad \mathbf{g_L^2} = \mathbf{1}\,,\ \mathbf{a^{-1}} \approx \mathbf{2}\,\mathrm{GeV}\,,\ \mathbf{a} \approx \mathbf{0.1}\mathrm{fermi}$$

The relative magnitude of $\mathbf{\Lambda}_{\mathrm{QCD}}$ with respect to the quark masses has important physical significance. In particular, it is very important that the value of $\mathbf{\Lambda}_{\mathrm{QCD}}$ is much bigger than the value of the light quark masses $\mathbf{m_u}, \mathbf{m_d}$ . It is also important that in high energy experiments the typical energy scale of is much higher than the value of $\mathbf{\Lambda}_{\mathrm{QCD}}$.

# Lattice QCD defines the theory with infrared and ultraviolet cut-off

30 years of effort of a large community




S. Dürr *et. al.*, Science 322:1224-1227,2008

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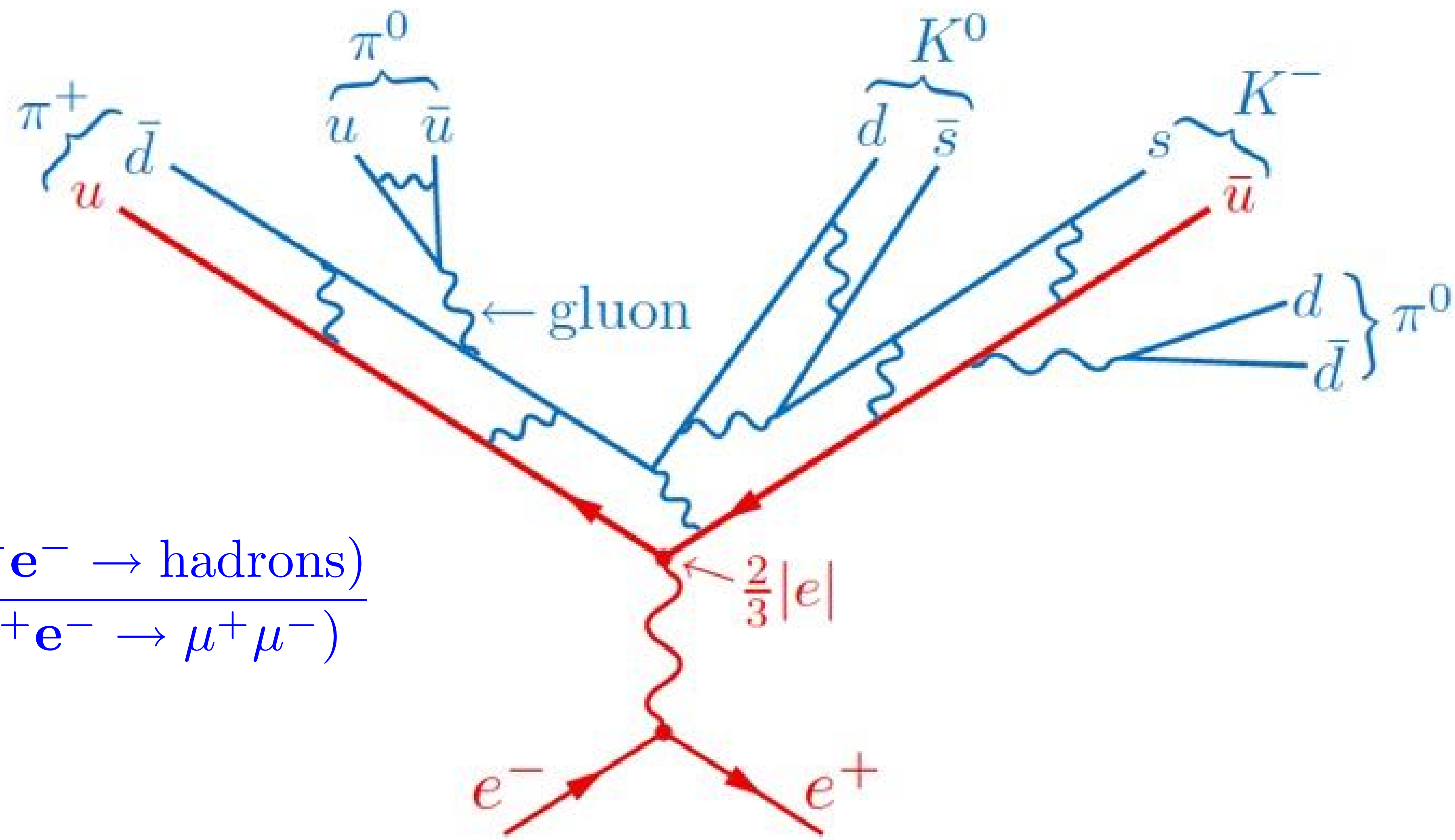
$$\sum_{\mathbf{h}} |\mathbf{h} >< \mathbf{h}| = \sum_{\mathbf{q,g}} |\mathrm{gluons, quarks} >< \mathrm{gluons, quarks}|$$

$$\sqrt{\mathbf{s}} = \mathbf{Q}, \quad \mathbf{Q}\,, \gg, \mathbf{m_a}, \quad \mathbf{a} = \mathbf{u}, \mathbf{d}, \mathbf{s}, ...$$

# $e^+e^- \longrightarrow q\bar{q} \longrightarrow$ hadrons




$$\mathbf{R(Q^2)_{e^+e^-}} = \frac{\sigma(\mathbf{e^+e^-} \to \text{hadrons})}{\sigma(\mathbf{e^+e^-} \to \mu^+\mu^-)}$$

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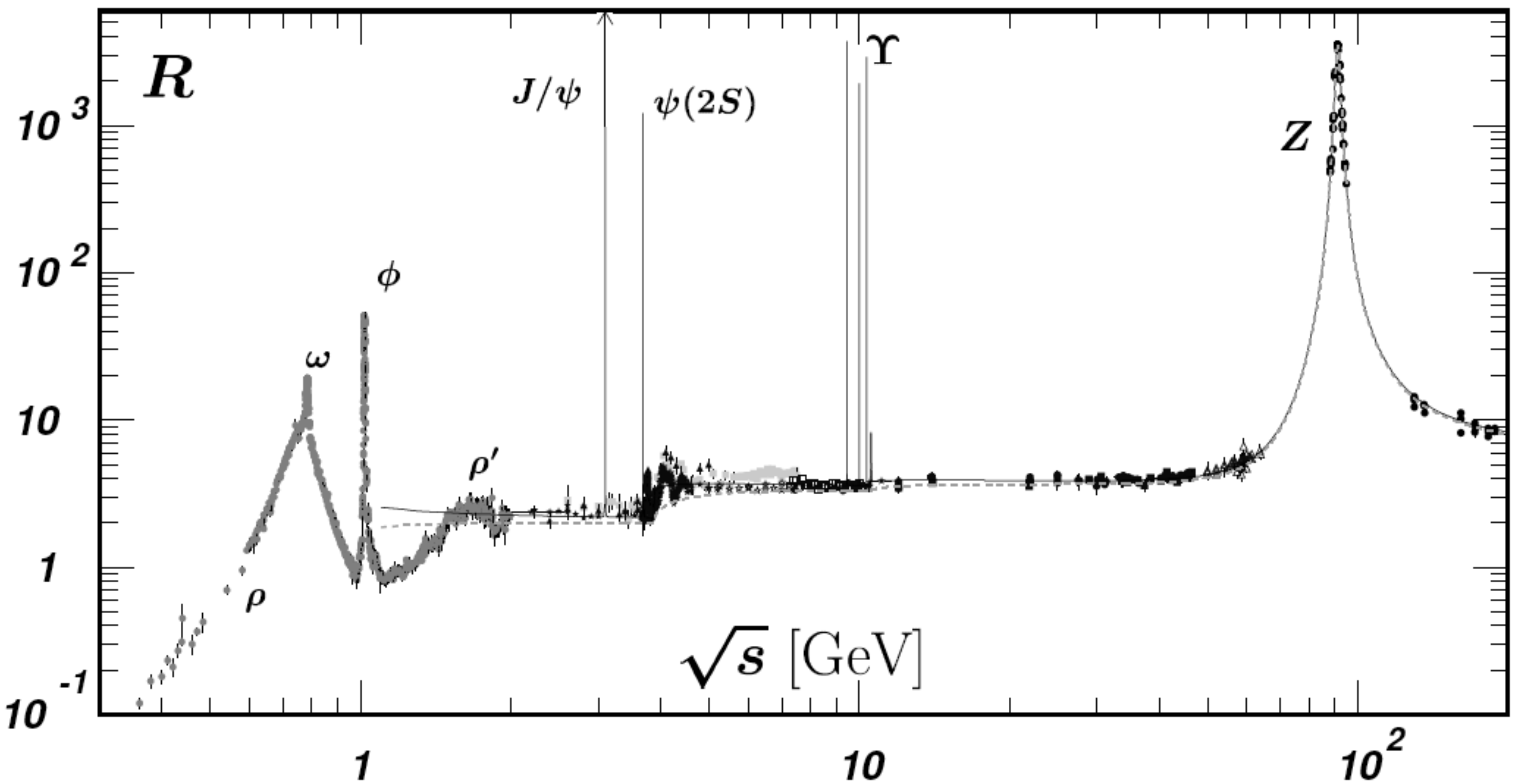
$$\mathbf{R(s)} = \mathbf{12}\,\pi \,\mathrm{Im}\, \mathbf{\Pi}^{\mathrm{EM}}(\mathbf{s}) = \mathbf{3} \sum_{\mathbf{i=1}}^{\mathbf{n_f}} \mathbf{Q_i^2} \left(\mathbf{1} + \frac{\alpha_\mathbf{s}}{\pi} + \ldots\right)$$

The measured values of $\mathbf{R(Q^2)_{e^+e^-}} = \frac{\sigma(\mathbf{e^+e^-} \to \mathrm{hadrons})}{\sigma(\mathbf{e^+e^-} \to \mu^+\mu^-)}$



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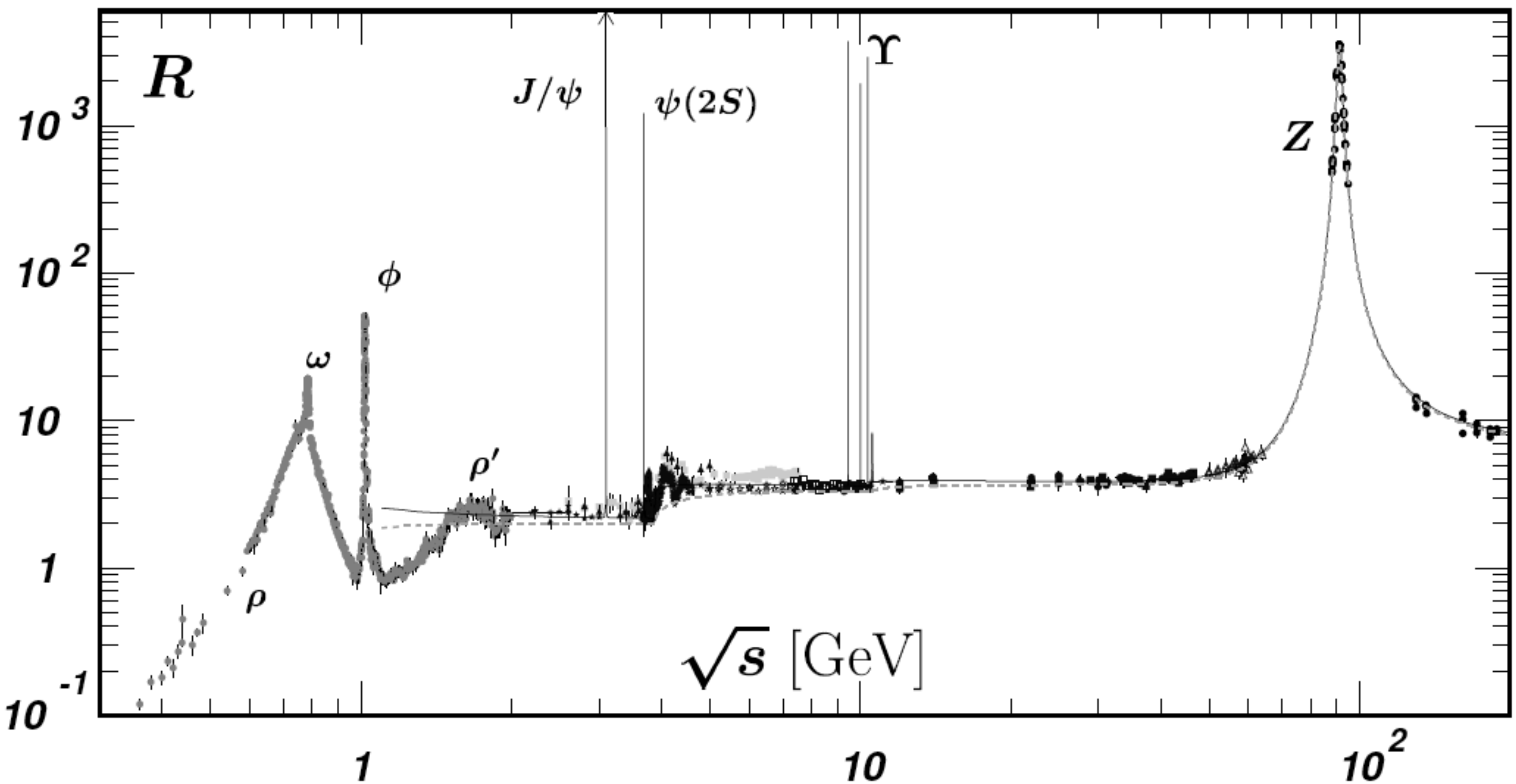


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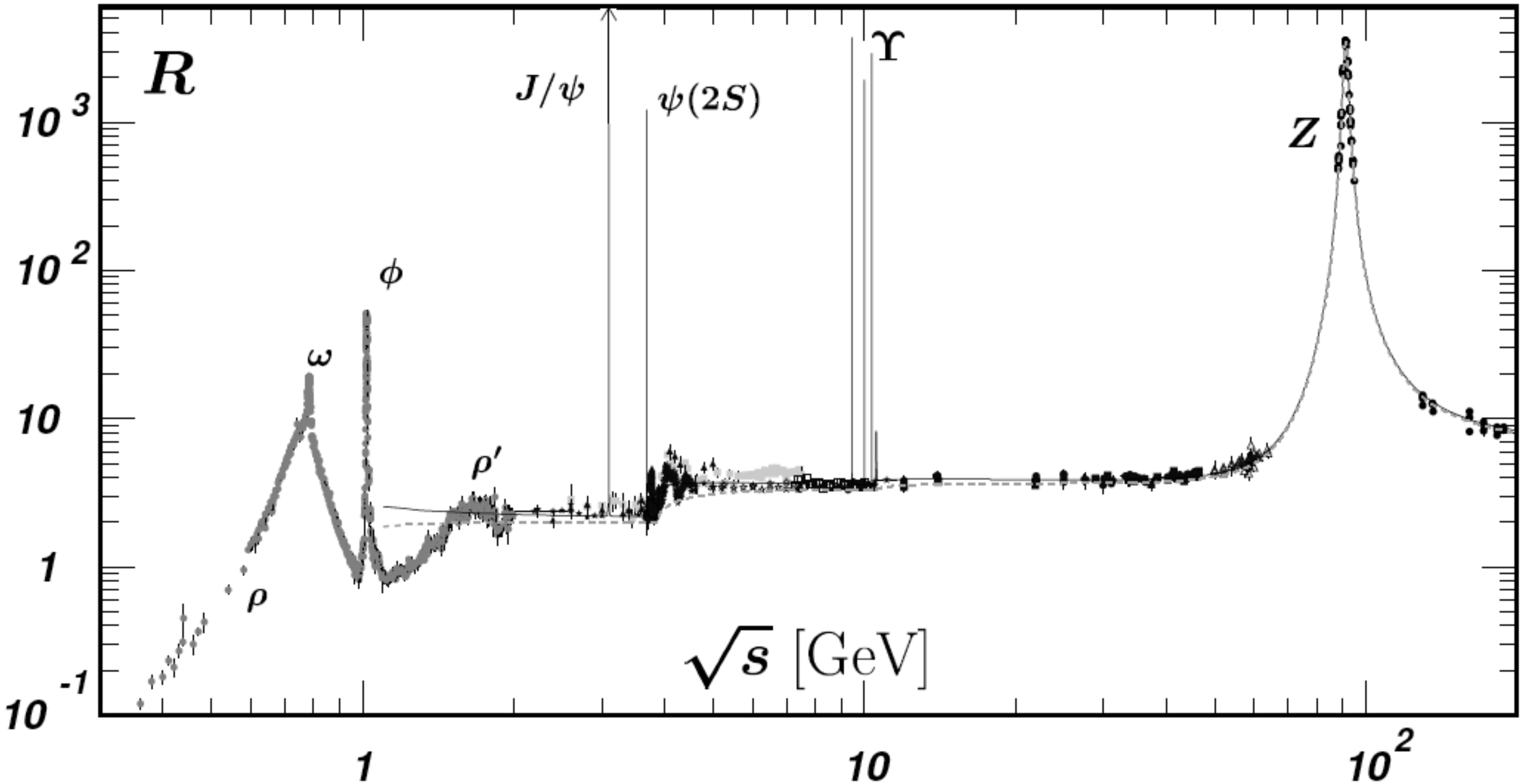


and its precise theoretical calculation to NNNLO order in perturbative QCD

$$R(s) = 1 + a_s + 1.4097a_s^2 - 12.76703a_s^3 - 80.0075a_s^4..$$

$$\alpha_s(M_z)^{\mathrm{NNNLO}} = 0.1190 \pm 0.0026_{\mathrm{exp}}$$

Baikov, Chetyrkin,Kuhn

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parton number densities
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hard parton x-section expanded
in the QCD coupling constant

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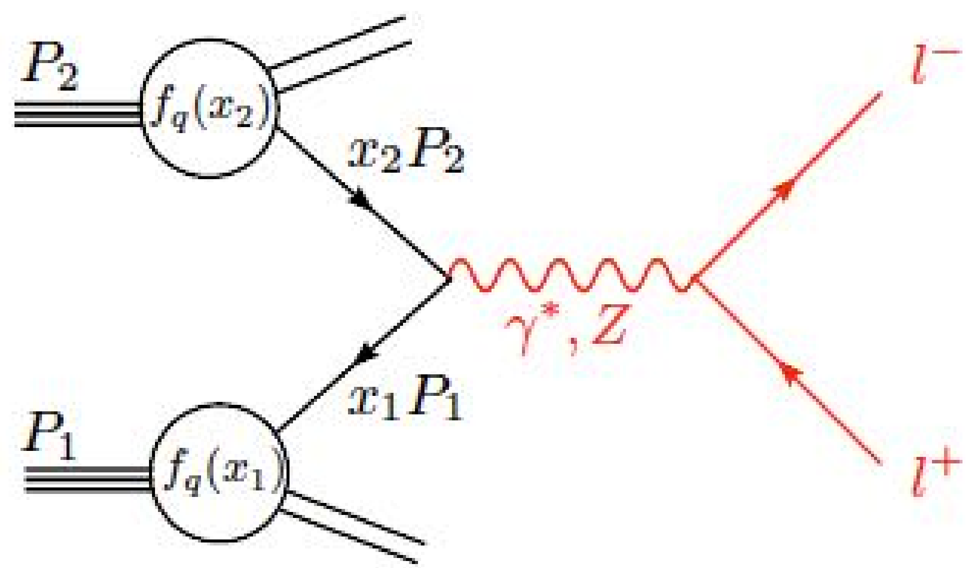


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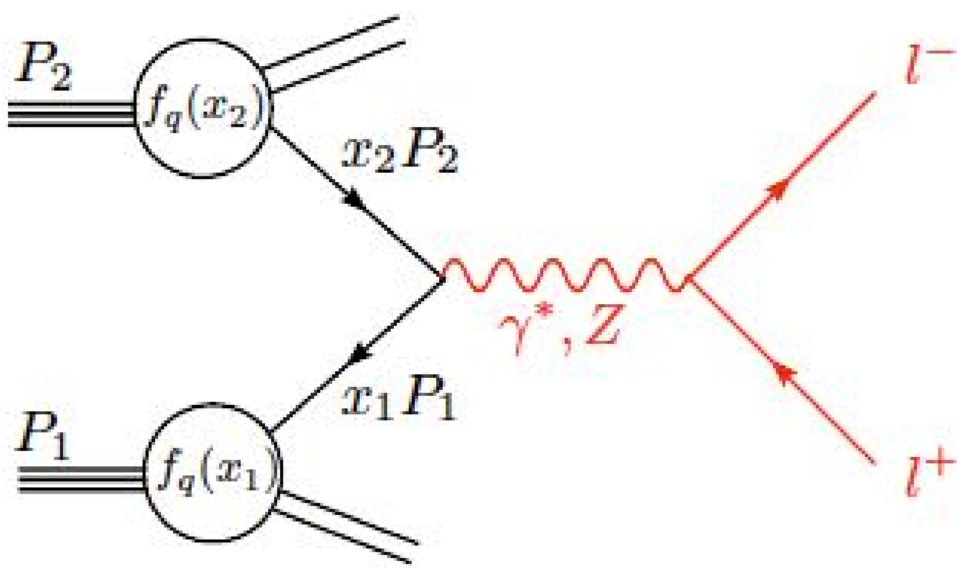
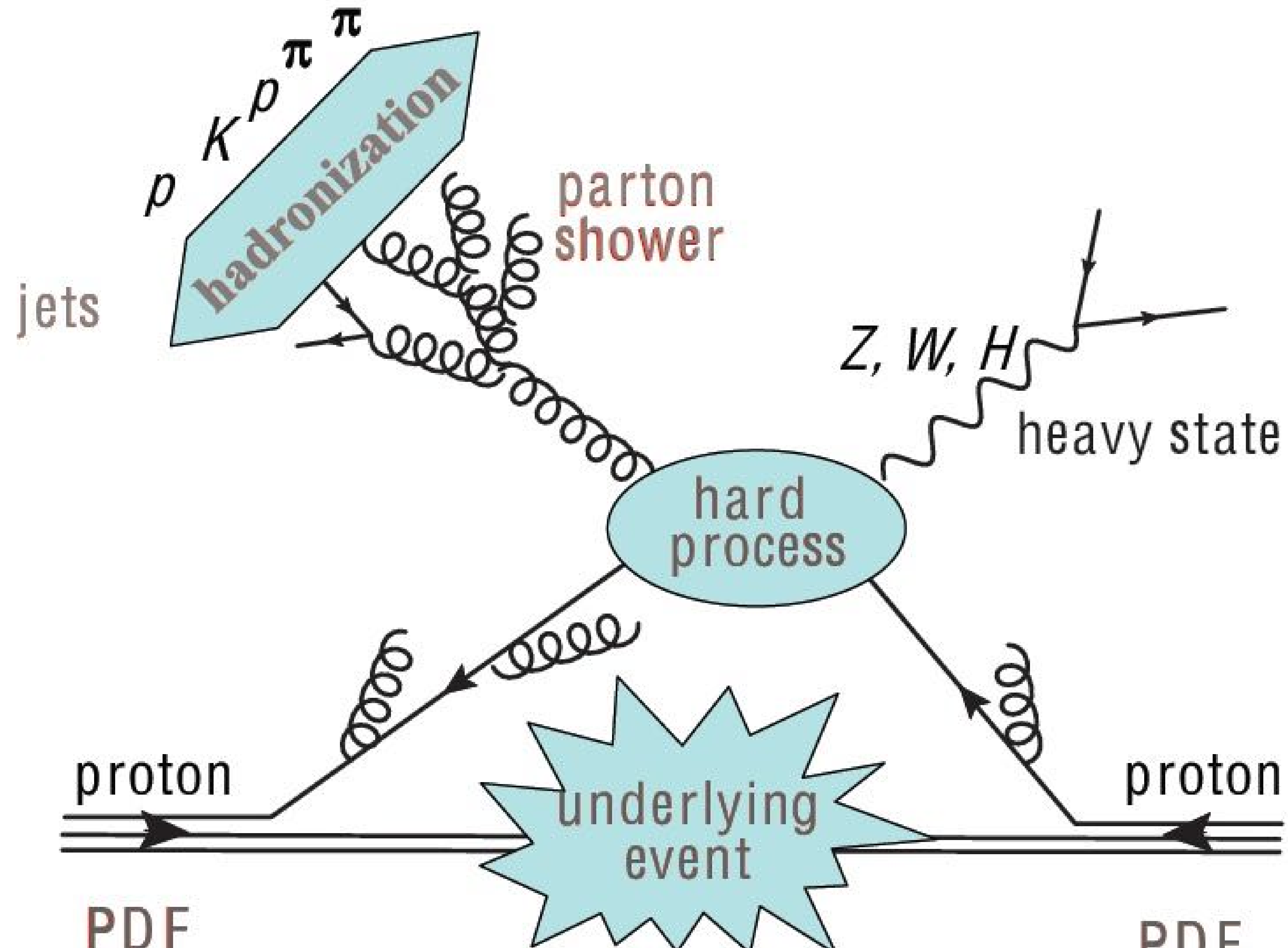



Drell-Yan process

**Many processes, many different multi particle final states**

# Schematic view of hard processes at the LHC




Z. Bern

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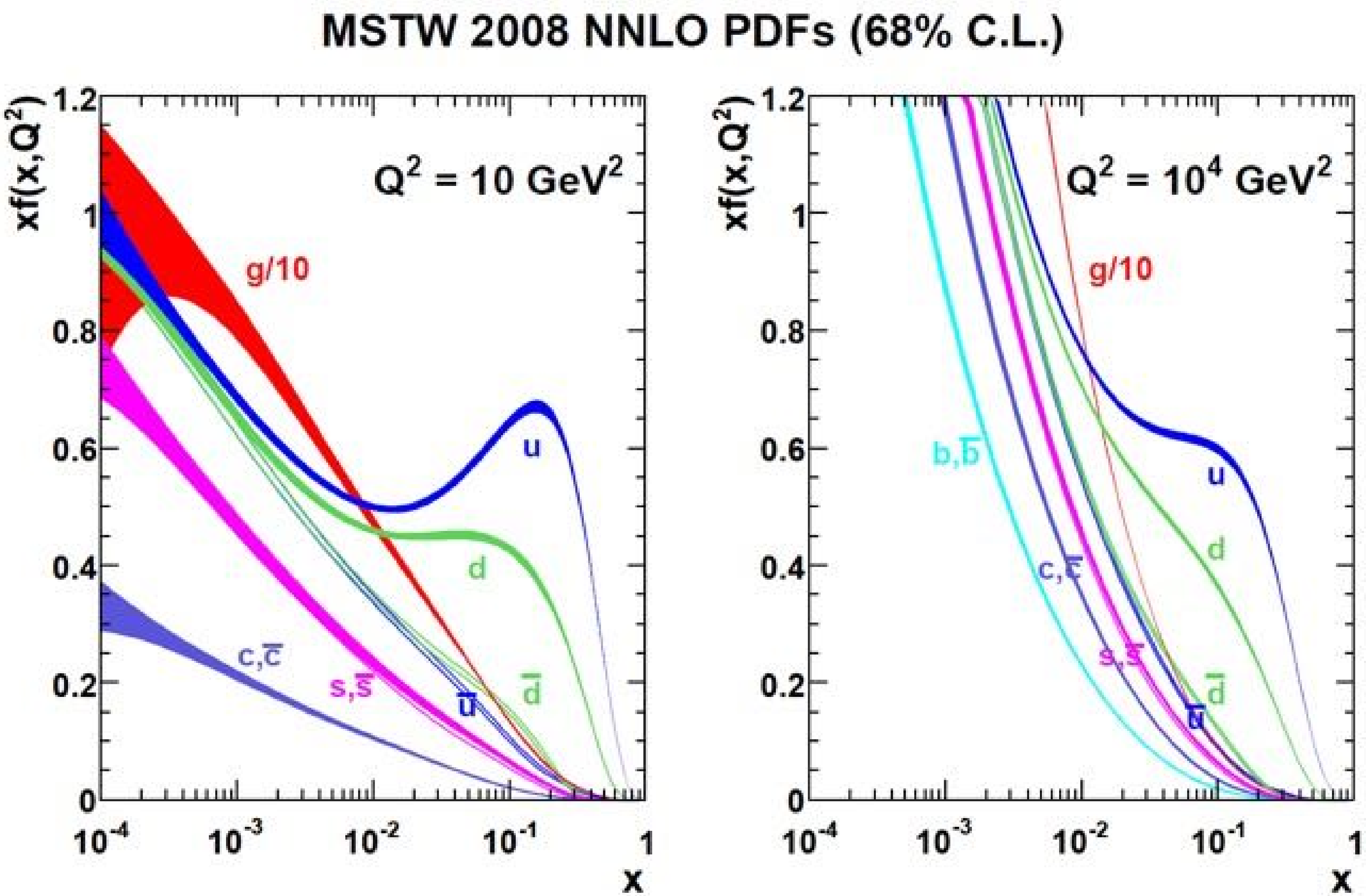


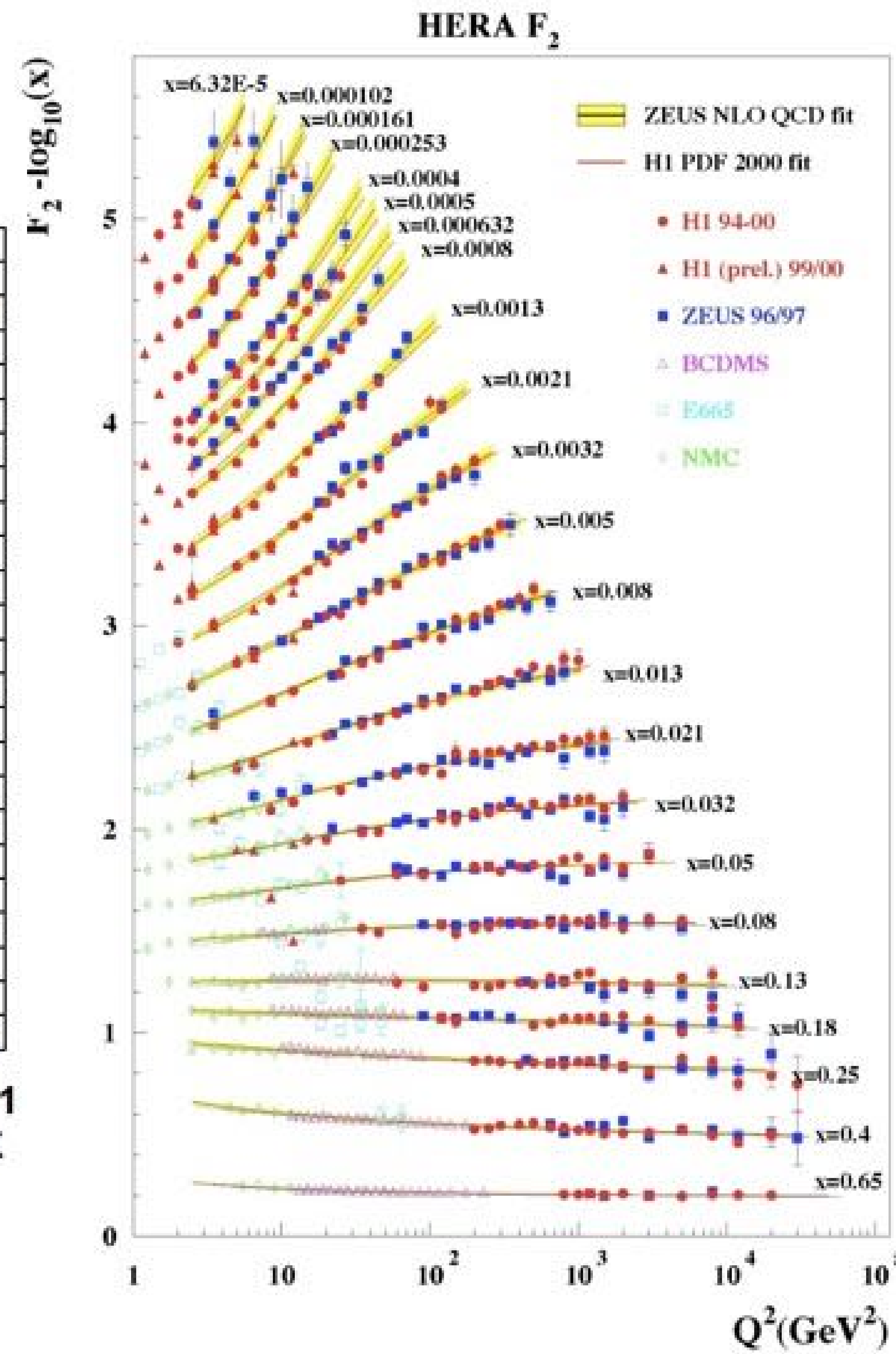

## The hard parton cross-section:

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$$\mathbf{d\sigma_{ab}^{(bare)}(p_a,p_b,d)=\sum_{ij}\int_0^1 dx_a\int_0^1 dx_b\Gamma_{i/a}(x_a,\epsilon)\Gamma_{j/b}(x_b,\epsilon)d\hat{\sigma}_{ij}(q,x_ap_a,x_bp_b,\alpha_s(\mu),\mu)}$$

$$\mathbf{\Gamma_{i/a}(x)=\delta_{ia}\delta(1-x)-\frac{\alpha_s}{2\pi}\left(\frac{1}{\bar{\epsilon}}P_{i/a}(x,0)-K_{i/a}(x)\right)+\mathcal{O}\left(\alpha_s^2\right)},$$

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**Physical cross section:** the scale dependence of the parton number densitiesis cancelled by the scale dependence of the hard scattering cross-section

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$$\mathbf{d\sigma_{pp->\mu^+\mu^-+hadrons} = \int f_{a/A}(x_a, \mu) f_{b/B}(x_b, \mu) d\hat{\sigma}_{ab}(x_a x_b s, \mu, \alpha_s(\mu))}$$

## Cancellation and factorization theorems:

They constitute the theoretical basis of description of scattering processes of hadrons in perturbative QCD. These theorems are valid for infrared safe measurables.

An observable is infrared safe if it is insensitive to collinear splittings of partons and/or emission of soft gluons.

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**In the case of infrared safe quantities the perturbative QCD predictions are well defined in terms of partons.**

**The fundamental assumption of the QCD improved parton model:**

**Predictions made for infrared save quantities in terms of partons partons give good approximation to the same quantities measured in terms of hadrons (the power corrections are small at high momentum scales).**

## The spectacular success of perturbative QCD:

In all the cases considered, the comparison of observables that are calculable in perturbative QCD with experimental results improved if next-to-leading order (NLO) QCD computations were used.  This fact establishes perturbative QCD as a systematic framework to describe the physics of hard hadronic collisions.  It also suggests that, ideally, the theoretical toolkit for the LHC should contain next-to-leading computation  for a large variety of processes.

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- ☑ Reduced theoretical uncertainties due to more meaningful scale dependence and more precisely predicted rates and shapes
- ☑ Data at HERA,LEP,Tevatron and LHC fully validate the improvements of the agreement between theory and experiment if NLO corrections are included
- ☑ Smaller uncertainties in extrapolating measured background cross-sections into signal regions
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### Further improvements at NNLO

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# Hard parton cross-section at NLO

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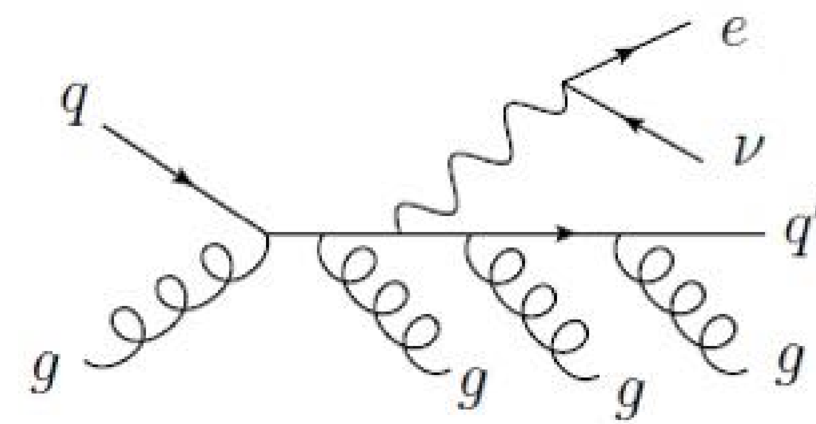

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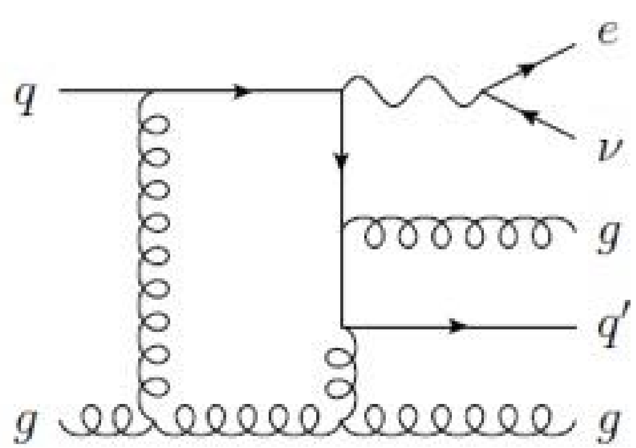
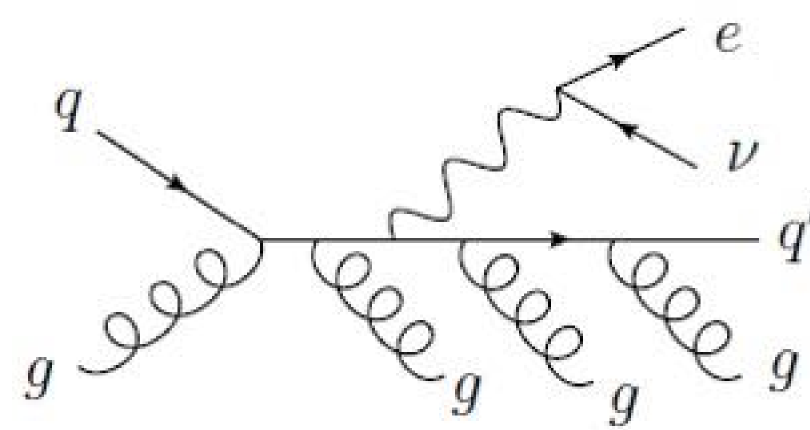
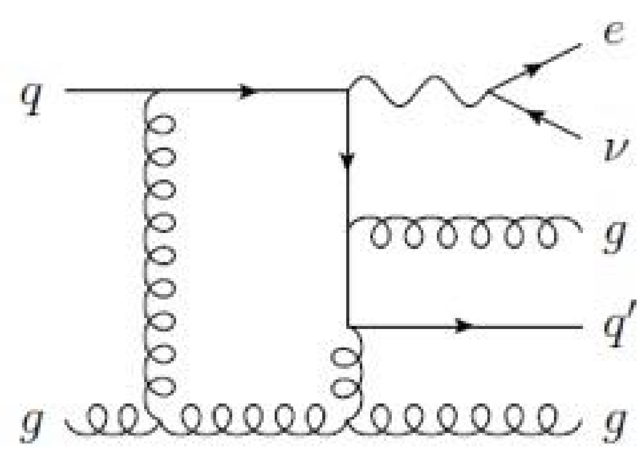
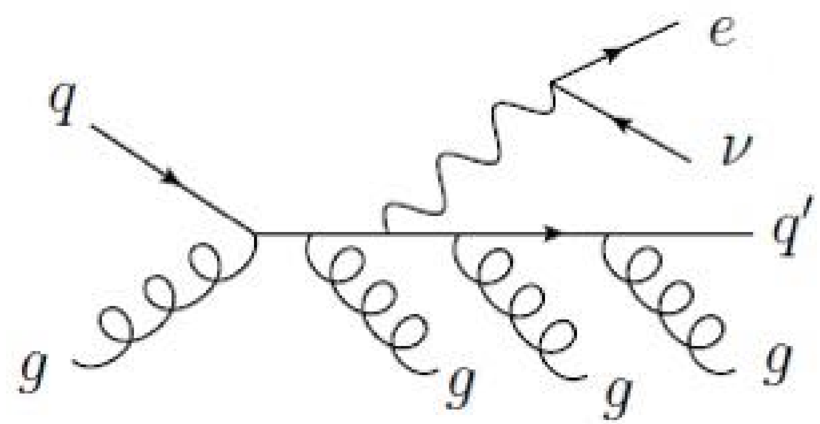


# Hard parton cross-section at NLO

Soft and collinear singularities of virtual and real contributions are cancelled by subtraction method



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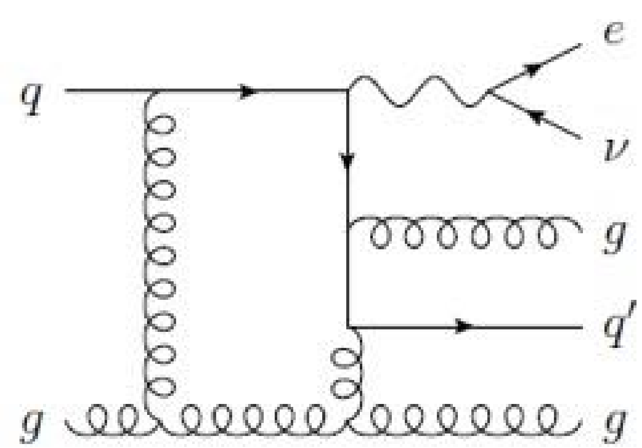
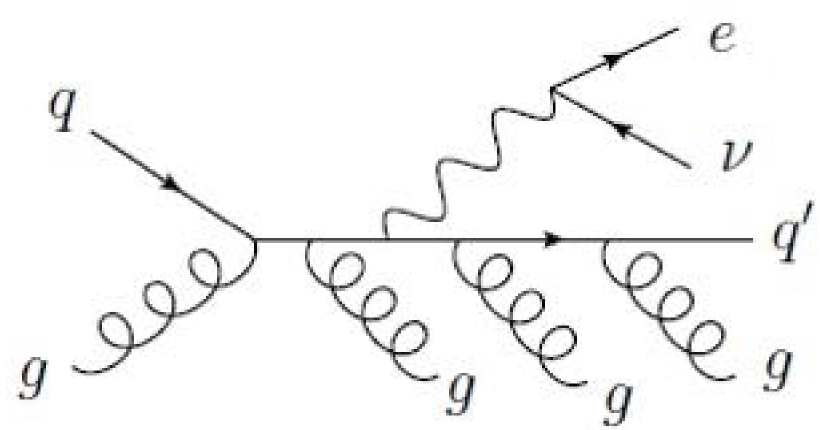



..times tree amplitude




squared amplitude

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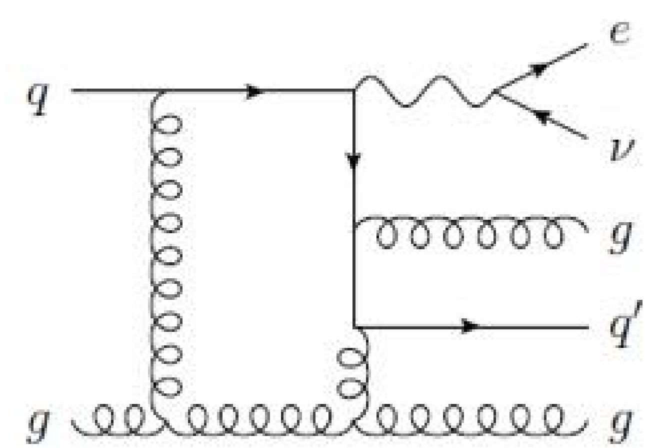


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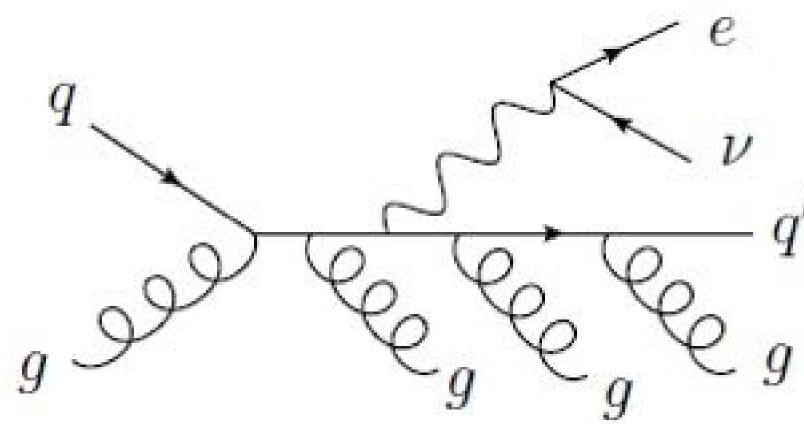


squared amplitude

$$\mathbf{d}\sigma_{\mathbf{n}}^{(\mathbf{1})} \approx |\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\mathbf{\Phi}_{\mathbf{n-2}} + \mathbf{2}\mathrm{Re}(\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})\dagger}\mathbf{M}_{\mathbf{n}}^{(\mathbf{1})})\mathbf{d}\mathbf{\Phi}_{\mathbf{n-2}} + |\mathbf{M}_{\mathbf{n+1}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\mathbf{\Phi}_{\mathbf{n-1}}$$

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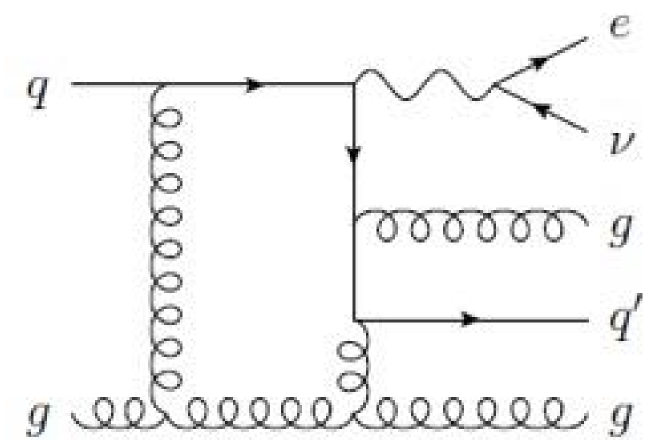


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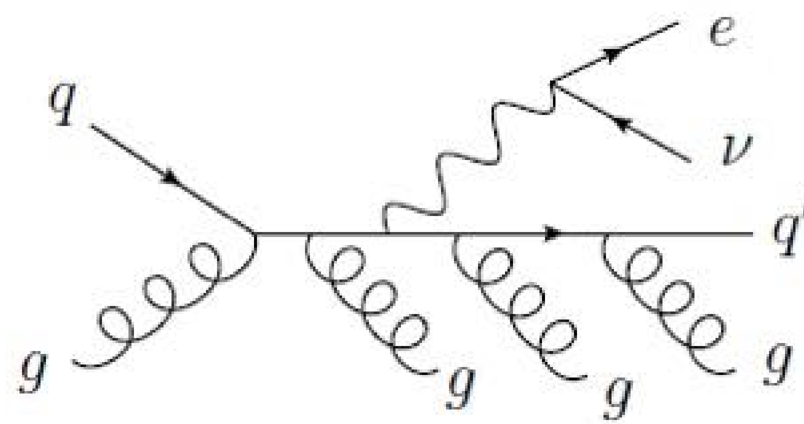


squared amplitude

Born

$$\mathbf{d}\sigma_{\mathbf{n}}^{(\mathbf{1})} \approx |\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-2}} + \mathbf{2}\mathrm{Re}(\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})\dagger}\mathbf{M}_{\mathbf{n}}^{(\mathbf{1})})\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-2}} + |\mathbf{M}_{\mathbf{n+1}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-1}}$$

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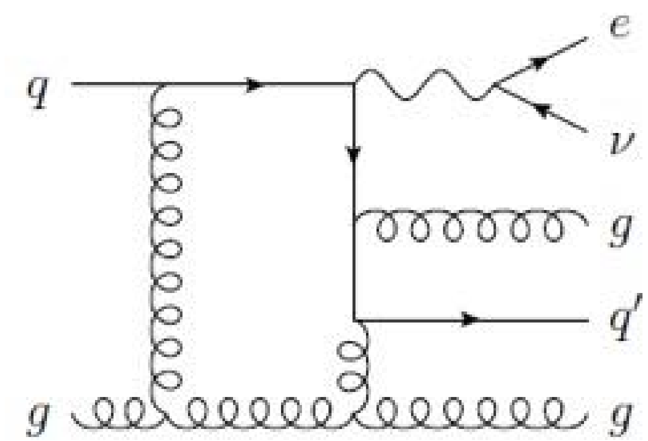
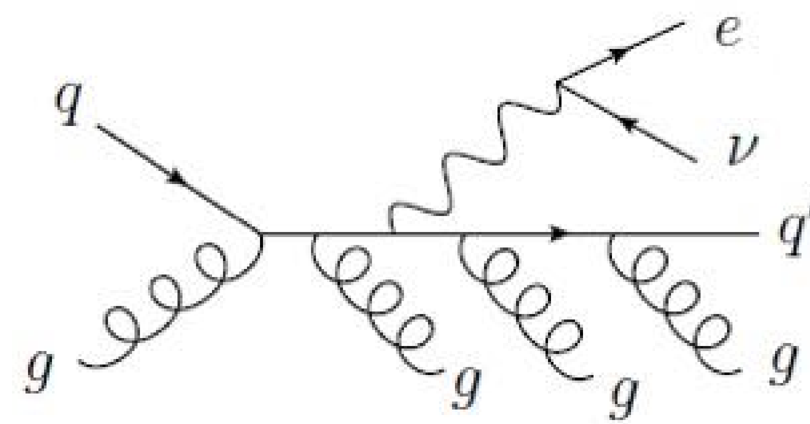



..times tree amplitude




squared amplitude

Born

Virtual

$$\mathbf{d}\sigma_{\mathbf{n}}^{(\mathbf{1})} \approx |\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\mathbf{\Phi}_{\mathbf{n-2}} + \mathbf{2}\mathrm{Re}(\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})\dagger}\mathbf{M}_{\mathbf{n}}^{(\mathbf{1})})\mathbf{d}\mathbf{\Phi}_{\mathbf{n-2}} + |\mathbf{M}_{\mathbf{n+1}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\mathbf{\Phi}_{\mathbf{n-1}}$$

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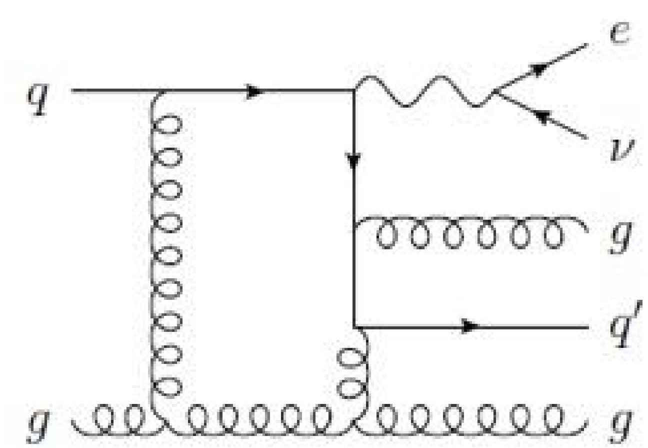


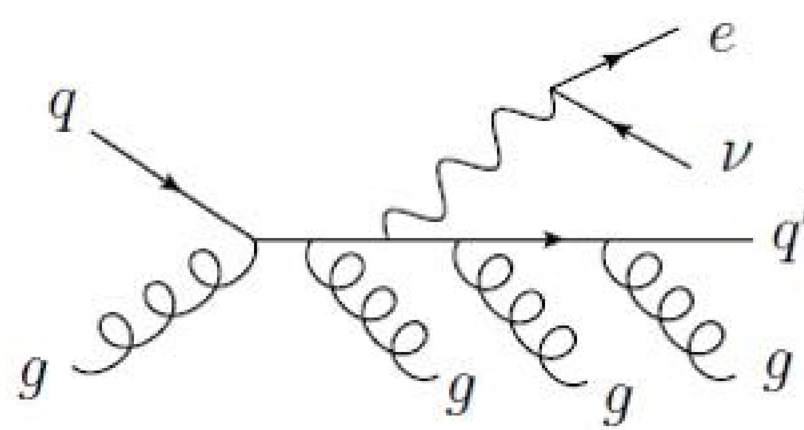


..times tree amplitude

squared amplitude

Born | Virtual | Real

$$\mathbf{d}\sigma_{\mathbf{n}}^{(\mathbf{1})} \approx |\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-2}} + \mathbf{2}\mathrm{Re}(\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})\dagger}\mathbf{M}_{\mathbf{n}}^{(\mathbf{1})})\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-2}} + |\mathbf{M}_{\mathbf{n+1}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-1}}$$

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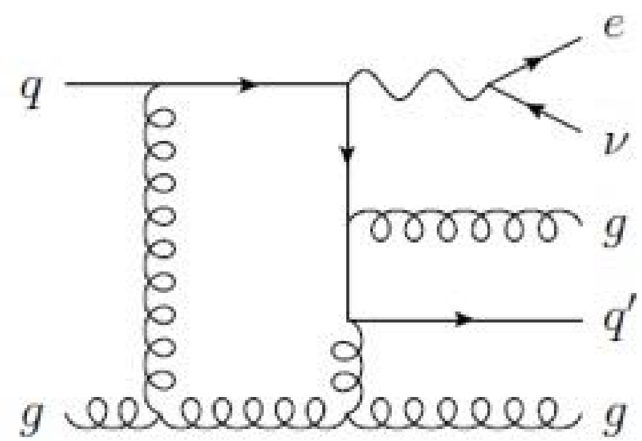


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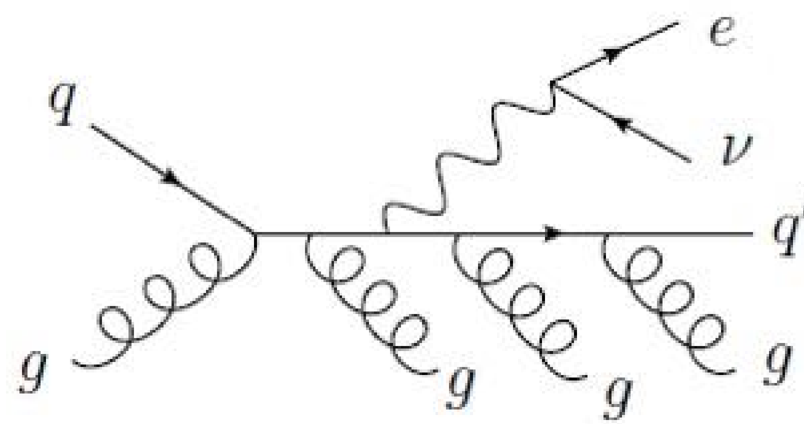


squared amplitude

Born Virtual Real

$$\mathbf{d}\sigma_{\mathbf{n}}^{(\mathbf{1})} \approx |\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-2}} + \mathbf{2}\mathrm{Re}(\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})\dagger}\mathbf{M}_{\mathbf{n}}^{(\mathbf{1})})\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-2}} + |\mathbf{M}_{\mathbf{n+1}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-1}}$$

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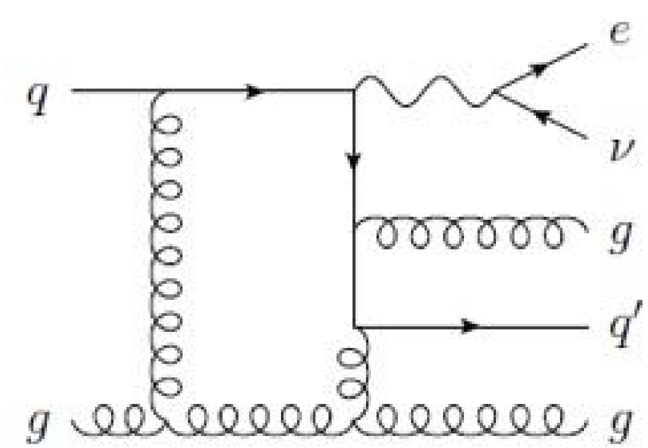


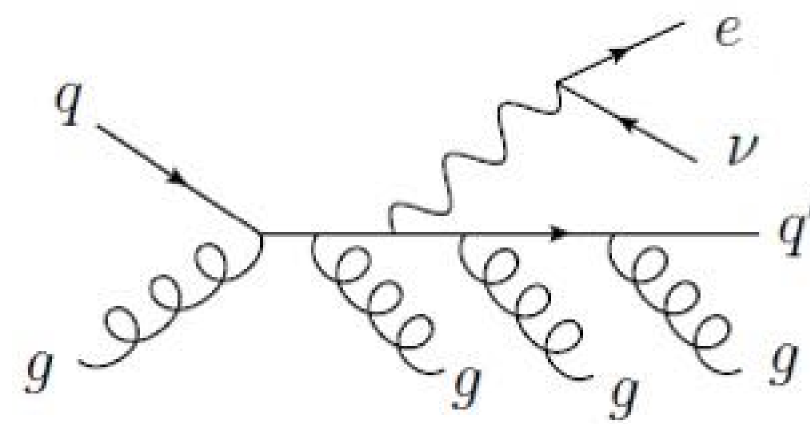


..times tree amplitude

squared amplitude

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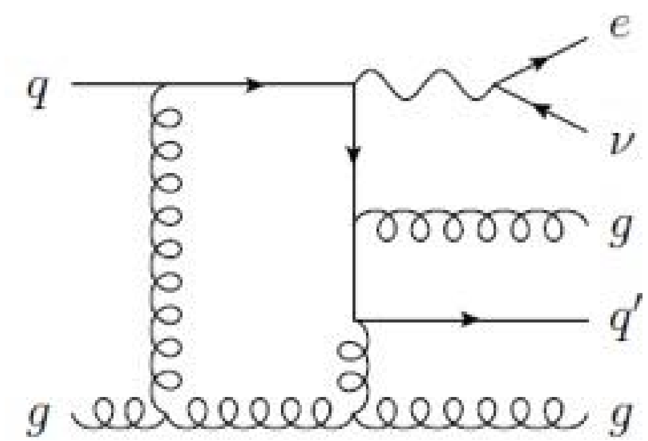
$$\mathbf{d}\hat{\sigma}_{\mathbf{a_1a_2}}^{\mathrm{hard}} = \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{\mathrm{born}} + \left\{ \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{\mathrm{virt}} + \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{\mathrm{soft}} + \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{(\mathrm{coll,initial})} + \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{(\mathrm{coll,final})} \right\}$$

# Hard parton cross-section at NLO

Soft and collinear singularities of virtual and real contributions are cancelled by subtraction method

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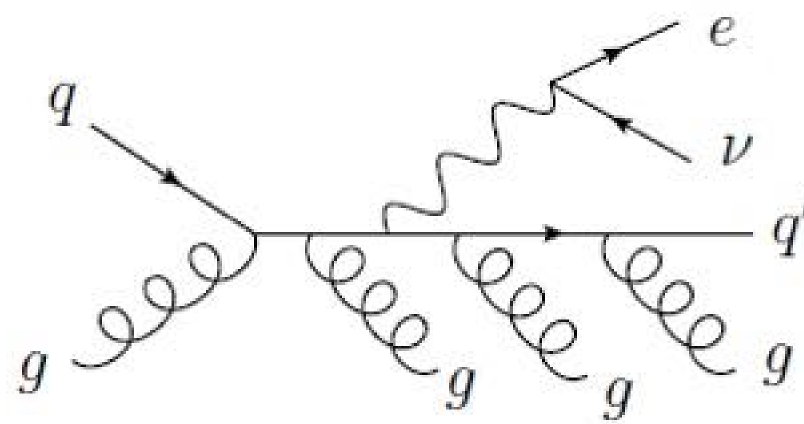


squared amplitude

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$$+ \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{(\mathrm{coll,counter})} + \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{(\mathrm{real,subtracted})}$$

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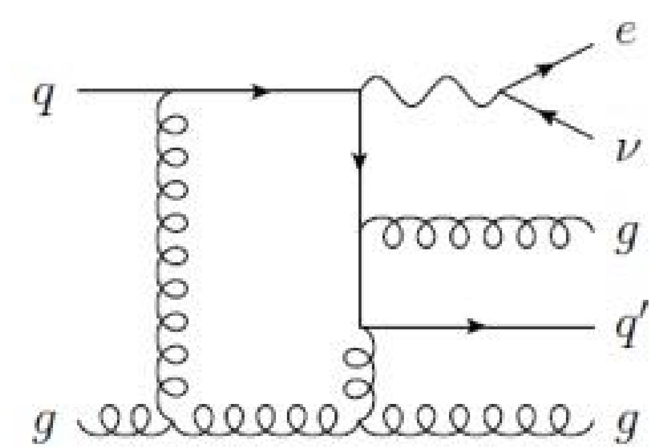
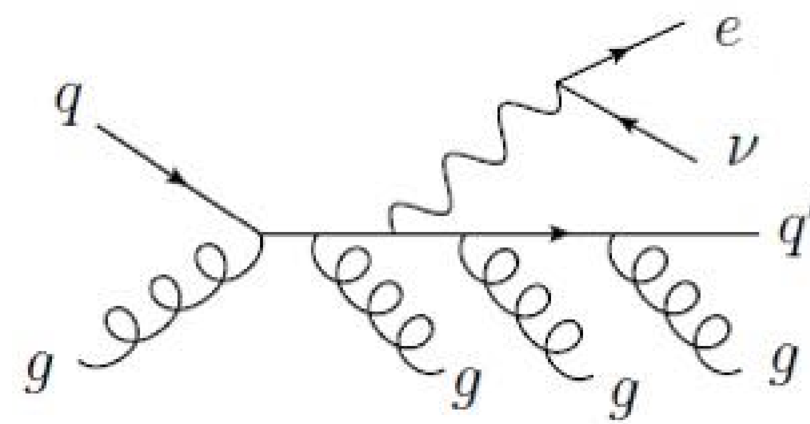
Soft and collinear singularities of virtual and real contributions are cancelled by subtraction method




..times tree amplitude

squared amplitude

Born Virtual Real

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$$+ \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{(\mathrm{coll,counter})} + \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{(\mathrm{real,subtracted})}$$

Finite

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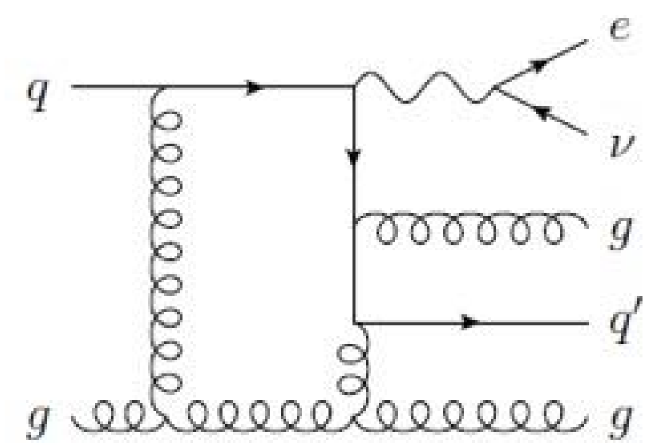


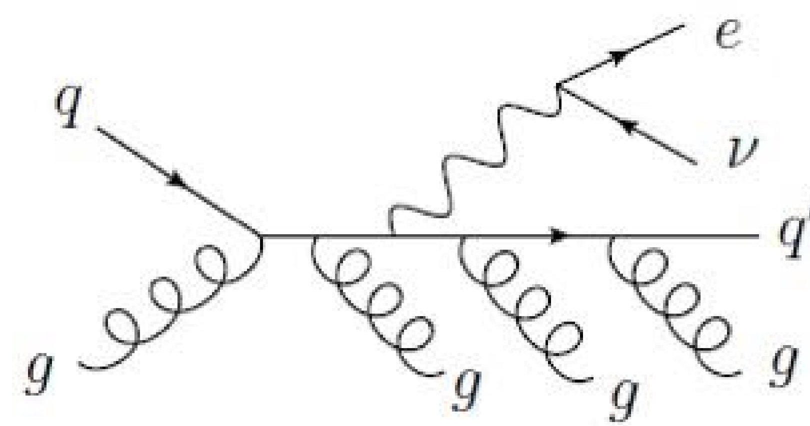


..times tree amplitude

squared amplitude

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$$\mathbf{d}\sigma_{\mathbf{n}}^{(\mathbf{1})} \approx |\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-2}} + \mathbf{2}\mathrm{Re}(\mathbf{M}_{\mathbf{n}}^{(\mathbf{0})\dagger}\mathbf{M}_{\mathbf{n}}^{(\mathbf{1})})\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-2}} + |\mathbf{M}_{\mathbf{n+1}}^{(\mathbf{0})}|^{\mathbf{2}}\mathbf{d}\boldsymbol{\Phi}_{\mathbf{n-1}}$$

$$\mathbf{d}\hat{\sigma}_{\mathbf{a_1a_2}}^{\text{hard}} = \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{\text{born}} + \Big\{\mathbf{d}\sigma_{\mathbf{a_1a_2}}^{\text{virt}} + \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{\text{soft}} + \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{(\text{coll,initial})} + \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{(\text{coll,final})}\Big\}$$

$$+ \mathbf{d}\sigma_{\mathbf{a_1a_2}}^{(\text{coll,counter})} + \underbrace{\mathbf{d}\sigma_{\mathbf{a_1a_2}}^{(\text{real,subtracted})}}_{\text{Finite}}$$

Finite

# Calculating higher order corrections with many legs and loop is hard:

# Calculating higher order corrections with many legs and loop is hard:

- Traditional Feynman-diagram approach with Passarino-Veltman reduction has factorial growth with the number of the legs

- Major improvements in the last 15 years

  recursion relations, double box loop integral, IBP identities and Lorenz invariance identities for loop integrals, Laporta algorithm, understanding the structure of soft and collinear singularities, unitarity method, OPP reduction....

- NNLO and multi-leg NLO spectacular progress

- NNLO evolution of patron densities (Moch, Vermaseren,Vogt,2005)

- Subtraction methods for the real contributions (diploe, FKS)

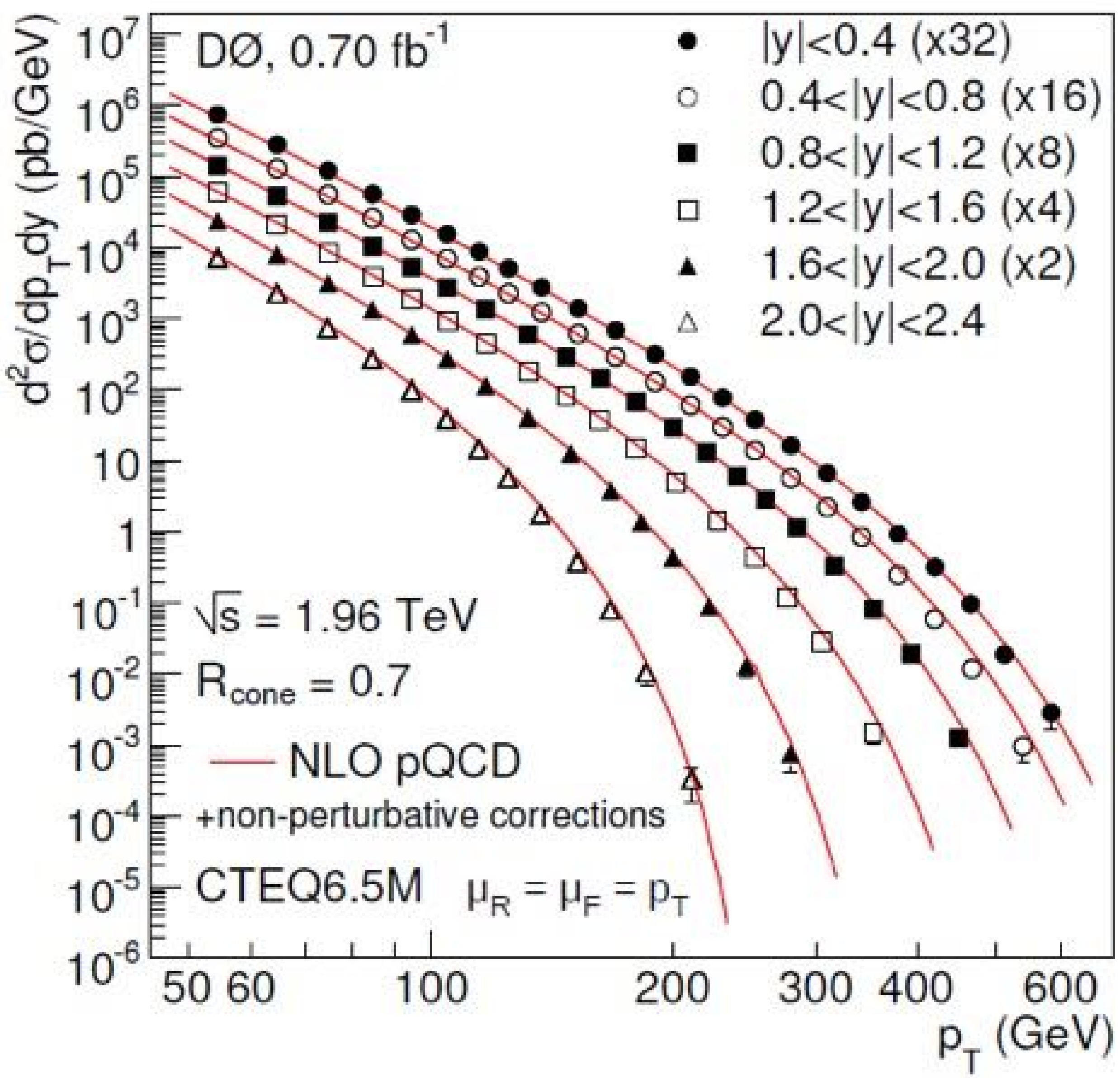
- Embedding into shower Monte Carlo programs

# Engineering progress for NLO:

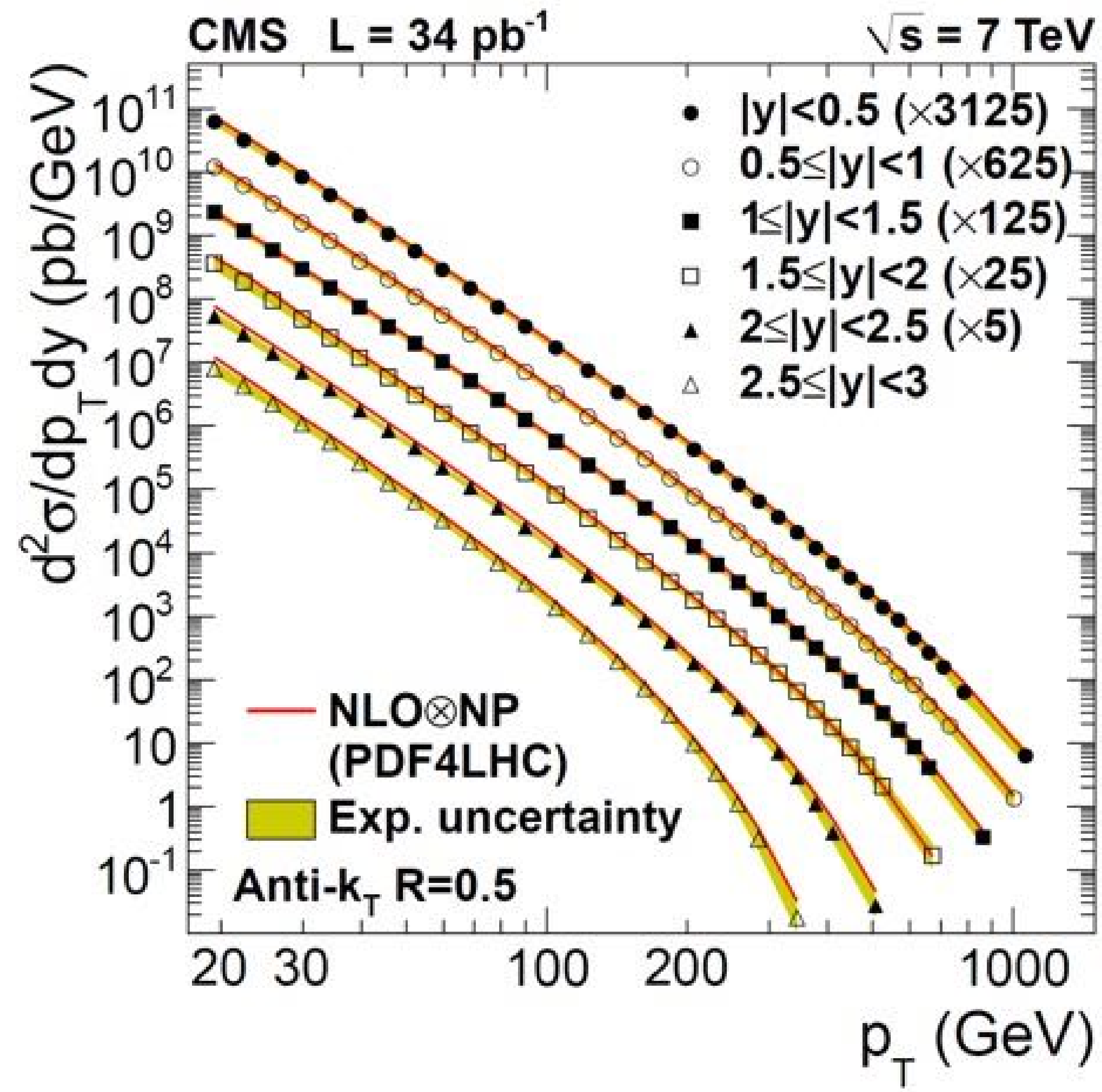
* Application of unitarity to the calculation of one loop diagrams.
* Improvements in traditional techniques for Passarino-Veltman reduction.
* Generation of one loop integrand by parametric fit, (in dimensions D, OPP).
* Automatic procedures for generation of graphs and the consequent integrands.
* Automatic procedures for the generation of counter terms to implement the real virtual subtraction.
* Tabulation and numerical implementation of all integrals.

# Jet production at Tevatron and the LHC

$p\bar{p} \to$ jet + X     Tevatron




$pp \to$ jet + X     LHC




- NLO QCD predictions (Ellis, ZK, Soper 1990)

# 5-loop NNLO calculation, 2-loop NNLO, 1-loop NLO multi-leg

Hadronic Z and tau decays, Baikov, Chetyrkin,Kuhn (2008)

$$R(s) = 1 + a_s + 1.4097a_s^2 - 12.76703a_s^3 - 80.0075a_s^4..$$

$$\alpha_s(M_z)^{\mathrm{NNNLO}} = 0.1190 \pm 0.0026_{\mathrm{exp}}$$

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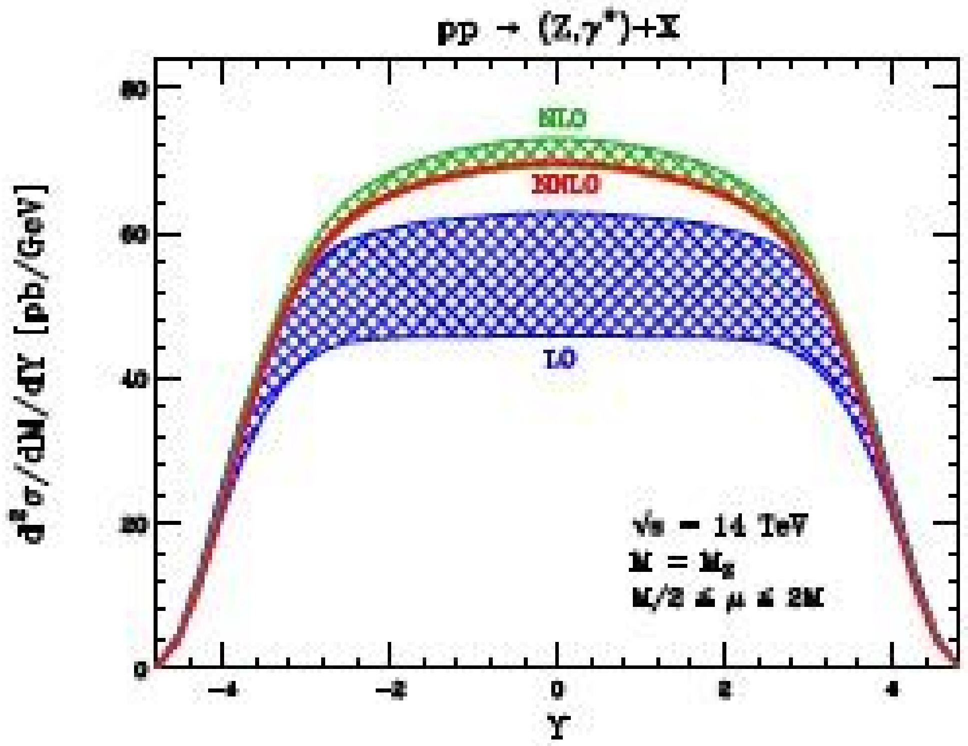
Drell - Yan production
Anastasiou, Dixon, Melnikov
Petriello (2004)



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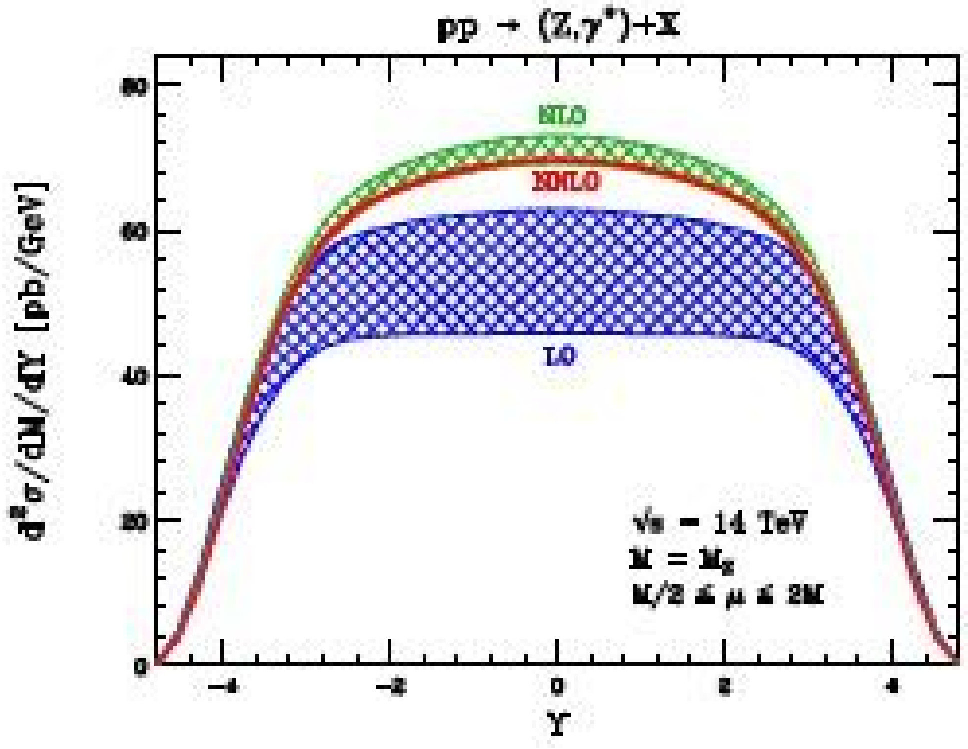
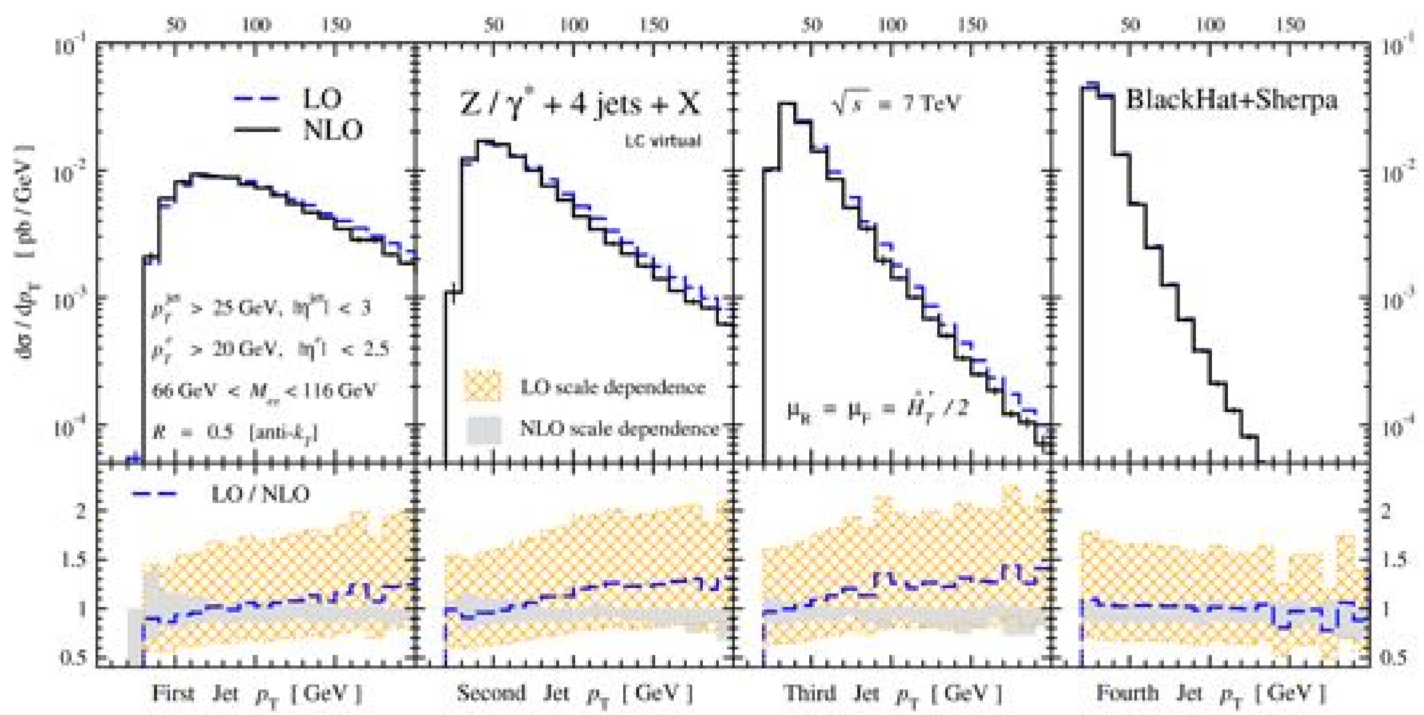
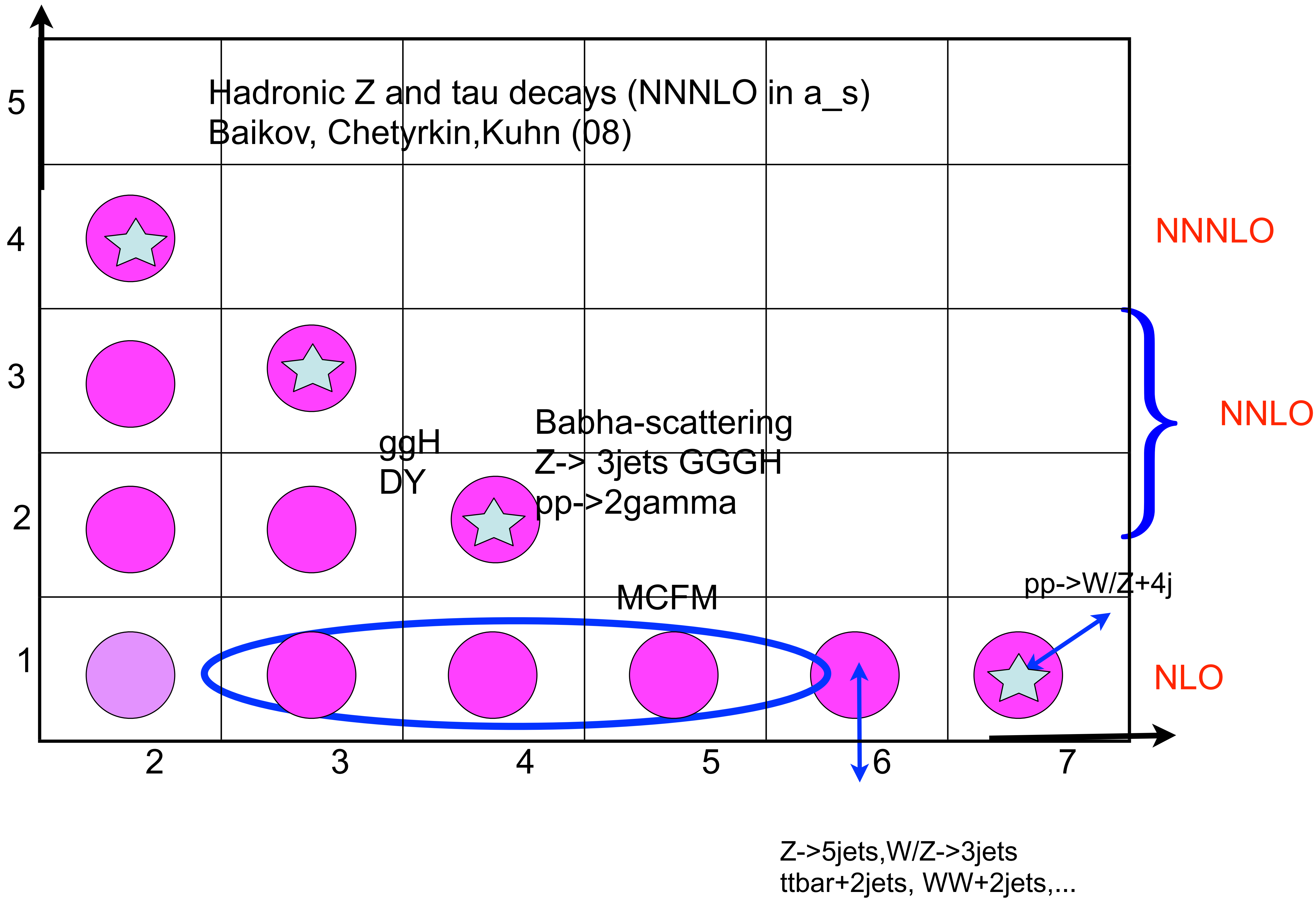



$Z/\gamma^* + 4$jet production
BlackHat (2011)



# What is available?



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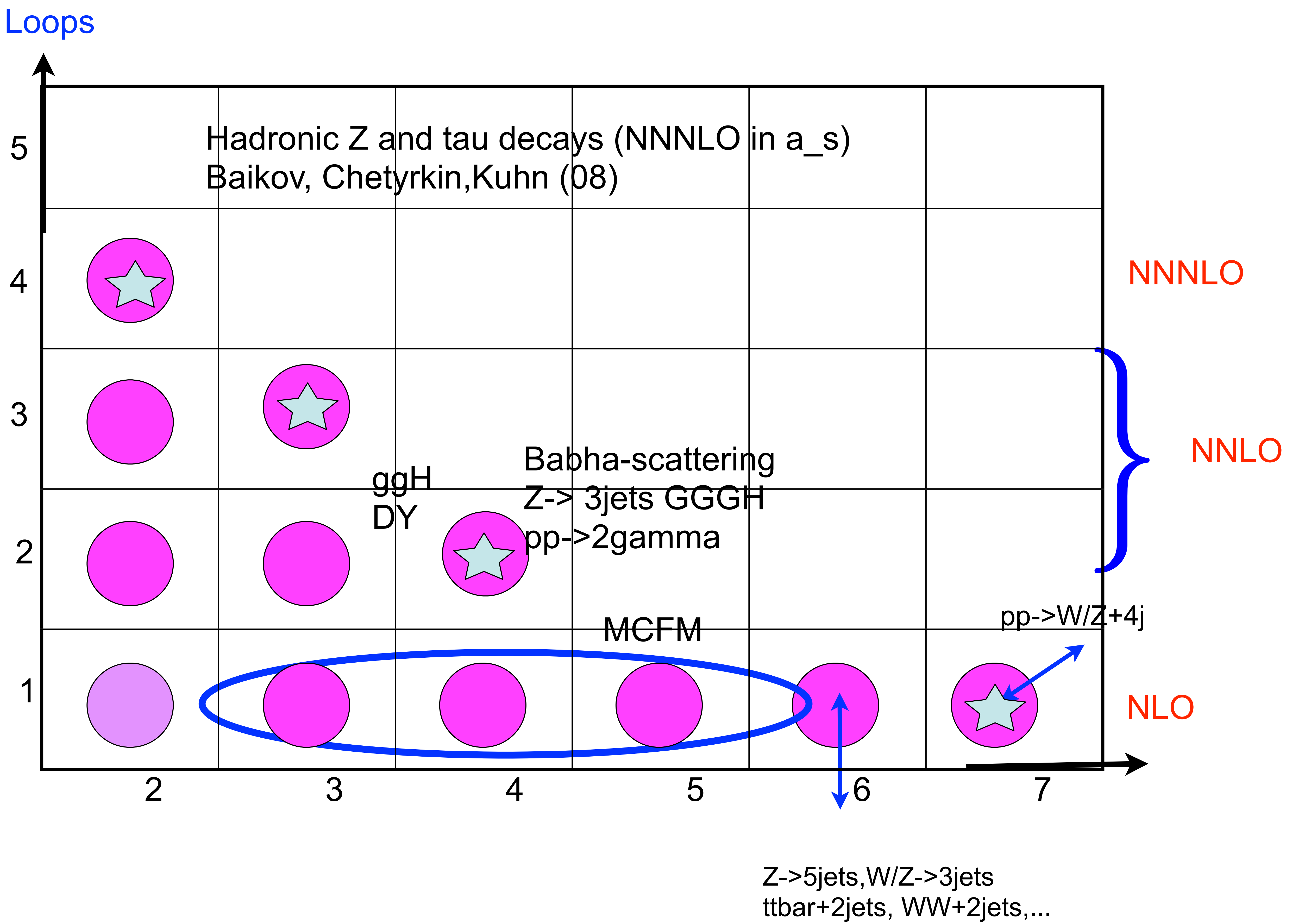

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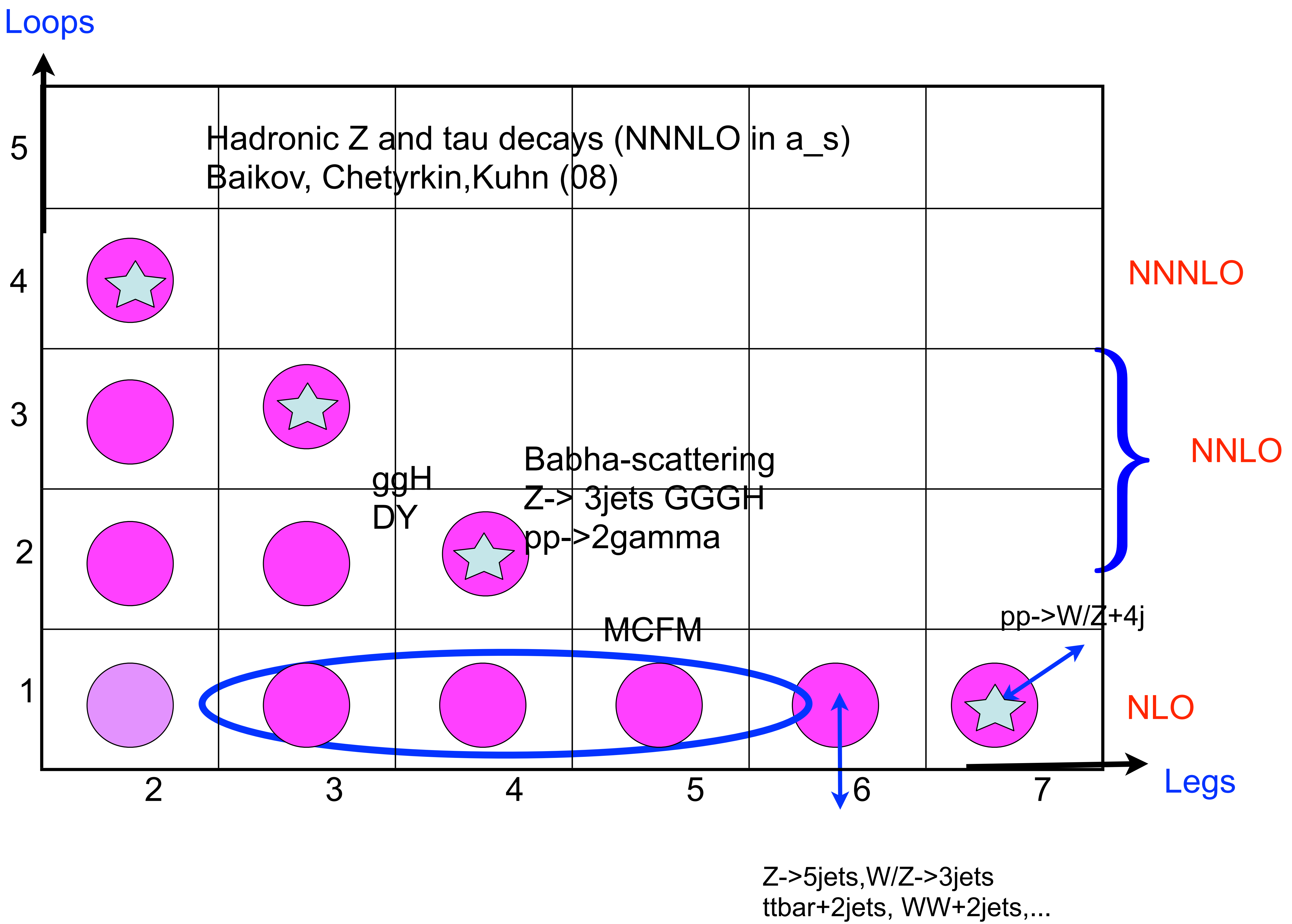
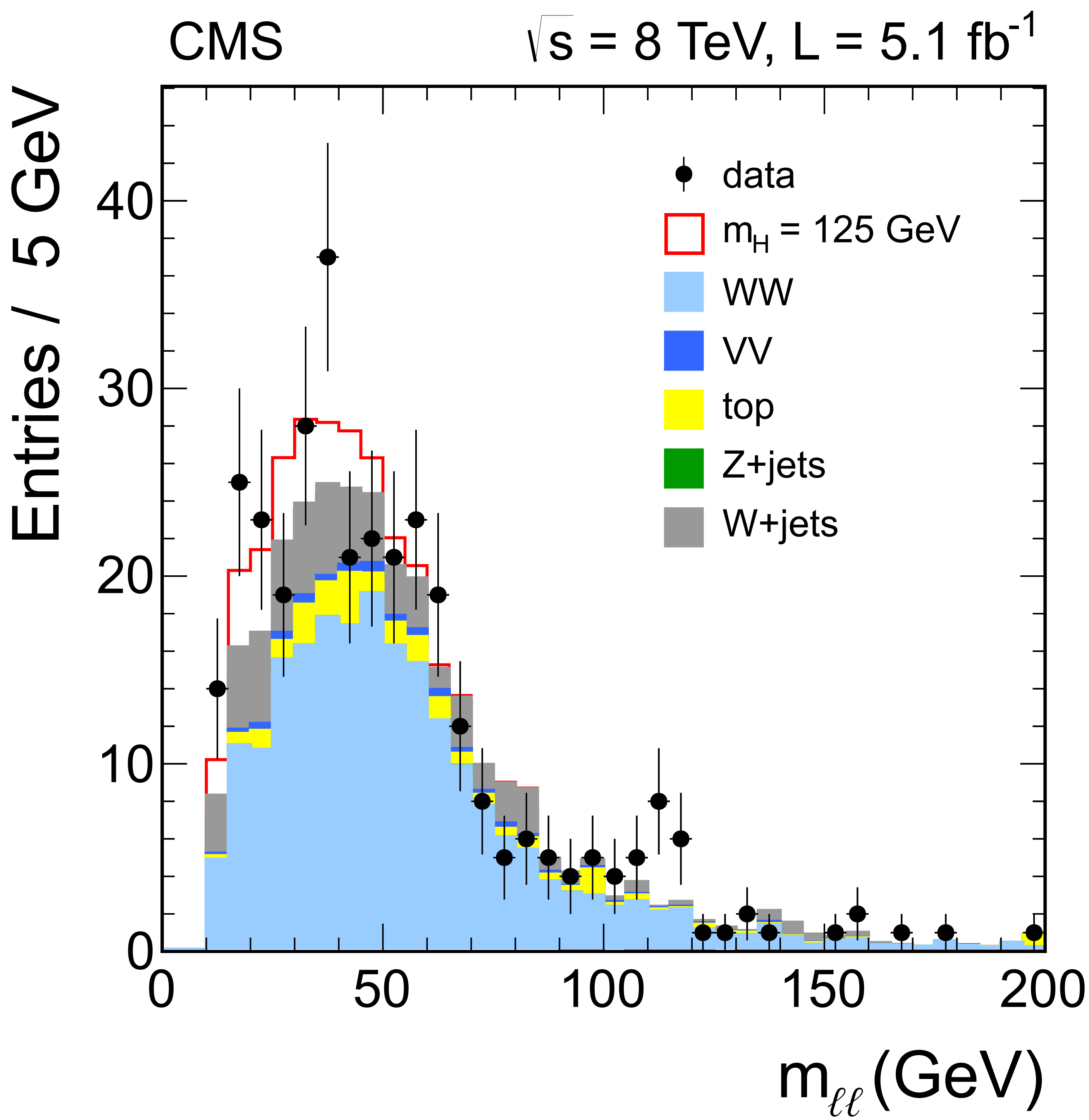


# Higgs search by CMS



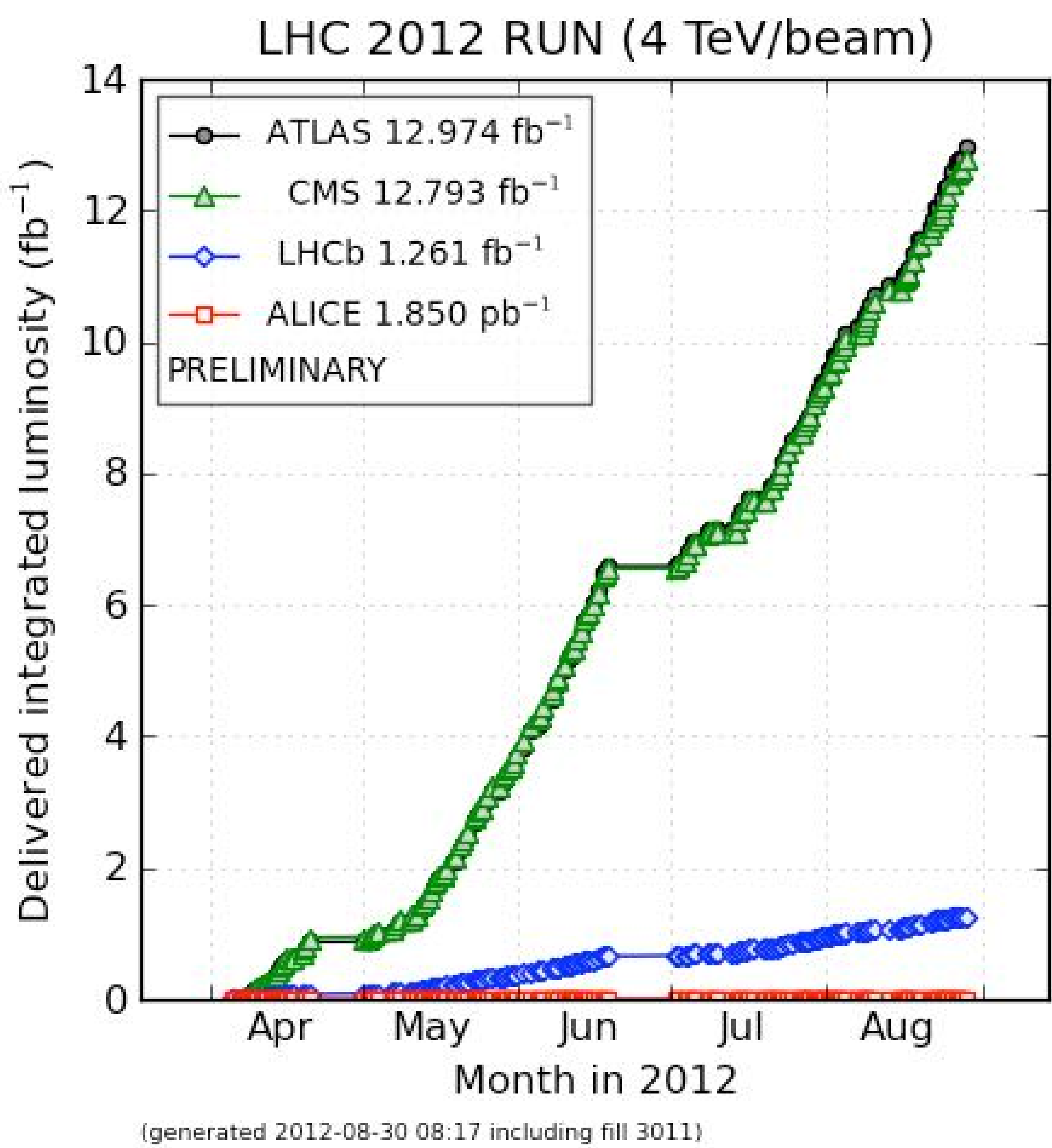




Figure 7: Distribution of $m_{\ell\ell}$ for the zero-jet e$\mu$ category in the H $\to$ WW search at 8 TeV. The signal expected from a Higgs boson with a mass $m_{\mathrm{H}} = 125$ GeV is shown added to the background.

# New physics search, complex final states

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SUSY searches

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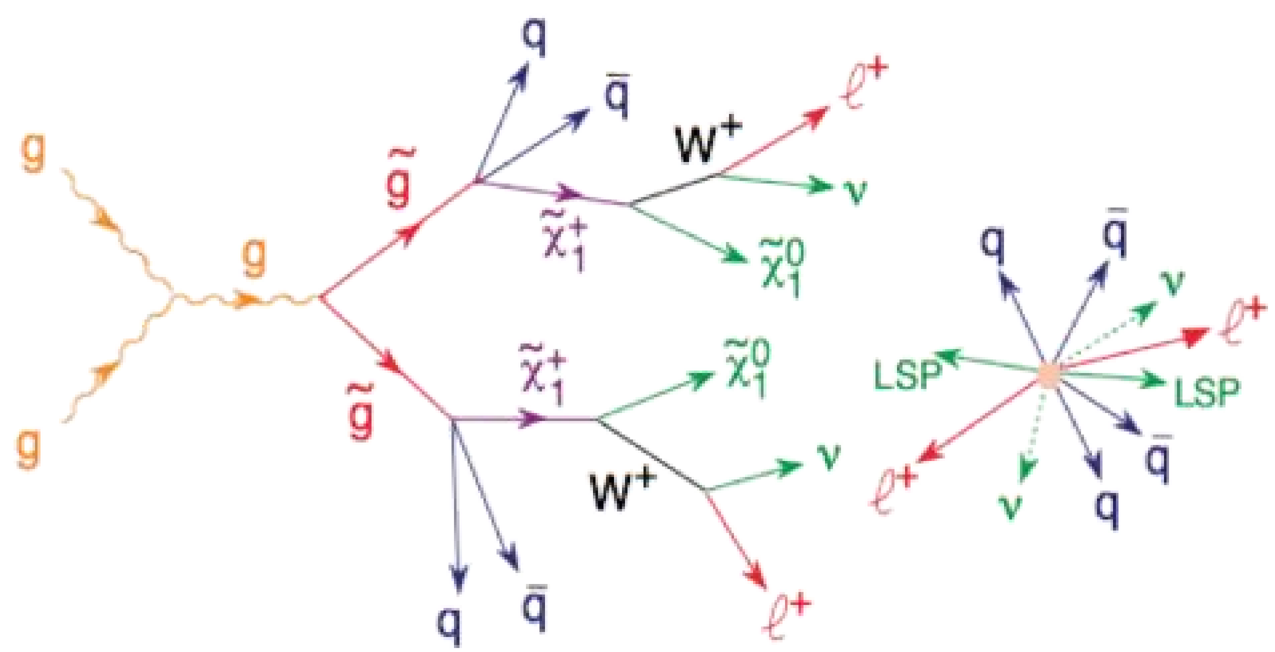
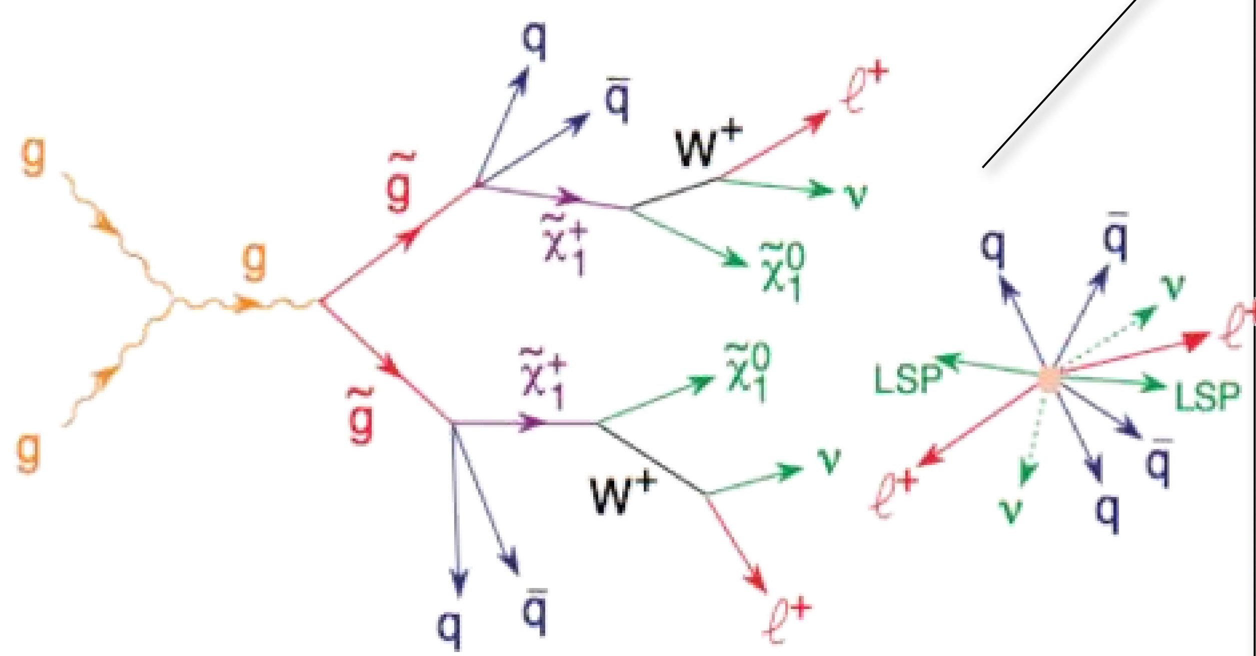
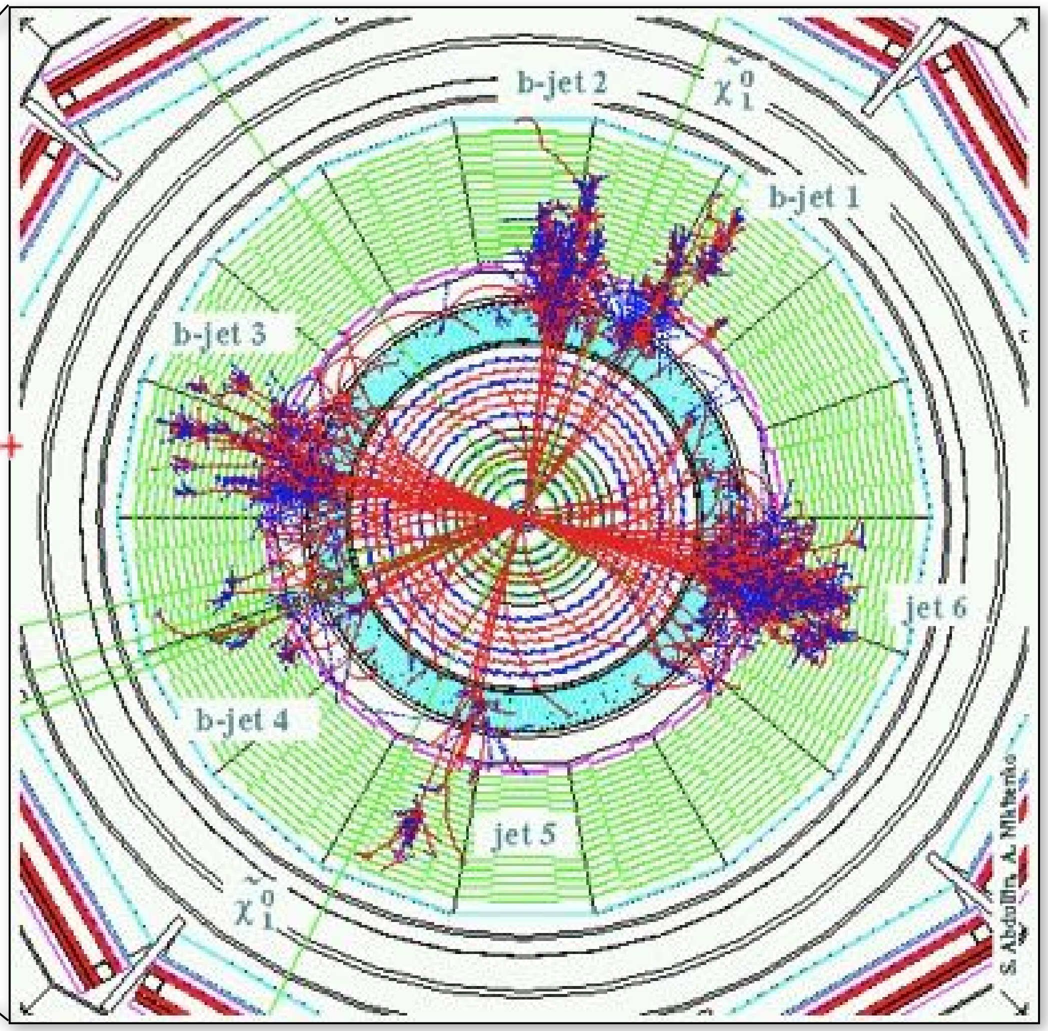


# New physics search, complex final states

SUSY searches

Typical event signature in CMS at LHC:

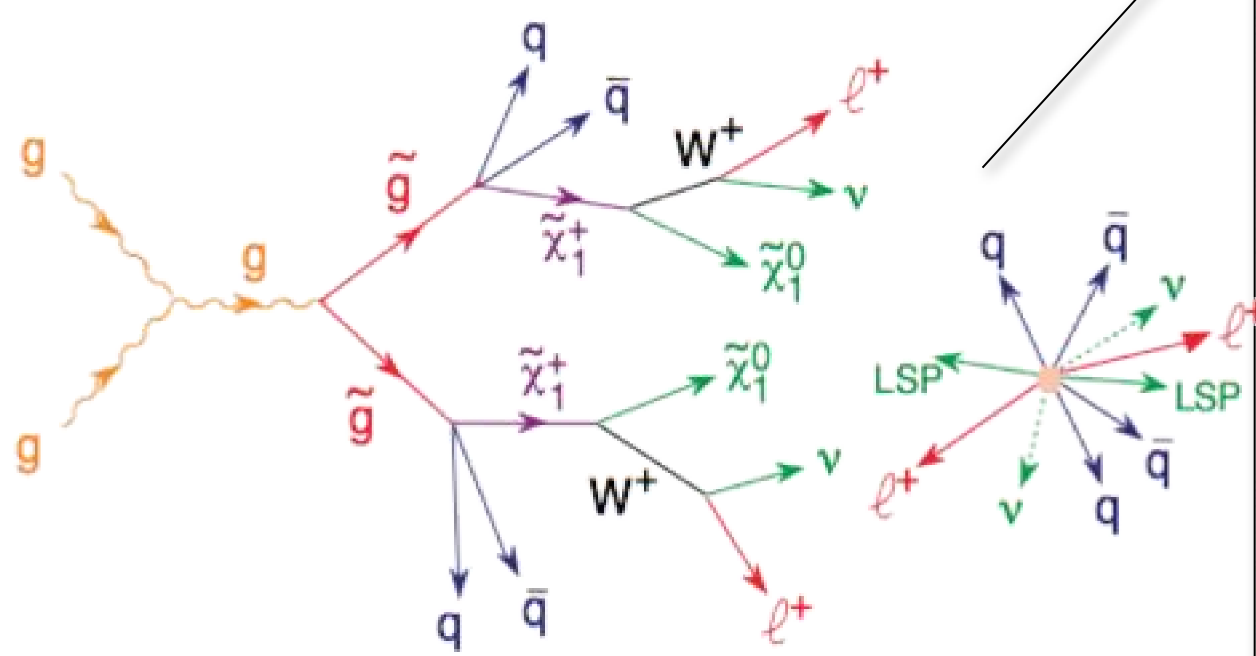



n leptons + n jets + missing $E_T$

# New physics search, complex final states

SUSY searches

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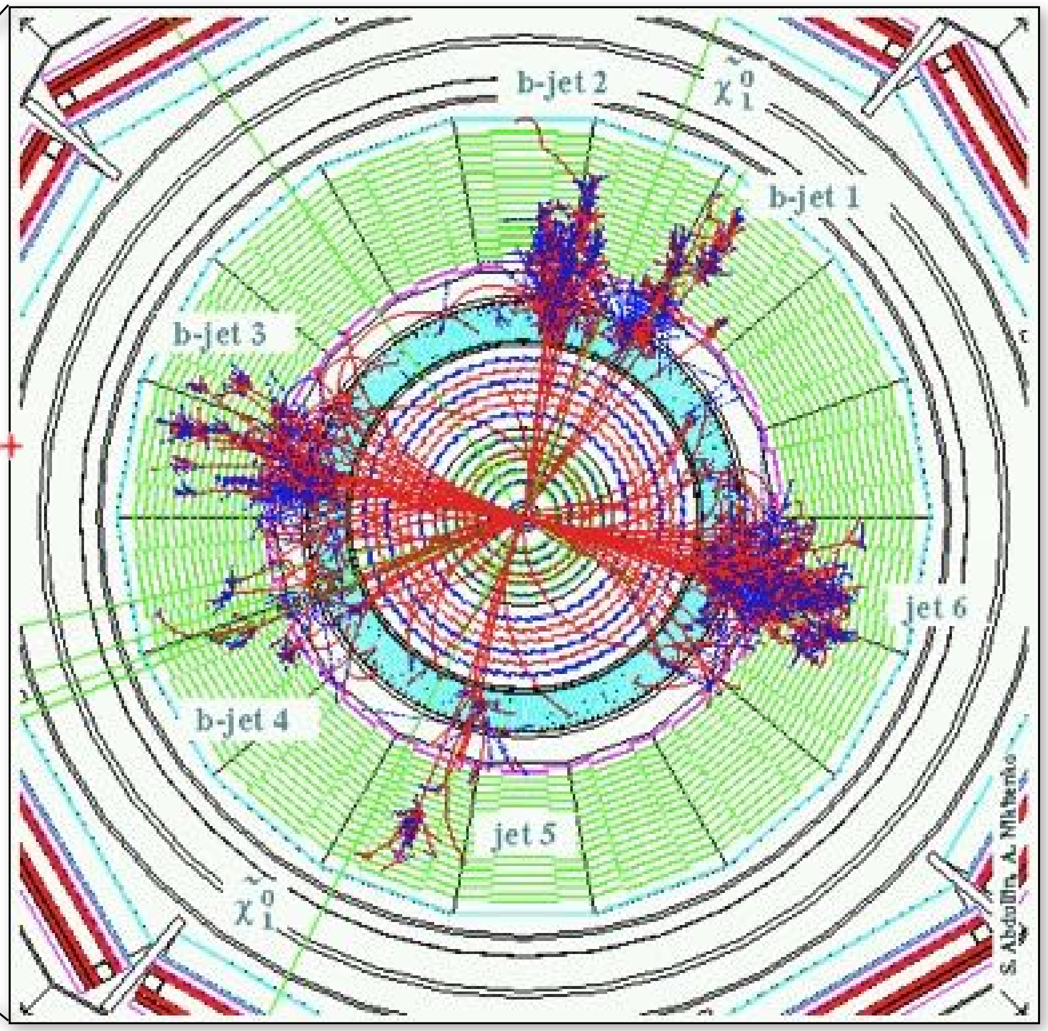


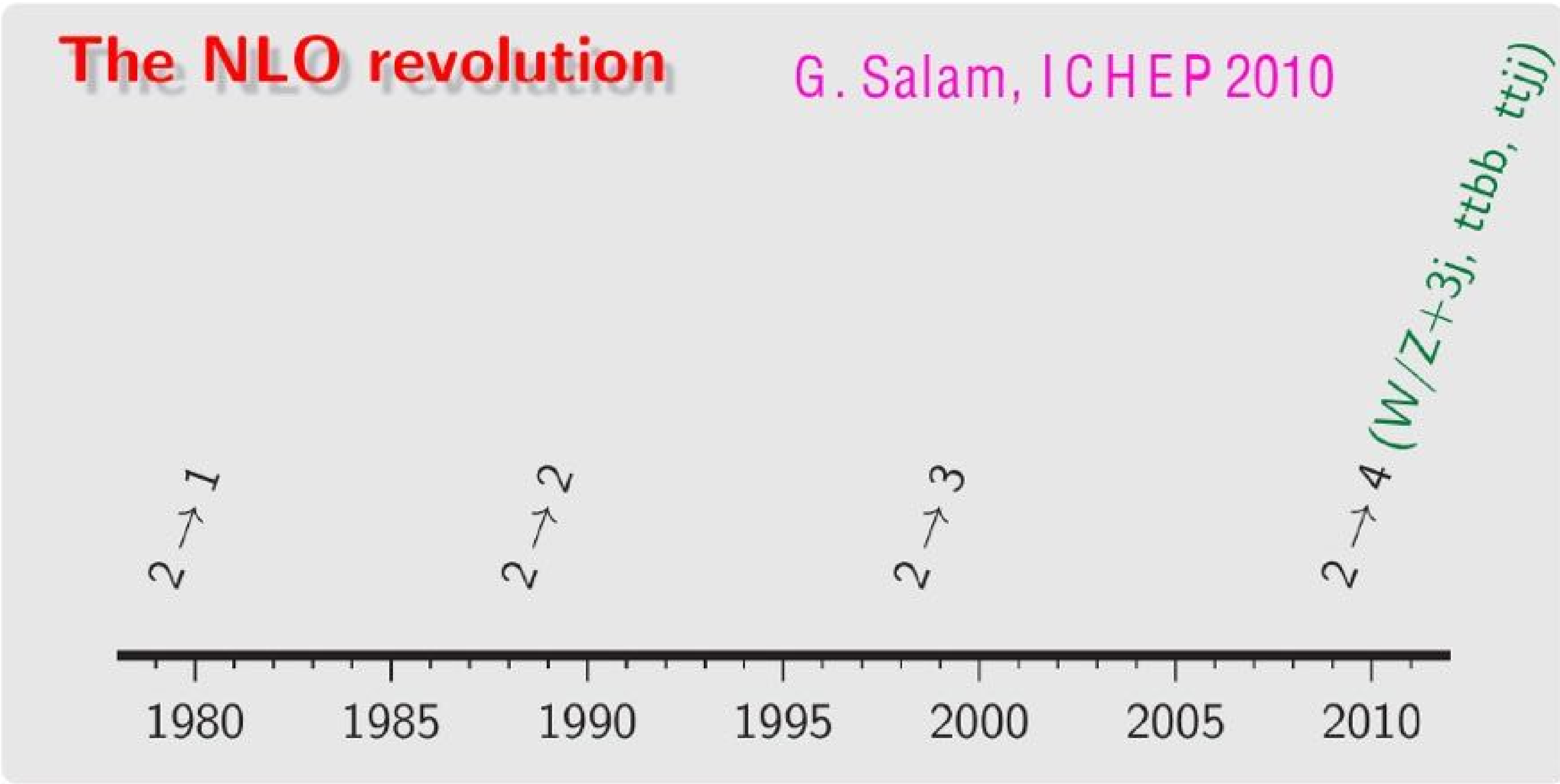

$* \quad p + p \to 1, 2, \ldots 7 \text{ jets, inclusive}$

$* \quad p + p \to W/Z + 1, 2, \ldots 5 \text{ jets,}$

$* \quad p + p \to t + \bar{t} + 1,\ 2,\ 3 \text{ jets}$

n leptons + n jets + missing $E_T$

# Schematic view of hard processes at the LHC




2009: NLO $W$+3$j$ [Rocket: Ellis, Melnikov & Zanderighi] [unitarity]
2009: NLO $W$+3$j$ [BlackHat: Berger et al] [unitarity]
2009: NLO $t\bar{t}b\bar{b}$ [Bredenstein et al] [traditional]
2009: NLO $t\bar{t}b\bar{b}$ [HELAC-NLO: Bevilacqua et al] [unitarity]
2009: NLO $q\bar{q} \to b\bar{b}b\bar{b}$ [Golem: Binoth et al] [traditional]
2010: NLO $t\bar{t}jj$ [HELAC-NLO: Bevilacqua et al] [unitarity]
2010: NLO $Z$+3$j$ [BlackHat: Berger et al] [unitarity]

# Les Houches Experimenters' Wish List

2010

| process wanted at NLO | background to |
| --- | --- |
| 1. $pp \to VV +$ jet | $t\bar{t}H$, new physics<br>Dittmaier, Kallweit, Uwer; Campbell, Ellis, Zanderighi |
| 2. $pp \to H + 2$ jets | $H$ in VBF BCDEGMRSW; Campbell, Ellis, Williams<br>Campbell, Ellis, Zanderighi; Ciccolini, Denner Dittmaier |
| 3. $pp \to t\bar{t}b\bar{b}$ | $t\bar{t}H$ Bredenstein, Denner Dittmaier, Pozzorini;<br>Bevilacqua, Czakon, Papadopoulos, Pittau, Worek |
| 4. $pp \to t\bar{t} + 2$ jets | $t\bar{t}H$ Bevilacqua, Czakon, Papadopoulos, Worek |
| 5. $pp \to VVb\bar{b}$ | VBF $\to H \to VV$, $t\bar{t}H$, new physics |
| 6. $pp \to VV + 2$ jets | VBF $\to H \to VV$<br>VBF: Bozzi, Jäger, Oleari, Zeppenfeld |
| 7. $pp \to V + 3$ jets | new physics<br>Berger Bern, Dixon, Febres Cordero, Forde, Gleisberg, Ita,<br>Kosower, Maitre; Ellis, Melnikov, Zanderighi |
| 8. $pp \to VVV$ | SUSY trilepton<br>Lazopoulos, Melnikov, Petriello; Hankele, Zeppenfeld;<br>Binoth, Ossola, Papadopoulos, Pittau |
| 9. $pp \to b\bar{b}b\bar{b}$ | Higgs, new physics GOLEM |

Feynman diagram methods

now joined by

on-shell methods

table courtesy of C. Berger

# N=4 sYM Lagrangian

The field content of the model:
one gauge field $\mathbf{A}_\mu$, four Majorana fermions $\psi_\mathbf{i}$ , three real scalars ,$\mathbf{X_p}$ and three real pseudo-scalars $\mathbf{Y_q}$ . All fields are in the adjoint representation.

$$\mathcal{L} = \mathrm{tr}\Big\{-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} + \bar{\psi}_\mathbf{i}\not{\mathbf{D}}\psi_\mathbf{i} + \mathbf{D}^\mu\mathbf{X_p}\mathbf{D}_\mu\mathbf{X_p} + \mathbf{D}^\mu\mathbf{Y_q}\mathbf{D}_\mu\mathbf{Y_q}$$
$$- \mathbf{ig}\bar{\psi}_\mathbf{i}\alpha^\mathbf{p}_\mathbf{ij}[\mathbf{X_p},\psi_\mathbf{j}] + \mathbf{g}\bar{\psi}_\mathbf{i}\gamma_\mathbf{5}\beta^\mathbf{q}_\mathbf{ij}[\mathbf{Y_q},\psi_\mathbf{j}]$$
$$+\frac{\mathbf{g^2}}{\mathbf{2}}\Big([\mathbf{X_l},\mathbf{X_k}][\mathbf{X_l},\mathbf{X_k}] + [\mathbf{Y_l},\mathbf{Y_k}][\mathbf{Y_l},\mathbf{Y_k}] + \mathbf{2}[\mathbf{X_l},\mathbf{Y_k}][\mathbf{X_l},\mathbf{Y_k}]\Big)\Big\},$$

$$\alpha^\mathbf{1} = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix},\quad \alpha^\mathbf{2} = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix},\quad \alpha^\mathbf{3} = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix},$$

$$\beta^\mathbf{1} = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix},\quad \beta^\mathbf{2} = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix},\quad \beta^\mathbf{3} = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}$$

$$\beta(\mathbf{g}) = \mu^\mathbf{2}\frac{\partial}{\partial\mu^\mathbf{2}} = -\frac{\mathbf{g^3}}{\mathbf{16}\pi^\mathbf{2}}\left(\frac{\mathbf{11}}{\mathbf{3}} - \frac{\mathbf{2}}{\mathbf{3}}\mathbf{n_f} - \frac{\mathbf{1}}{\mathbf{6}}\mathbf{n_s}\right)\mathbf{C_A}$$

$$\beta(\mathbf{g}) = \mu^\mathbf{2}\frac{\partial}{\partial\mu^\mathbf{2}} = -\frac{\mathbf{g^3}}{\mathbf{16}\pi^\mathbf{2}}\left(\frac{\mathbf{11}}{\mathbf{3}} - \frac{\mathbf{2}}{\mathbf{3}}\mathbf{4} - \frac{\mathbf{1}}{\mathbf{6}}\mathbf{6}\right)\mathbf{C_A} = \mathbf{0}$$

Beta function vanishes to all orders in PT.

# N=4 sYM Lagrangian

- The scale invariance of the classical Lagrange density remains a symmetry at the quantum level.

- Nc limit, ‘t Hooft coupling $\lambda = N_c g^2$, relation to string theory?

- It has a much reacher set of symmetries:
  superconformal symmetry, integrability (hidden symmetry)

- Local gauge invariant composite operators (observables) with calculable anonalous dimensions

- AdS/CFT relates it to D=2 integrability, string theory in AdS, strong and weak coupling

# On-shell methods of calculating amplitudes in QCD and N=4 sYM at large N

Helicity method
unitarity cuts (BDDK), OPP, cuts in D-dim.
soft and collinear limits
Anomalous dimensions at NNLO
Regge limit
Automated NLO codes for phenomenology (MCFM, BH, Sherpa, Powheg)

Complexification of external moment with twistors
CSW and BCFW recursion relations
BDS Ansatz; AdS/CFT
Amplitudes and Wilson loops
Dual conformal invariance,
Yangian symmetry of tree amplitudes,
Integrability, Bethe Ansatz
Relation to N=8 supergratvity

QCD $\longleftrightarrow$ N=4 sYM

# Planar N=4 sYM and QCD in perturbation theory

- Tree-level amplitudes calculated in N=4 SYM could be carried over to QCD since all possible spins appear
- Generalized unitarity was invented in the simple case of
  N=4 SYM
  Any 4D gauge theory can be viewed as N=4 sYM with some particles
  and interactions added and removed
- Regge-limit and transcendentality in N=4 sYM and QCD
- New tree level recursion relations BCFW
- Dual conformal invariance, Yangian symmetry for tree amplitudes,
  2- loop results for 5, 6 point amplitudes

# Parma International School of Theoretical Physics

## September 3 - September 8, 2012

## Parma, Italy

***Scattering Amplitudes in QCD, Supersymmetric Gauge Theories and Supergravity***

# OUTLINE

Lecture 1: QCD and review its main features.

Lecture 2: One loop tensor integrals and their reduction

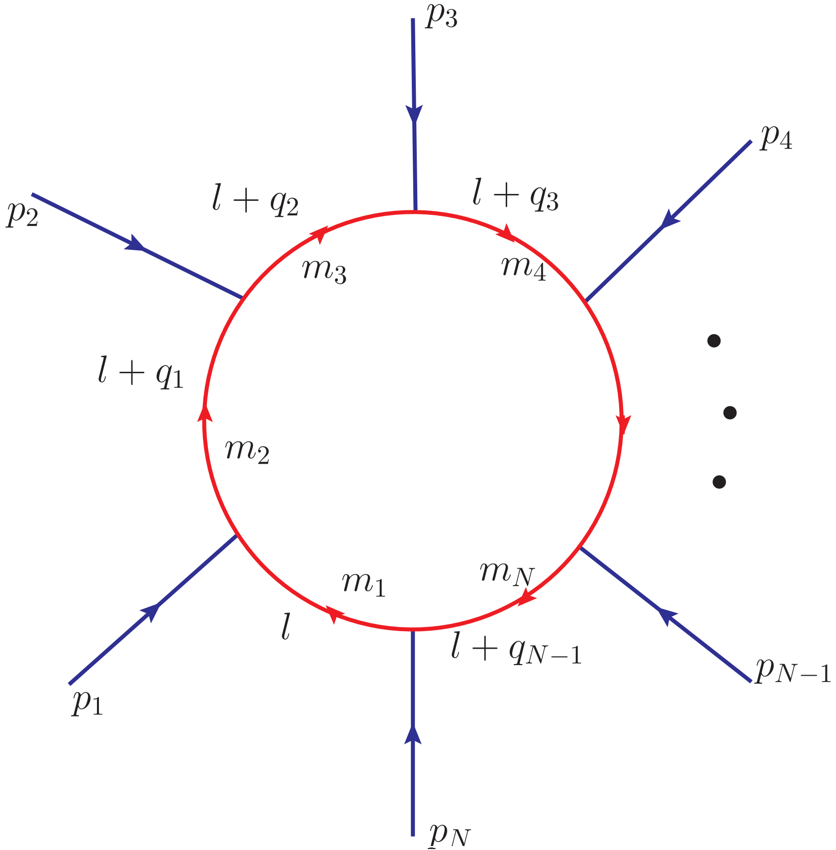
Lecture 3: Unitarity method and amplitudes

Lecture 4: Analytic and numerical computations

Lecture 5: Different implementations, Outlook

**One-loop calculations in quantum field theory: from Feynman diagrams to unitarity cuts.**
R.Keith Ellis,,Zoltan Kunszt, Kirill Melnikov, Giulia Zanderighi,
**to appear in Phys. Rep. arXiv:1105.4319** [hep-ph]

# One loop calculation with traditional methods



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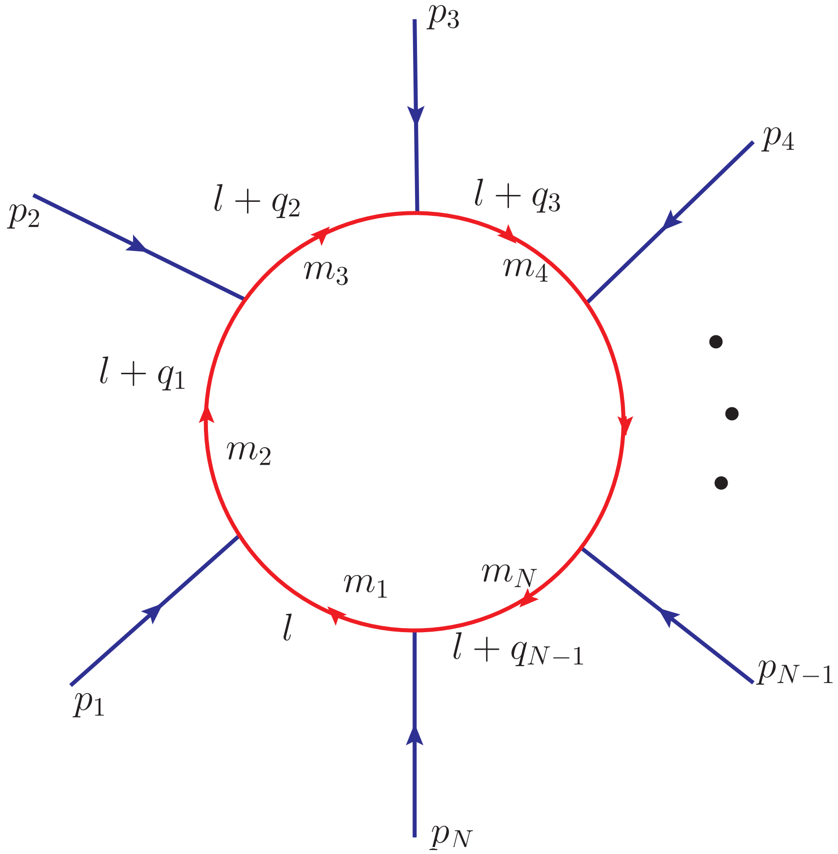


$\mathcal{N}(\mathbf{l})$ =1 one loop scalar, otherwise one loop tensor integral

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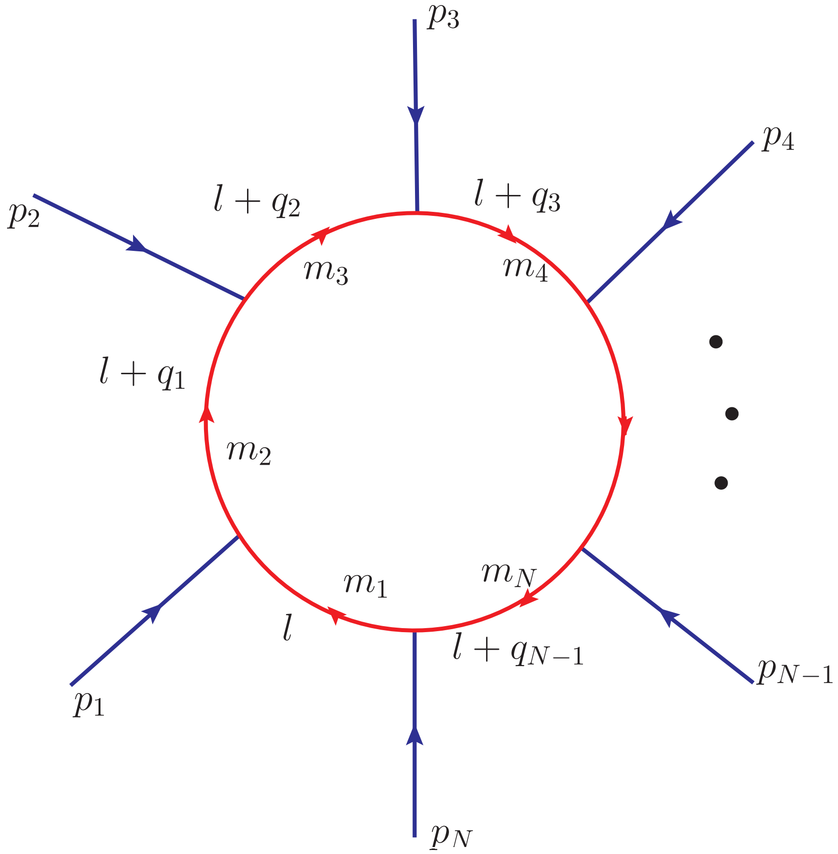


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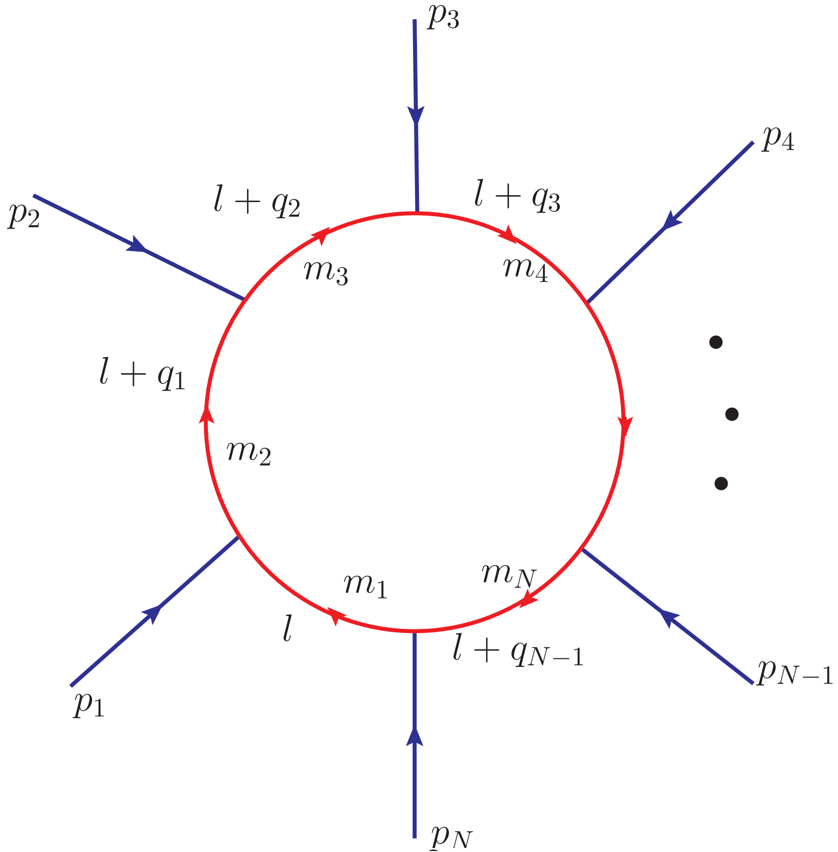


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$$\mathbf{I_N} \sim \int \frac{\mathrm{d}^4\mathbf{l}}{(\mathbf{2}\pi)^4} \frac{\mathcal{N}(\mathbf{l})}{((\mathbf{l}+\mathbf{q_0})^2 - \mathbf{m_1^2})((\mathbf{l}+\mathbf{q_1})^2 - \mathbf{m_2^2})....((\mathbf{l}+\mathbf{q_{N-1}})^2 - \mathbf{m_N^2})}$$

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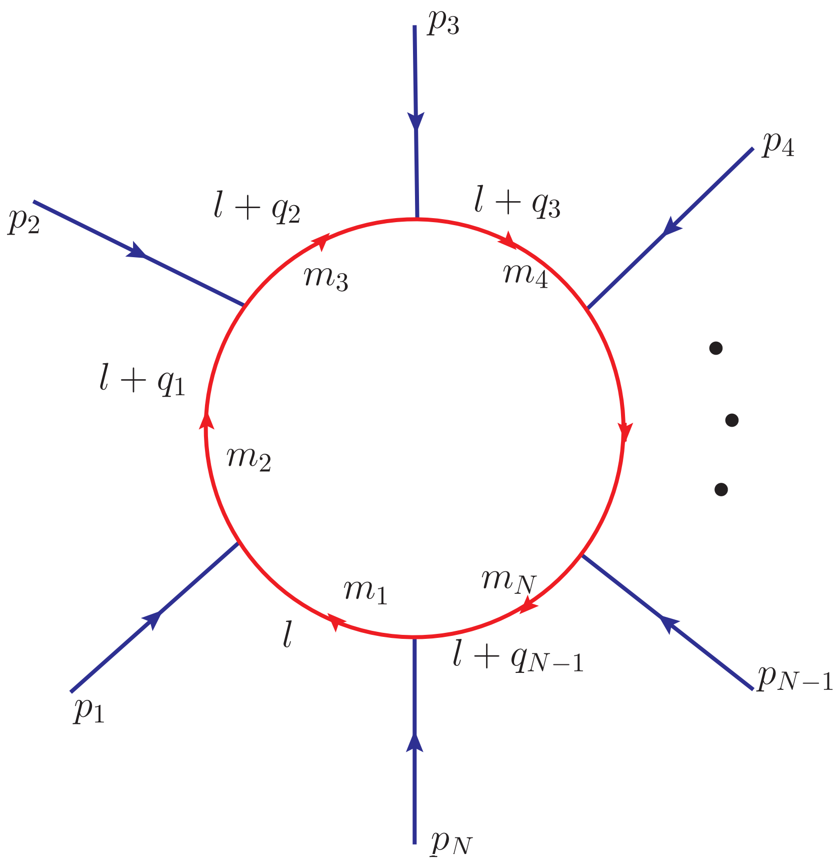


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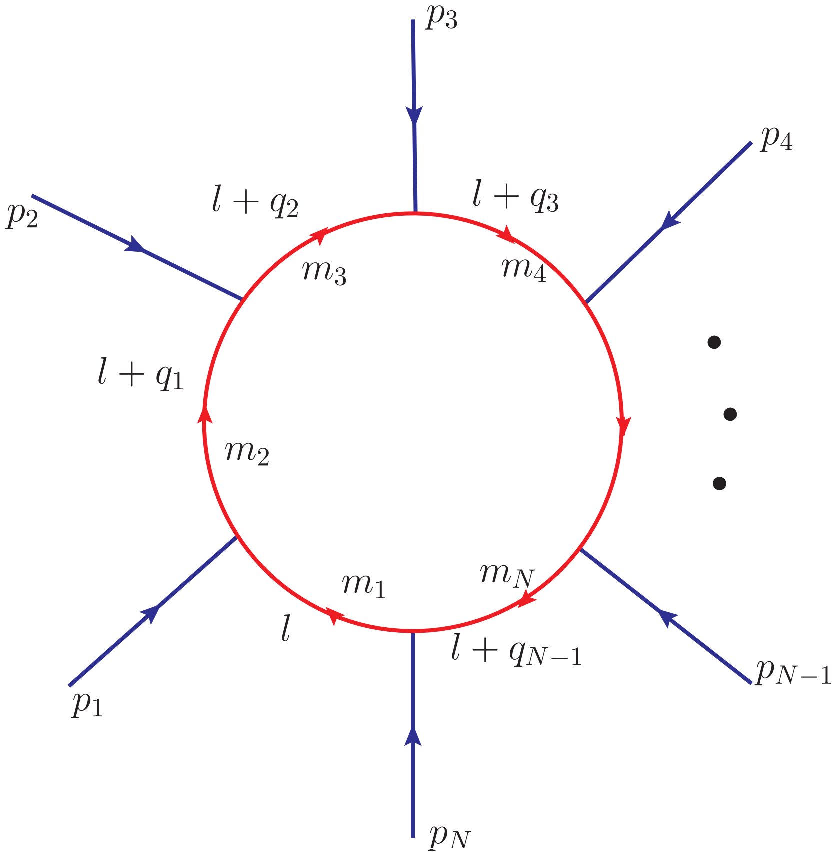
UV divergent if $\quad \mathbf{r \geq 2N-4}$

# One loop calculation with traditional methods

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Only one-, two-, three- and four-point one-loop integrals can be UV-divergent.

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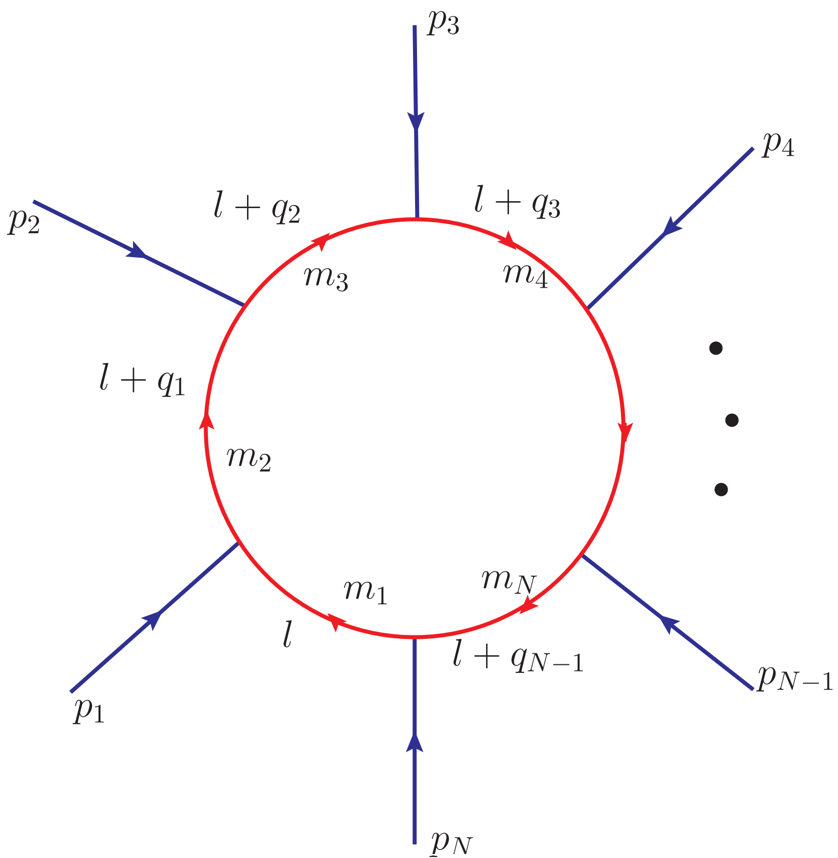


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$$\frac{\mathrm{d}^4\mathbf{l}}{(\mathbf{2}\pi)^4} \rightarrow \frac{\mathrm{d}^\mathbf{D}\mathbf{l}}{(\mathbf{2}\pi)^\mathbf{D}}$$

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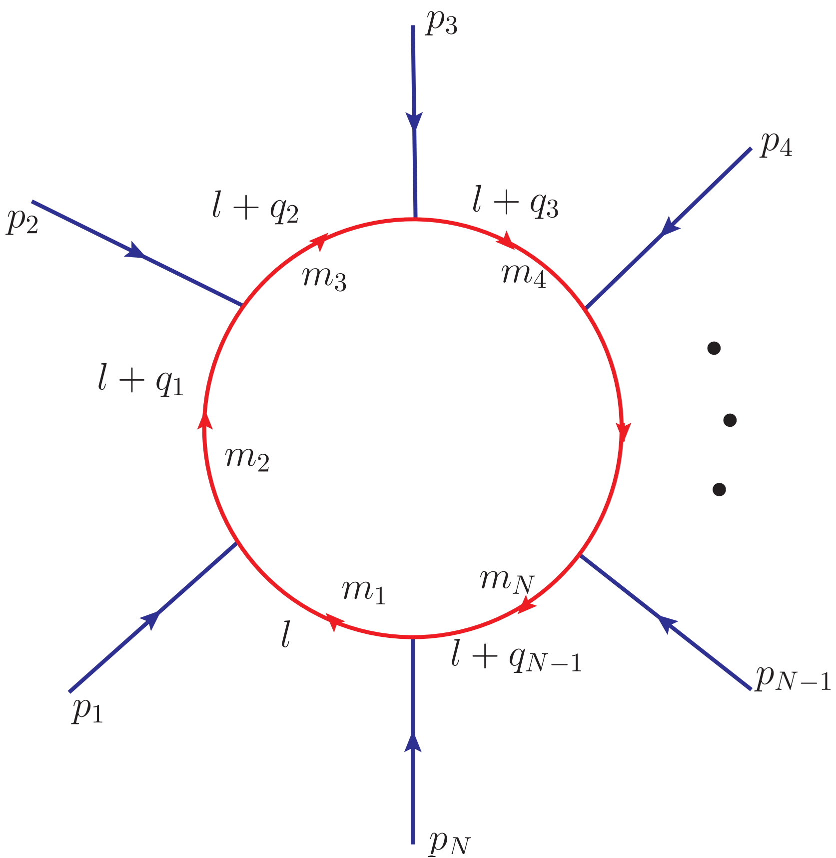


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$$\frac{\mathrm{d}^4\mathbf{l}}{(2\pi)^4} \rightarrow \frac{\mathrm{d}^\mathbf{D}\mathbf{l}}{(2\pi)^\mathbf{D}} \qquad , \quad \mathbf{d}(\mathbf{l}) = \mathbf{l}^2 - \mu^2$$

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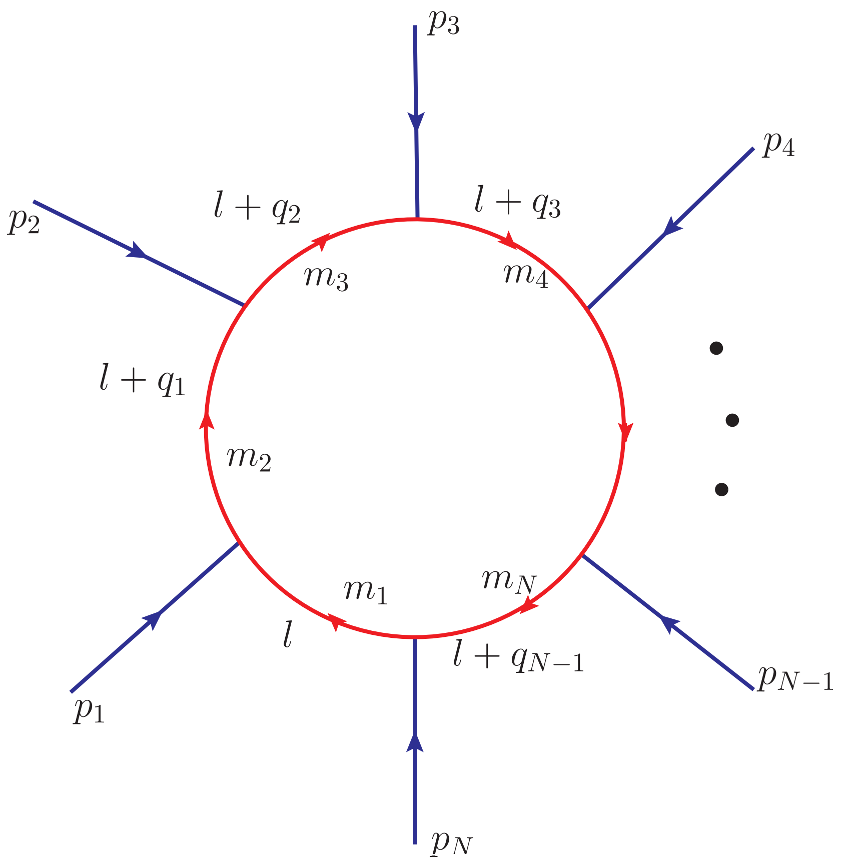


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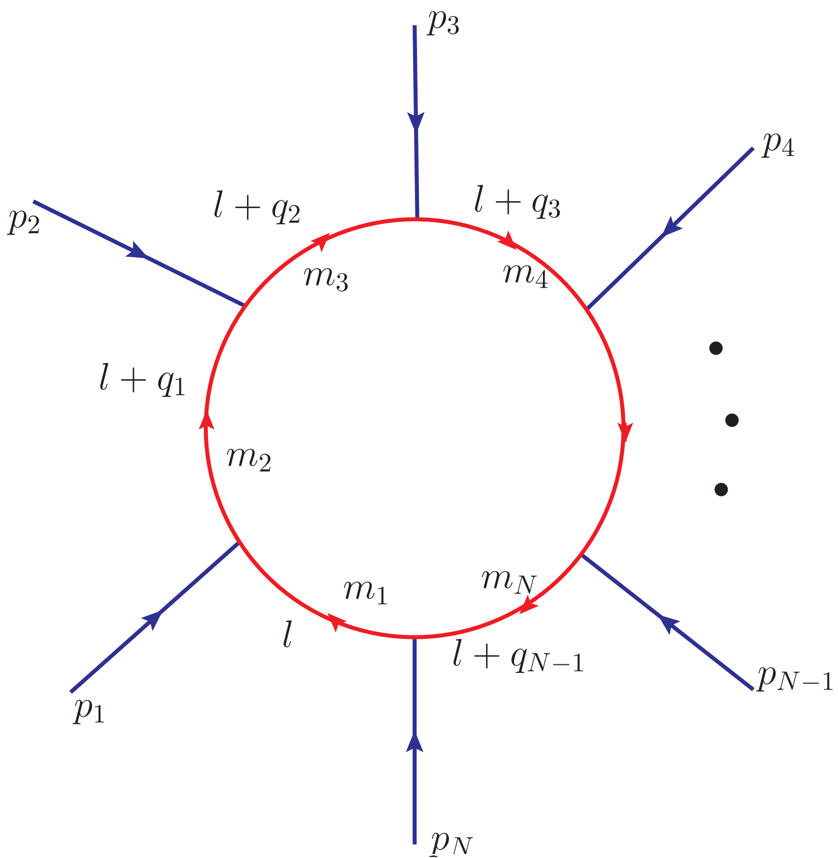
$$\mathbf{I_4} \to \int \frac{\mathrm{d}^D\mathbf{l}}{(2\pi)^D} \frac{\mathbf{l}_\mu \mathbf{l}_\nu \mathbf{l}_\rho \mathbf{l}_\delta}{[\mathbf{d}(\mathbf{l})]^4} = \frac{(\mathbf{g}_{\mu\nu}\mathbf{g}_{\rho\delta} + \mathbf{g}_{\mu\rho}\mathbf{g}_{\nu\delta} + \mathbf{g}_{\mu\delta}\mathbf{g}_{\nu\rho})}{\mathbf{D}(\mathbf{D}+\mathbf{2})} \int \frac{\mathrm{d}^D\mathbf{l}}{(2\pi)^D} \frac{\mathbf{l}^4}{[\mathbf{d}(\mathbf{l})]^4}$$

# One loop calculation with traditional methods

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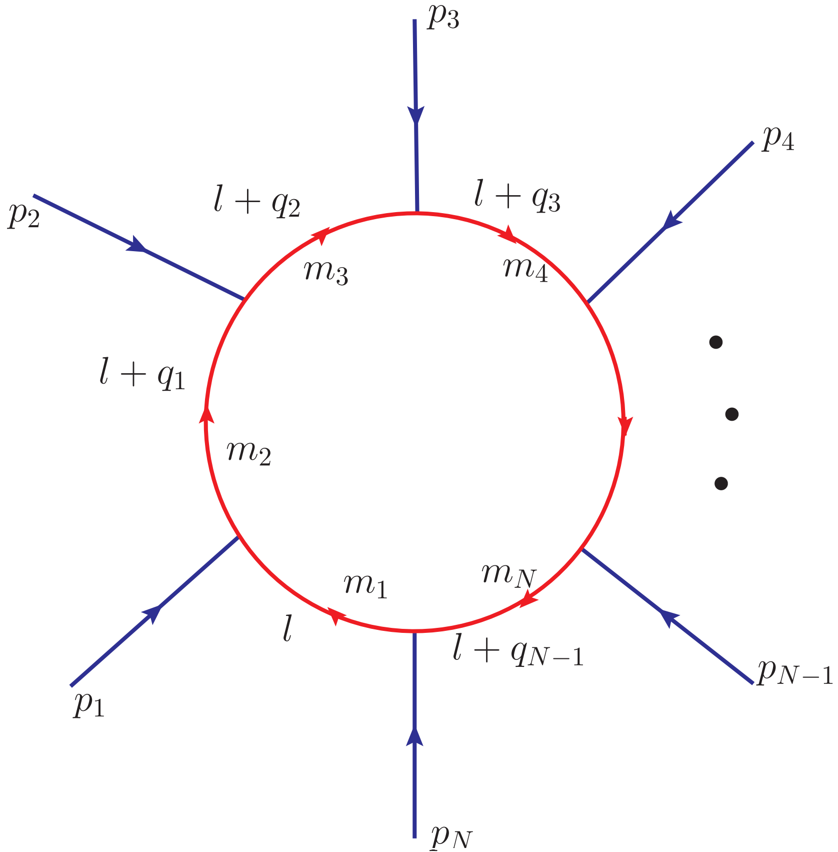
# One loop calculation with traditional methods

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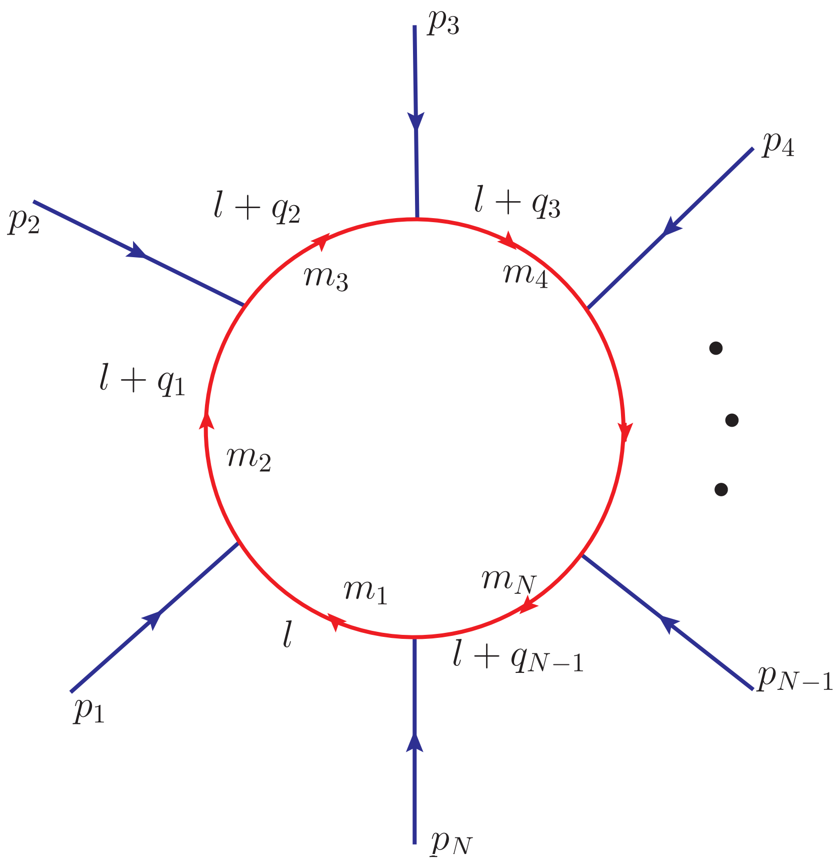


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Finite result for $\quad \mathbf{D \neq 4\,, m=4\,, s=2}$

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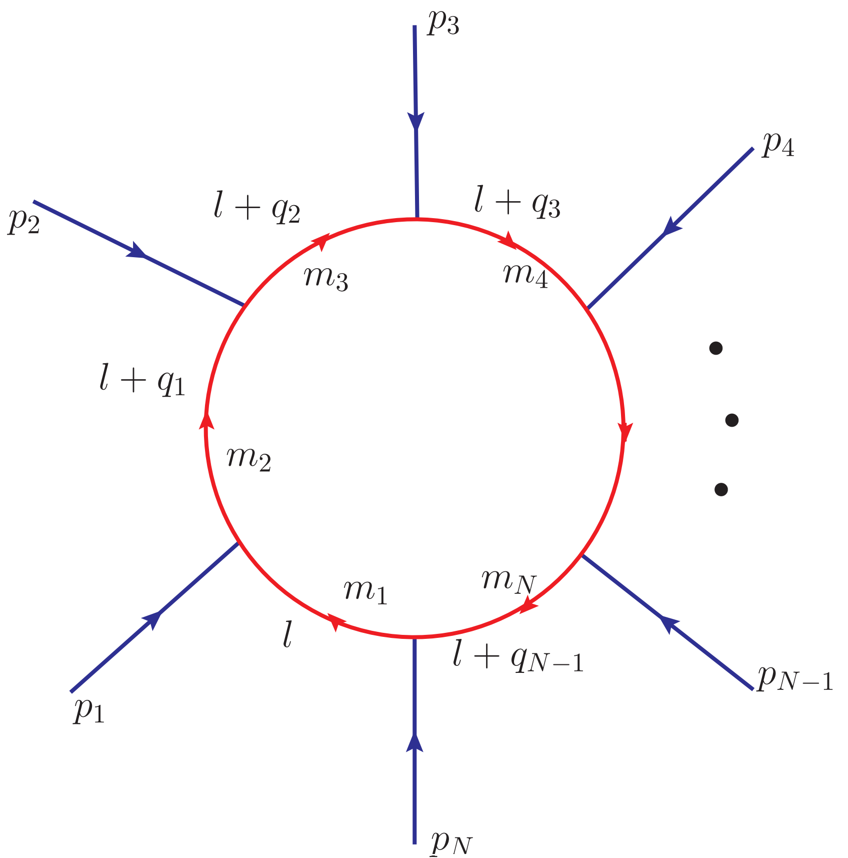


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IR divergent if sufficient number of propagators can go to mass-shell. Soft and collinear singularities.

# Integral basis in the limit $\mathrm{D} \rightarrow 4$

## Integral basis in the limit $D \to 4$

Any one-loop integral can be written as linear combination of scalar one-loop integrals and rational terms:

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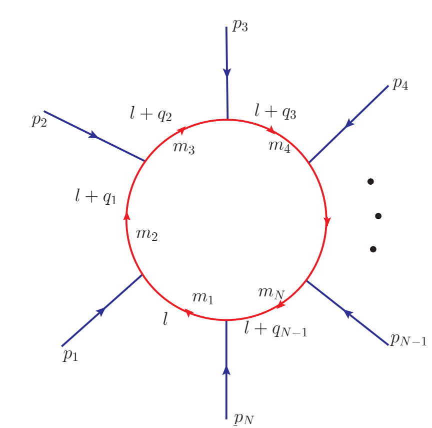


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$$+ \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4}} \mathbf{I}_{\mathbf{i_1 i_2 i_3 i_4}} + \sum_{\mathbf{i_1,i_2,i_3}} \tilde{\mathbf{c}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3}} \mathbf{I}_{\mathbf{i_1 i_2 i_3}} + \sum_{\mathbf{i_1,i_2}} \tilde{\mathbf{b}}^{(\mathbf{0})}_{\mathbf{i_1,i_2}} \mathbf{I}_{\mathbf{i_1 i_2}} + \sum_{\mathbf{i_1}} \tilde{\mathbf{a}}^{(\mathbf{0})}_{\mathbf{i_1}} \mathbf{I}_{\mathbf{i_1}} + \mathcal{R}$$

Non-vanishing master integrals with $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2$ factors in the numerator

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = -\frac{D-4}{2} I^{D+2}_{i_1 i_2 i_3 i_4},$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^4}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = \frac{(D-2)(D-4)}{4} I^{D+4}_{i_1 i_2 i_3 i_4},$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2 i_3},$$

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$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3 i_4} = 0,$$

$$\lim_{D\to 4} \frac{(D-4)(D-2)}{4} I^{(D+4)}_{i_1 i_2 i_3 i_4} = -\frac{1}{3},$$

$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3} = \frac{1}{2},$$

$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2} = -\frac{m_{i_1}^2 + m_{i_2}^2}{2} + \frac{1}{6}(q_{i_1} - q_{i_2})^2$$

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they contribute to the rational terms

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$$\mathbf{I_N} = \int \frac{\mathrm{d}^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathrm{Num}(\mathbf{l})}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} = \sum_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \tilde{\mathbf{e}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \mathbf{I_{i_1i_2i_3i_4i_5}} + \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4}} \mathbf{I_{i_1i_2i_3i_4}} + \sum_{\mathbf{i_1,i_2,i_3}} \tilde{\mathbf{c}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3}} \mathbf{I_{i_1i_2i_3}} + \sum_{\mathbf{i_1,i_2}} \tilde{\mathbf{b}}^{(\mathbf{0})}_{\mathbf{i_1,i_2}} \mathbf{I_{i_1i_2}} + \sum_{\mathbf{i_1}} \tilde{\mathbf{a}}^{(\mathbf{0})}_{\mathbf{i_1}} \mathbf{I_{i_1}} + \mathcal{R}$$

Non-vanishing master integrals with $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2$ factors in the numerator

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$$\mathcal{R} = -\sum_{\mathbf{i_1,i_2,i_3,i_4}} \frac{\tilde{\mathbf{d}}^{(4)}_{\mathbf{i_1i_2i_3i_4}}}{\mathbf{6}} + \sum_{\mathbf{i_1,i_2,i_3}} \frac{\tilde{\mathbf{c}}^{(7)}_{\mathbf{i_1i_2i_3}}}{\mathbf{2}} + \sum_{\mathbf{i_1,i_2}} \left[\frac{\mathbf{m^2_{i_1}}+\mathbf{m^2_{i_2}}}{\mathbf{2}} - \frac{(\mathbf{q_{i_1}}-\mathbf{q_{i_2}})^2}{\mathbf{6}}\right] \tilde{\mathbf{b}}^{(9)}_{\mathbf{i_1i_2}}.$$

## PV reduction of one-loop tensor integrals (coming from individual Feynman diagrams):

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$$\mathbf{A_0(m_1) = \frac{1}{i\pi^{D/2}} \int d^D l \, \frac{1}{d_1},}$$

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$$\mathbf{C_0; C^{\mu}; C^{\mu\nu}; C^{\mu\nu\alpha}(p_1, p_2, m_1, m_2, m_3) = \frac{1}{i\pi^{D/2}} \int d^D l \, \frac{1; l^{\mu}; l^{\mu} l^{\nu}; l^{\mu} l^{\nu} l^{\alpha}}{d_1 d_2 d_3},}$$

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Reduction with the method of form factor expansions (Passarino, Veltman, 1979):

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Reduction with the method of form factor expansions (Passarino, Veltman, 1979):

Examples:

$$C^\mu = p_1^\mu C_1 + p_2^\mu C_2\,, \qquad C^{\mu\nu} = g^{\mu\nu} C_{00} + \sum_{i,j=1}^{2} p_i^\mu p_j^\nu C_{ij}\,, \qquad C_{12} = C_{21}$$

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$$\mathbf{B_0; B^\mu; B^{\mu\nu}(p_1, m_1, m_2) = \frac{1}{i\pi^{D/2}} \int d^D l \, \frac{1; l^\mu; l^\mu l^\nu}{d_1 d_2}},$$

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rank 1 and 2 tensor three point integrals

Reduction of $\mathbf{C}^{\mu} = \mathbf{p}_{\mathbf{1}}^{\mu}\mathbf{C}_{\mathbf{1}} + \mathbf{p}_{\mathbf{2}}^{\mu}\mathbf{C}_{\mathbf{2}}\,,$

Reduction of $\mathbf{C}^{\mu} = \mathbf{p_1^{\mu} C_1} + \mathbf{p_2^{\mu} C_2}\,,$

$$\mathbf{G_2} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \langle l \cdot p_1 \rangle \\ \langle l \cdot p_2 \rangle \end{pmatrix} = \begin{pmatrix} R_1^{[c]} \\ R_2^{[c]} \end{pmatrix},$$

trace with $\mathbf{p_1}$ and $\mathbf{p_2}$

$$\langle \mathbf{l} \cdot \mathbf{p_j} \rangle = \int \frac{\mathbf{d^D l}}{\mathbf{i}\pi^{\mathbf{D/2}}} \frac{\mathbf{l} \cdot \mathbf{p_j}}{\mathbf{d_1 d_2 d_3}}, \quad \text{for } \mathbf{j} = \mathbf{1}, \mathbf{2}$$

$$\mathbf{G_2} = \begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2 \cdot p_2 \end{pmatrix},$$

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$$\mathbf{R_1^{[c]}} = \frac{\mathbf{1}}{\mathbf{2}} (\mathbf{f_1 C_0}(\mathbf{1}, \mathbf{2}, \mathbf{3}) + \mathbf{B_0}(\mathbf{1}, \mathbf{3}) - \mathbf{B_0}(\mathbf{2}, \mathbf{3})),$$

$$\mathbf{G_2} = \begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2 \cdot p_2 \end{pmatrix},$$

$$\mathbf{R_2^{[c]}} = \frac{\mathbf{1}}{\mathbf{2}} (\mathbf{f_2 C_0}(\mathbf{1}, \mathbf{2}, \mathbf{3}) + \mathbf{B_0}(\mathbf{1}, \mathbf{2}) - \mathbf{B_0}(\mathbf{1}, \mathbf{3}))$$

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$$\langle \mathbf{l \cdot p_j} \rangle = \int \frac{\mathbf{d^D l}}{\mathbf{i\pi^{D/2}}} \frac{\mathbf{l \cdot p_j}}{\mathbf{d_1 d_2 d_3}}, \quad \text{for } \mathbf{j = 1, 2}$$

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$$\mathbf{R_1^{[c]} = \frac{1}{2}(f_1 C_0(1,2,3) + B_0(1,3) - B_0(2,3))},$$

$$\mathbf{R_2^{[c]} = \frac{1}{2}(f_2 C_0(1,2,3) + B_0(1,2) - B_0(1,3))}$$

cancel terms in numerator with denominator factors

$$\mathbf{l \cdot p_1 = \frac{1}{2}(f_1 + d_2 - d_1), \quad f_1 = m_2^2 - m_1^2 - p_1^2,}$$

$$\mathbf{l \cdot p_2 = \frac{1}{2}(f_2 + d_3 - d_2), \quad f_2 = m_3^2 - m_2^2 - p_2^2 - 2p_1 \cdot p_2\,.}$$

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$$\mathbf{l \cdot p_1} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{f_1 + d_2 - d_1}), \quad \mathbf{f_1 = m_2^2 - m_1^2 - p_1^2},$$

$$\mathbf{l \cdot p_2} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{f_2 + d_3 - d_2}), \quad \mathbf{f_2 = m_3^2 - m_2^2 - p_2^2 - 2p_1 \cdot p_2}\,.$$

$$\mathbf{B_0(2,3) \equiv B_0(p_2, m_2, m_3)} = \int \frac{\mathbf{d^D l}}{\mathbf{i}\pi^{\mathbf{D/2}}} \frac{\mathbf{1}}{\mathbf{(l^2 - m_2^2)((l + p_2)^2 - m_3^2)}}$$

## Reduction of $\mathbf{C}^\mu = \mathbf{p}_1^\mu \mathbf{C}_1 + \mathbf{p}_2^\mu \mathbf{C}_2\,,$

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cancel terms in numerator with denominator factors

$$\mathbf{l}\cdot\mathbf{p_1} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{f_1} + \mathbf{d_2} - \mathbf{d_1}),\quad \mathbf{f_1} = \mathbf{m_2^2} - \mathbf{m_1^2} - \mathbf{p_1^2},$$

Solve for $\mathbf{C_1}$ and $\mathbf{C_2}$

$$\mathbf{l}\cdot\mathbf{p_2} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{f_2} + \mathbf{d_3} - \mathbf{d_2}),\quad \mathbf{f_2} = \mathbf{m_3^2} - \mathbf{m_2^2} - \mathbf{p_2^2} - \mathbf{2p_1}\cdot\mathbf{p_2}\,.$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \mathbf{G_2^{-1}}\begin{pmatrix} R_1^{[c]} \\ R_2^{[c]} \end{pmatrix}$$

$$\mathbf{B_0(2,3)} \equiv \mathbf{B_0(p_2,m_2,m_3)} = \int \frac{\mathbf{d^D l}}{\mathbf{i}\pi^{\mathbf{D/2}}}\,\frac{\mathbf{1}}{(\mathbf{l^2}-\mathbf{m_2^2})((\mathbf{l}+\mathbf{p_2})^{\mathbf{2}}-\mathbf{m_3^2})}$$

## Reduction of $\mathbf{C^\mu = p_1^\mu C_1 + p_2^\mu C_2}$ ,

$$\mathbf{G_2} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \langle l \cdot p_1 \rangle \\ \langle l \cdot p_2 \rangle \end{pmatrix} = \begin{pmatrix} R_1^{[c]} \\ R_2^{[c]} \end{pmatrix},$$

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$$\langle \mathbf{l \cdot p_j} \rangle = \int \frac{\mathbf{d^D l}}{\mathbf{i\pi^{D/2}}} \frac{\mathbf{l \cdot p_j}}{\mathbf{d_1 d_2 d_3}}, \quad \text{for } \mathbf{j = 1, 2}$$

$$\mathbf{R_1^{[c]}} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{f_1 C_0(1,2,3) + B_0(1,3) - B_0(2,3)}),$$

$$\mathbf{G_2} = \begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2 \cdot p_2 \end{pmatrix},$$

$$\mathbf{R_2^{[c]}} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{f_2 C_0(1,2,3) + B_0(1,2) - B_0(1,3)})$$

cancel terms in numerator with denominator factors

$$\mathbf{l \cdot p_1} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{f_1 + d_2 - d_1}), \quad \mathbf{f_1 = m_2^2 - m_1^2 - p_1^2},$$

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$$\mathbf{p_1}_{\mu}\mathbf{C}^{\mu\nu} = \mathbf{p}_1^\nu(\mathbf{p_1.p_1C_{11}}+\mathbf{p_1.p_2C_{12}}+\mathbf{C_{00}}) + \mathbf{p}_2^\nu(\mathbf{p_1.p_1C_{12}}+\mathbf{p_1.p_2C_{22}}),$$
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trace with $\mathbf{p_1}$ and $\mathbf{p_2}$

$$\mathbf{\langle l\cdot p_j\rangle = \int \frac{d^D l}{i\pi^{D/2}}\,\frac{l\cdot p_j}{d_1 d_2 d_3}}, \quad \text{for } \mathbf{j = 1, 2}$$

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$$\mathbf{p_{2\,\mu} C^{\mu\nu} = p_1^\nu(p_1.p_2 C_{11} + p_2.p_2 C_{12}) + p_2^\nu(p_1.p_2 C_{12} + p_2.p_2 C_{22} + C_{00})}\,.$$

trace with $\mathbf{p_1}$ and $\mathbf{p_2}$

$$\mathbf{C^\nu_\mu = B_0(2,3) + m_1^2 C_0(1,2,3) = D C_{00} + p_1^2 C_{11} + 2p_1.p_2 C_{12} + p_2^2 C_{22}}$$

trace with $\mathbf{g^{\mu\nu}}$

## Reduction chains for Passarino-Veltman procedure.

| | | |
|---|---|---|
| $\mathbf{D_{ijkl}}$ | $\longrightarrow$ | $D_{00ij}, D_{ijk}, C_{ijk}, C_{ij}, C_i, C_0$ |
| $\mathbf{D_{00ij}}$ | $\longrightarrow$ | $D_{ijk}, D_{ij}, C_{ij}, C_i$ |
| $\mathbf{D_{0000}}$ | $\longrightarrow$ | $D_{00i}, D_{00}, C_{00}$ |
| $\mathbf{D_{ijk}}$ | $\longrightarrow$ | $D_{00i}, D_{ij}, C_{ij}, C_i$ |
| $\mathbf{D_{00i}}$ | $\longrightarrow$ | $D_{ij}, D_i, C_i, C_0$ |
| $\mathbf{D_{ij}}$ | $\longrightarrow$ | $D_{00}, D_i, C_i, C_0$ |
| $\mathbf{D_{00}}$ | $\longrightarrow$ | $D_i, D_0, C_0$ |
| $\mathbf{D_{i}}$ | $\longrightarrow$ | $D_0, C_0$ |
| $\mathbf{C_{ijk}}$ | $\longrightarrow$ | $C_{00i}, C_{ij}, B_{ij}, B_i$ |
| $\mathbf{C_{00i}}$ | $\longrightarrow$ | $C_{ii}, C_i, B_i, B_0$ |
| $\mathbf{C_{ij}}$ | $\longrightarrow$ | $C_{00}, C_i, B_i, B_0$ |
| $\mathbf{C_{00}}$ | $\longrightarrow$ | $C_i, C_0, B_0$ |
| $\mathbf{C_{i}}$ | $\longrightarrow$ | $C_0, B_0$ |
| $\mathbf{B_{ii}}$ | $\longrightarrow$ | $B_{00}, B_i, A_0$ |
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## Reduction chains for Passarino-Veltman procedure.

For more then 4 particles it is cumbersome

$$
\begin{array}{lcl}
\mathbf{D_{ijkl}} & \rightarrow & D_{00ij}, D_{ijk}, C_{ijk}, C_{ij}, C_i, C_0 \\
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\mathbf{D_{0000}} & \rightarrow & D_{00i}, D_{00}, C_{00} \\
\mathbf{D_{ijk}} & \rightarrow & D_{00i}, D_{ij}, C_{ij}, C_i \\
\mathbf{D_{00i}} & \rightarrow & D_{ij}, D_i, C_i, C_0 \\
\mathbf{D_{ij}} & \rightarrow & D_{00}, D_i, C_i, C_0 \\
\mathbf{D_{00}} & \rightarrow & D_i, D_0, C_0 \\
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\hline
\mathbf{C_{ijk}} & \rightarrow & C_{00i}, C_{ij}, B_{ij}, B_i \\
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For more then 4 particles it is cumbersome

✦ The number of Feynman diagrams grows dramatically with the number of external particle (factorial).

✦ The number of terms generated during the reduction of tensor integrals grows rapidly with the number of external particles and with the rank of the integral.

✦ In cases with degenerate kinematics, the reduction procedure may lead to numerical instabilities.

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For 5-,6-, and more particles external momenta not linearly independent (additional input), Denner, Dittmaier 2006

# Singular regions in PV reduction:

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Example: rank 1 triangle

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$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \mathbf{G_2^{-1}} \begin{pmatrix} R_1^{[c]} \\ R_2^{[c]} \end{pmatrix}$$

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$$\mathbf{\Delta_2(p_1, p_2)} = \det[\mathbf{G_2}] = \mathbf{p_1^2 p_2^2} - (\mathbf{p_1 \cdot p_2})^{\mathbf{2}},$$

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We assume in the PV expansion that $\mathbf{p_1}, \mathbf{p_2}$ are linearly independent

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$$\mathbf{G_2^{-1}} = \frac{\begin{pmatrix} \mathbf{p_2} \cdot \mathbf{p_2} & -\mathbf{p_1} \cdot \mathbf{p_2} \\ -\mathbf{p_1} \cdot \mathbf{p_2} & \mathbf{p_1} \cdot \mathbf{p_1} \end{pmatrix}}{\mathbf{\Delta_2}(\mathbf{p_1}, \mathbf{p_2})},$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \mathbf{G_2^{-1}} \begin{pmatrix} R_1^{[c]} \\ R_2^{[c]} \end{pmatrix}$$

$$\mathbf{\Delta_2}(\mathbf{p_1}, \mathbf{p_2}) = \det[\mathbf{G_2}] = \mathbf{p_1^2 p_2^2} - (\mathbf{p_1} \cdot \mathbf{p_2})^{\mathbf{2}},$$

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if $\mathbf{p_1} || \mathbf{p_2}$ with $\mathbf{p_1^2} \neq \mathbf{0}$ then $\mathbf{\Delta_2} = \mathbf{0}$ and necessarily $\mathbf{R_1^{[c]}} = \mathbf{0}$ and $\mathbf{R_2^{[c]}} = \mathbf{0}$

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Exceptional points: vanishing Gram determinants, numerical instabilities
two and three point scalar integrals become linearly dependent
also higher powers of Gram-determinants

## Singular regions in PV reduction:

Example: rank 1 triangle

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Exceptional points: vanishing Gram determinants, numerical instabilities
two and three point scalar integrals become linearly dependent
also higher powers of Gram-determinants

Exercise: Investigate the collinear behaviour of reduction using form factors with

using $\mathbf{p_1^\mu}$ and $\mathbf{\tilde{p}_2^\mu} = \mathbf{p_2^\mu} - \frac{\mathbf{p_1 \cdot p_2}}{\mathbf{p_1^2}} \mathbf{p_1^\mu}$

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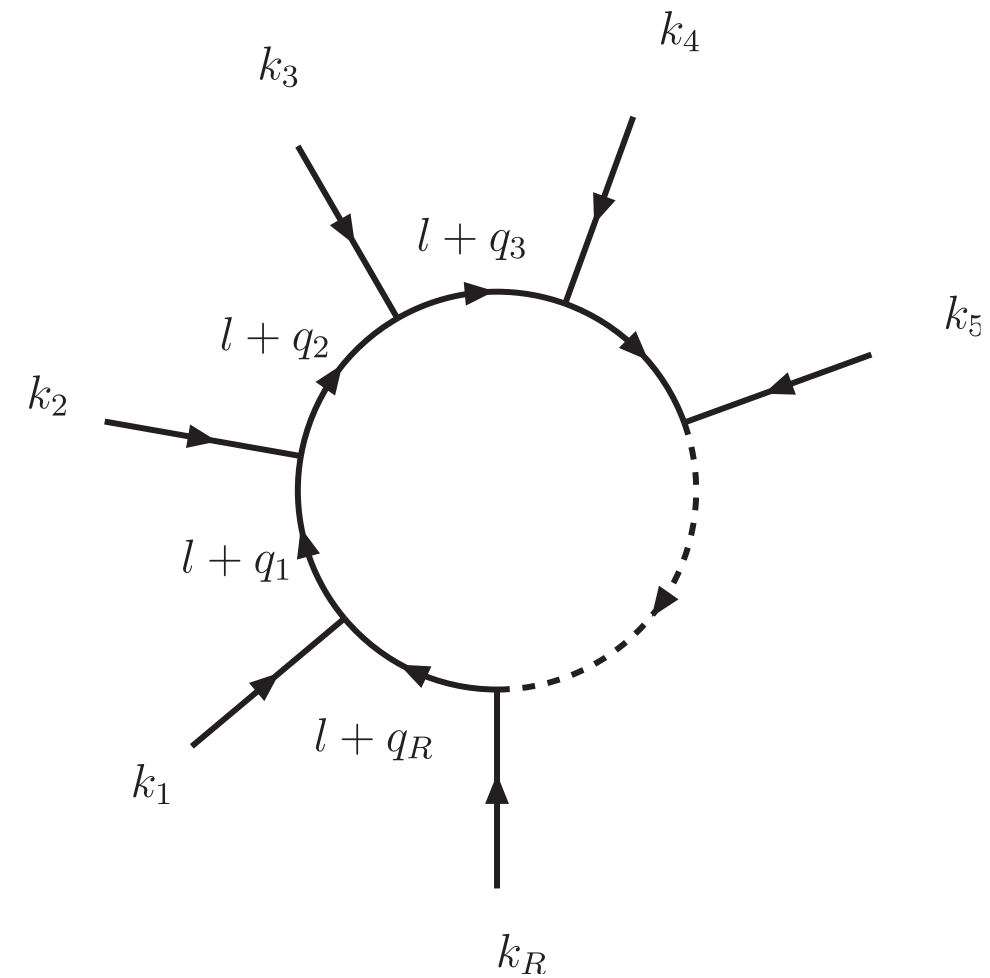
QFT in D-dimension, N-particle scattering amplitude, consider one one-loop Feynman-diagram with R loop-momentum dependent propagator. The integrand is a rational function of the loop momentum

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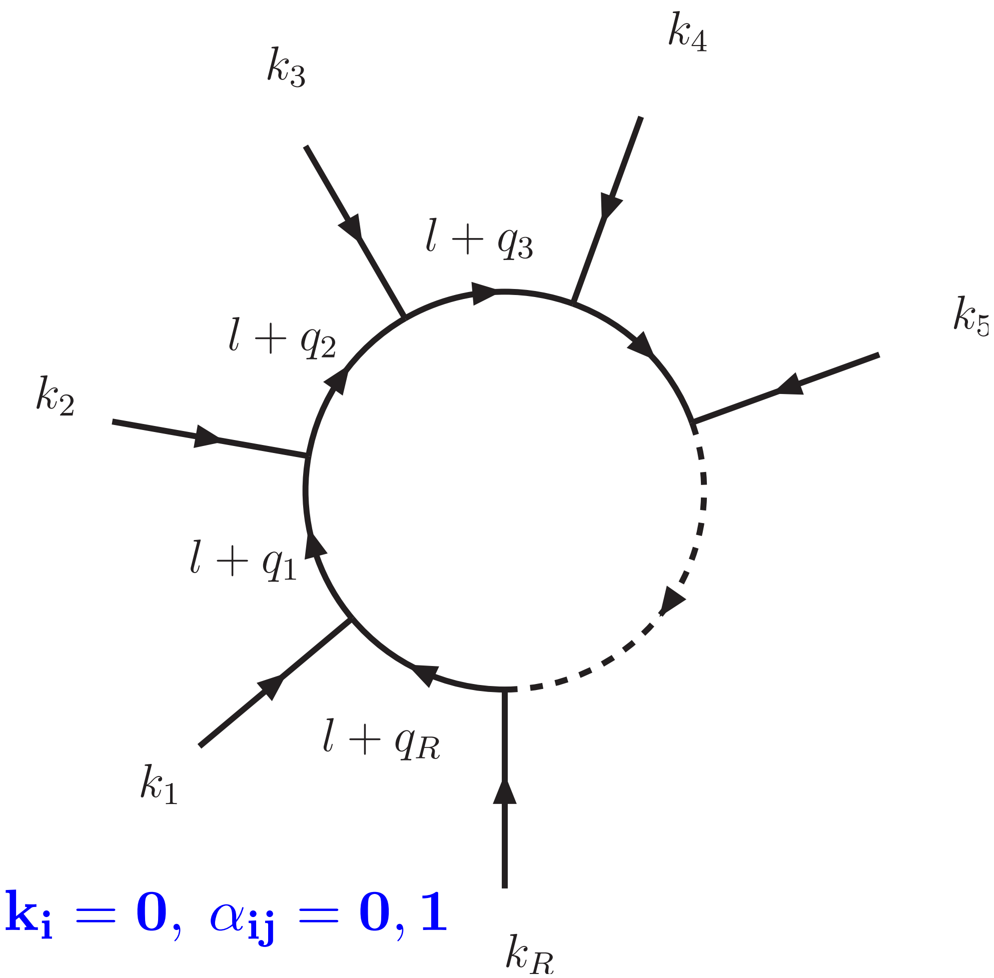


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The amplitude has R inflow momenta $\mathbf{k_1}, \ldots, \mathbf{k_R}$.

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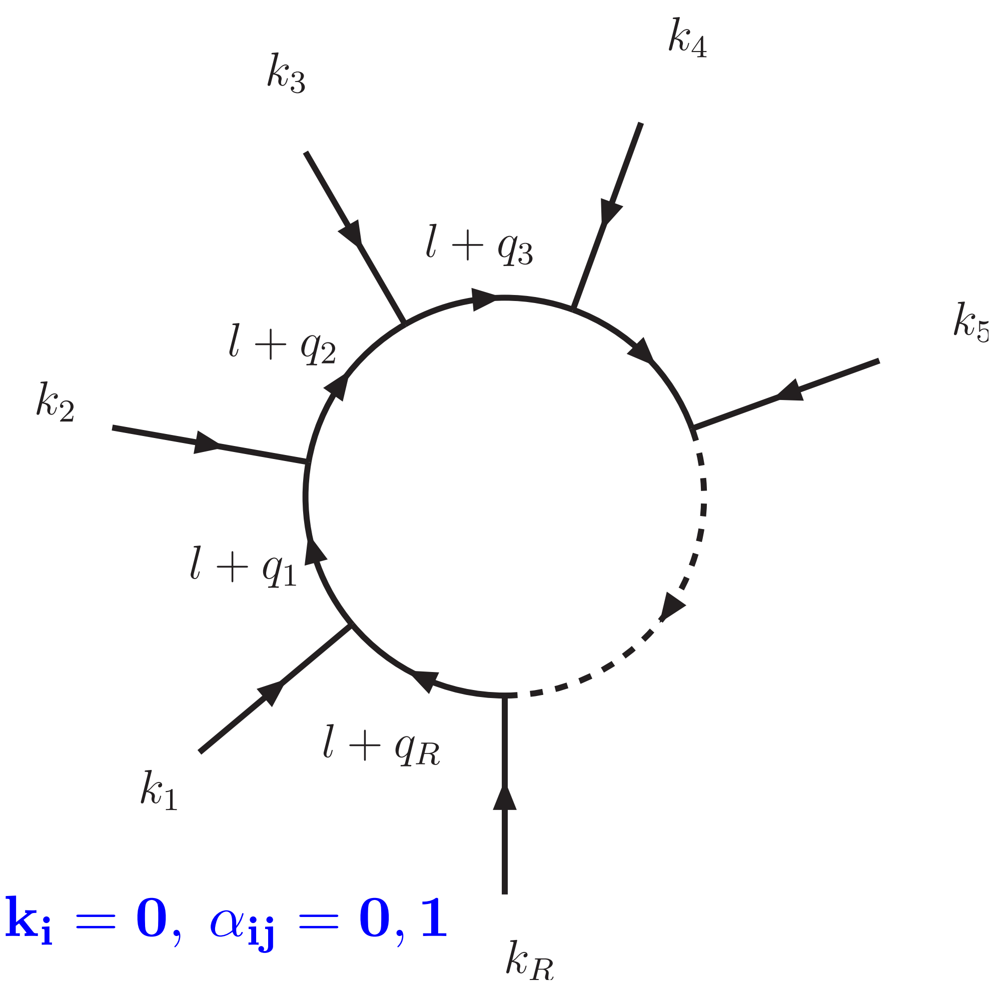
## Physical and transverse space

QFT in D-dimension, N-particle scattering amplitude, consider one one-loop Feynman-diagram with R loop-momentum dependent propagator. The integrand is a rational function of the loop momentum

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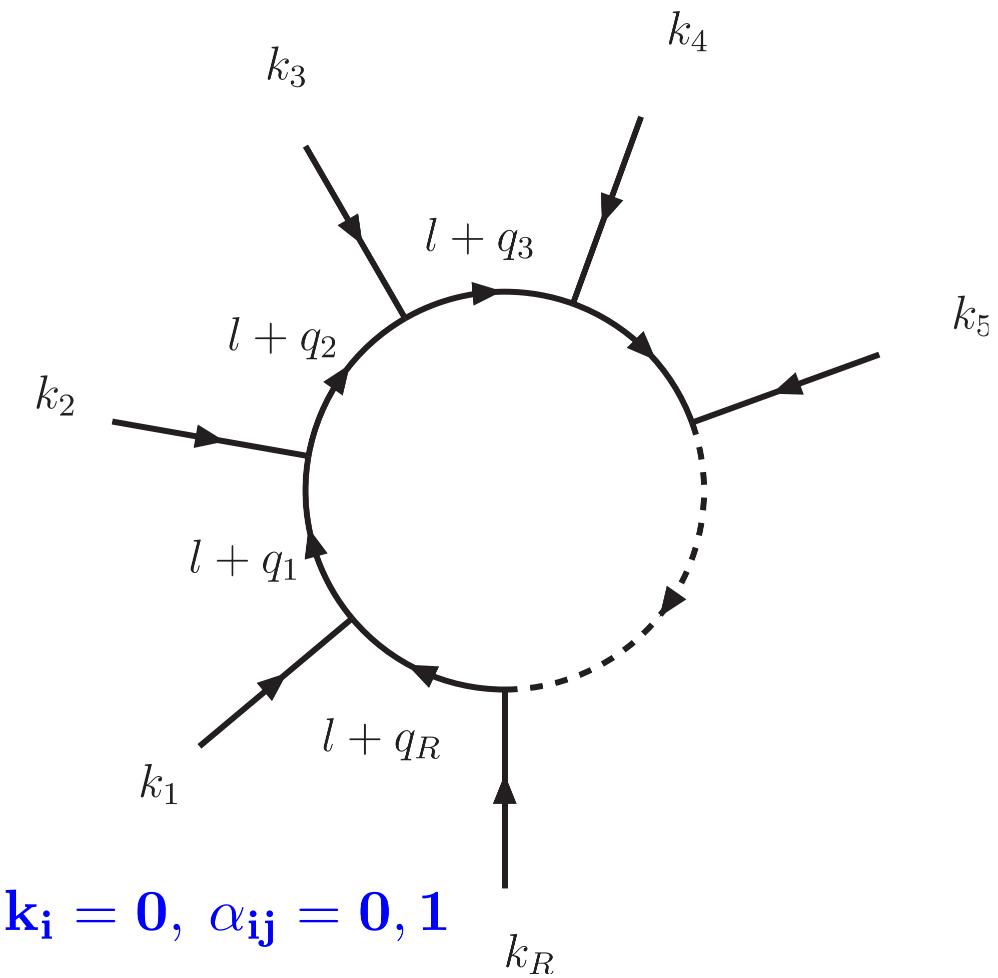


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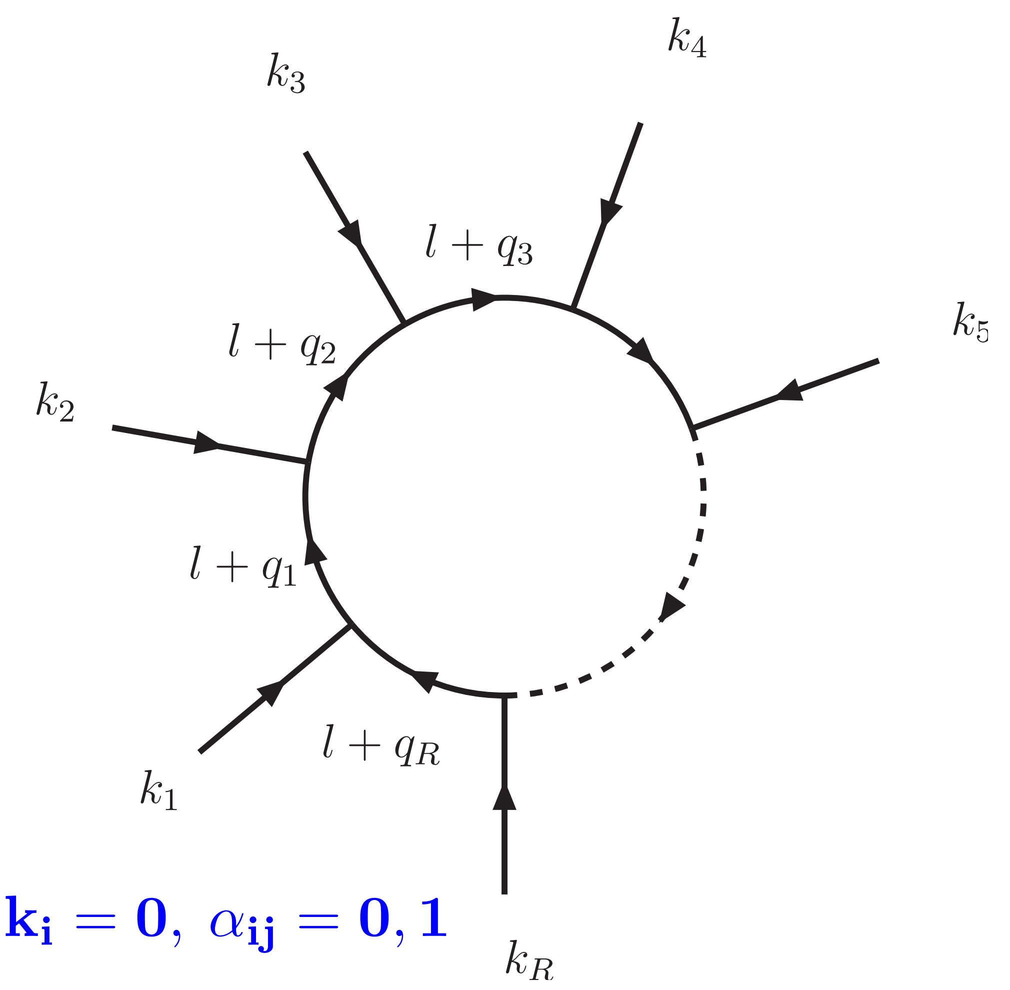


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If R>D, the transverse space is zero dimensional.

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The dual coordinates of the loop momentum vector provide us the reduction at the integrand level:

$$\mathbf{l}\cdot\mathbf{k}_i = \frac{1}{2}\left[\mathbf{d}_i - \mathbf{d}_{i-1} - (\mathbf{q}_i^2 - \mathbf{m}_i^2) + (\mathbf{q}_{i-1}^2 - \mathbf{m}_{i-1}^2)\right]$$

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$$\mathbf{l}^\mu = \mathbf{v}_1^\mu(\mathbf{l}\cdot\mathbf{q_1}) + \mathbf{v}_2^\mu(\mathbf{l}\cdot\mathbf{q_2}),\quad \mathbf{v}_1^\mu = \frac{\delta^{\mu\mathbf{q_2}}_{\mathbf{q_1 q_2}}}{\boldsymbol{\Delta}_2},\;\; \mathbf{v}_2^\mu = \frac{\delta^{\mathbf{q_1}\mu}_{\mathbf{q_1 q_2}}}{\boldsymbol{\Delta}_2},$$

$$\mathbf{l}\cdot\mathbf{q_i} = \frac{1}{2}(\mathbf{d_i} - \mathbf{d_0} - \mathbf{r_i}),\quad \mathbf{r_i} = \mathbf{q_i^2} - \mathbf{m_i^2} + \mathbf{m_0^2},\quad \mathbf{i} = 1, 2,$$

$$\mathbf{g}^{\mu\nu} = \sum_{\mathbf{j}=1}^{2} \mathbf{v}_i^\mu \mathbf{q}_i^\nu = \sum_{\mathbf{j}=1}^{2} \mathbf{v}_i^\nu \mathbf{q}_i^\mu$$

$$2(\mathbf{d_0}+\mathbf{m_0^2}) = \sum_{i=1}^{2}(\mathbf{l}\cdot\mathbf{v_i})(\mathbf{d_i}-\mathbf{d_0}) - \sum_{i=1}^{2}(\mathbf{l}\cdot\mathbf{v_i})\mathbf{r_i} = \sum_{i=1}^{2}(\mathbf{l}\cdot\mathbf{v_i})(\mathbf{d_i}-\mathbf{d_0}) - \mathbf{l}_\mu \sum_{i=1}^{2}\mathbf{v}_i^\mu \mathbf{r_i} = \sum_{i=1}^{2}(\mathbf{l}\cdot\mathbf{v_i})(\mathbf{d_i}-\mathbf{d_0}) - \mathbf{l}\cdot\mathbf{w}$$

$$= \sum_{i=1}^{2}(\mathbf{l}\cdot\mathbf{v_i})(\mathbf{d_i}-\mathbf{d_0}) - \sum_{i=1}^{2}(\mathbf{l}\cdot\mathbf{q_i})(\mathbf{v_i}\cdot\mathbf{w}) = \frac{1}{2}\sum_{i=1}^{2}(2\mathbf{l}\cdot\mathbf{v_i} - \mathbf{w}\cdot\mathbf{v_i})\mathbf{d_i} - (2\mathbf{l}\cdot\mathbf{v_i} - \mathbf{w}\cdot\mathbf{v_i})\mathbf{d_0} + \frac{1}{2}\mathbf{w}^2$$

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$$\frac{1}{\mathbf{d_0 d_1 d_2}} = \frac{1}{(4\mathbf{m_0^2} - \mathbf{w}^2)}\left\{\frac{2(\mathbf{l}\cdot\mathbf{v_1}) - (\mathbf{w}\cdot\mathbf{v_1})}{\mathbf{d_0 d_2}} + \frac{2(\mathbf{l}\cdot\mathbf{v_2}) - (\mathbf{w}\cdot\mathbf{v_2})}{\mathbf{d_0 d_1}} - \frac{4 + (2\mathbf{l}-\mathbf{w})\cdot(\mathbf{v_1}+\mathbf{v_2})}{\mathbf{d_1 d_2}}\right\}.$$

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In the right hand side the terms proportional to $\mathbf{l}_\mu$ depend only on $\mathbf{l}_\mathbf{T}^\mu$ and integrate to zero.

The inflow momenta for the three bubble denominators are different

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**Exercise:**

Any N-point scalar one-loop integrand function, for N > D, where D is the dimensionality of space-time, can be written as a linear combination of the D-point scalar and vector integrand functions.

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Result of the reduction:

$$\frac{(\hat{\mathbf{n}} \cdot \mathbf{l})^2}{\mathbf{d_1 d_2}} = -\frac{(\lambda^2 + \mu^2)}{\mathbf{d_1 d_2}} + \frac{\mathbf{1}}{\mathbf{4k^2}} \left[ \frac{\mathbf{r_1^2 - 2\,l \cdot k}}{\mathbf{d_1}} + \frac{\mathbf{r_2^2 + 2\,l \cdot k + 2k^2}}{\mathbf{d_2}} \right].$$

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$$\lambda^2 = \frac{\mathbf{k^4} - \mathbf{2k^2}(\mathbf{m_1^2} + \mathbf{m_2^2}) + (\mathbf{m_1^2} - \mathbf{m_2^2})^2}{\mathbf{4k^2}}$$

It follows the parametrization:

$$\frac{(\hat{\mathbf{n}} \cdot \mathbf{l})^2}{\mathbf{d_1 d_2}} = \frac{\mathbf{b_0} + \mathbf{b_1}(\hat{\mathbf{n}} \cdot \mathbf{l}) + \mathbf{b_2}(\mathbf{n}_\epsilon \cdot \mathbf{l})^2}{\mathbf{d_1 d_2}} + \frac{\mathbf{a_{1,0}} + \mathbf{a_{1,1}}(\mathbf{n} \cdot \mathbf{l}) + \mathbf{a_{1,2}}(\hat{\mathbf{n}} \cdot \mathbf{l})}{\mathbf{d_1}} + \frac{\mathbf{a_{2,0}} + \mathbf{a_{2,1}}(\mathbf{n} \cdot \mathbf{l}) + \mathbf{a_{2,2}}(\hat{\mathbf{n}} \cdot \mathbf{l})}{\mathbf{d_2}}.$$

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$$(\hat{\mathbf{n}}\cdot\mathbf{l})^{\mathbf{2}} = \left[\mathbf{b_0} + \mathbf{b_1}(\hat{\mathbf{n}}\cdot\mathbf{l}) + \mathbf{b_2}(\mathbf{n}_\epsilon\cdot\mathbf{l})^{\mathbf{2}}\right] + \left[\mathbf{a_{1,0}} + \mathbf{a_{1,1}}\cancel{(\mathbf{n}\cdot\mathbf{l})} + \mathbf{a_{1,2}}(\hat{\mathbf{n}}\cdot\mathbf{l})\right]\mathbf{d_2} \quad + \left[\mathbf{a_{2,0}} + \mathbf{a_{2,1}}\cancel{(\mathbf{n}\cdot\mathbf{l})} + \mathbf{a_{2,2}}(\hat{\mathbf{n}}\cdot\mathbf{l})\right]\mathbf{d_1}$$

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$$(\hat{\mathbf{n}}\cdot\mathbf{l})^2 = \left[\mathbf{b_0} + \mathbf{b_1}(\hat{\mathbf{n}}\cdot\mathbf{l}) + \mathbf{b_2}(\mathbf{n}_\epsilon\cdot\mathbf{l})^2\right] + \left[\mathbf{a_{1,0}} + \mathbf{a_{1,1}}\cancel{(\mathbf{n}\cdot\mathbf{l})} + \mathbf{a_{1,2}}(\hat{\mathbf{n}}\cdot\mathbf{l})\right]\mathbf{d_2} \quad + \left[\mathbf{a_{2,0}} + \mathbf{a_{2,1}}\cancel{(\mathbf{n}\cdot\mathbf{l})} + \mathbf{a_{2,2}}(\hat{\mathbf{n}}\cdot\mathbf{l})\right]\mathbf{d_1}$$

Calculate $\mathbf{b_0}, \mathbf{b_1}$ assuming $\mathbf{d_1}(\mathbf{l}) = \mathbf{d_2}(\mathbf{l}) = \mathbf{0}$ and $\mathbf{n}_\epsilon\cdot\mathbf{l} = \mathbf{0}$ :

$$\mathbf{l_c^\pm} = \alpha_\mathbf{c}\mathbf{n} \pm \mathbf{i}\beta_\mathbf{c}\hat{\mathbf{n}}$$

$$\mathbf{b_0} + \mathbf{b_1}\hat{\mathbf{n}}\cdot\mathbf{l_c^+} = -\lambda^2, \quad \mathbf{b_0} + \mathbf{b_1}\hat{\mathbf{n}}\cdot\mathbf{l_c^-} = -\lambda^2$$

$$\alpha_\mathbf{c} = -\frac{\mathbf{r_1^2}}{\mathbf{2}\sqrt{\mathbf{k^2}}}, \quad \beta_\mathbf{c} = \lambda$$

$$\mathbf{b_0} = -\lambda^2\,, \; \mathbf{b_1} = \mathbf{0}$$

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$$(\mathbf{\hat{n}\cdot l})^{\mathbf{2}} = \left[\mathbf{b_0} + \mathbf{b_1}(\mathbf{\hat{n}\cdot l}) + \mathbf{b_2}(\mathbf{n}_\epsilon\cdot\mathbf{l})^{\mathbf{2}}\right] + \left[\mathbf{a_{1,0}} + \mathbf{a_{1,1}}\cancel{(\mathbf{n\cdot l})} + \mathbf{a_{1,2}}(\mathbf{\hat{n}\cdot l})\right]\mathbf{d_2} \quad + \left[\mathbf{a_{2,0}} + \mathbf{a_{2,1}}\cancel{(\mathbf{n\cdot l})} + \mathbf{a_{2,2}}(\mathbf{\hat{n}\cdot l})\right]\mathbf{d_1}$$

Calculate $\mathbf{b_0}, \mathbf{b_1}$ assuming $\mathbf{d_1(l) = d_2(l) = 0}$ and $\mathbf{n}_\epsilon\cdot\mathbf{l} = \mathbf{0}$ :

$$\mathbf{l_c^{\pm}} = \alpha_\mathbf{c}\mathbf{n} \pm \mathbf{i}\beta_\mathbf{c}\mathbf{\hat{n}}$$

$$\mathbf{b_0} + \mathbf{b_1}\mathbf{\hat{n}}\cdot\mathbf{l_c^{+}} = -\lambda^{\mathbf{2}}, \quad \mathbf{b_0} + \mathbf{b_1}\mathbf{\hat{n}}\cdot\mathbf{l_c^{-}} = -\lambda^{\mathbf{2}}$$

$$\alpha_\mathbf{c} = -\frac{\mathbf{r_1^2}}{\mathbf{2}\sqrt{\mathbf{k^2}}}, \quad \beta_\mathbf{c} = \lambda$$

$$\mathbf{b_0} = -\lambda^{\mathbf{2}}\,, \ \mathbf{b_1} = \mathbf{0}$$

Calculate $\mathbf{b_2}$ assuming $\mathbf{d_1(l) = d_2(l) = 0}$ and $\mathbf{l\cdot\hat{n} = 0}$ :

## Connection to “unitarity” : evaluate coefficients with “on-shell” loop momenta

$$(\mathbf{\hat{n}\cdot l})^2 = \left[\mathbf{b_0} + \mathbf{b_1}(\mathbf{\hat{n}\cdot l}) + \mathbf{b_2}(\mathbf{n}_\epsilon\cdot \mathbf{l})^2\right] + \left[\mathbf{a_{1,0}} + \mathbf{a_{1,1}}\cancel{(\mathbf{n\cdot l})} + \mathbf{a_{1,2}}(\mathbf{\hat{n}\cdot l})\right]\mathbf{d_2} \quad + \left[\mathbf{a_{2,0}} + \mathbf{a_{2,1}}\cancel{(\mathbf{n\cdot l})} + \mathbf{a_{2,2}}(\mathbf{\hat{n}\cdot l})\right]\mathbf{d_1}$$

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$$\mathbf{0} = (\mathbf{1} + \mathbf{b_2})\lambda^\mathbf{2}$$

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Calculate $b_0, b_1$ assuming $d_1(l) = d_2(l) = 0$ and $n_\epsilon \cdot l = 0$ :

$$l_c^\pm = \alpha_c n \pm i\beta_c \hat{n}$$

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$$l_1 = \gamma_1 n + \gamma_2\hat{n}\,, \quad \gamma_1^2 + \gamma_2^2 = m_1^2$$

Choose $\gamma_1 = 0, \gamma_2 = \pm m_1$

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Calculate $\mathbf{b_0, b_1}$ assuming $\mathbf{d_1(l) = d_2(l) = 0}$ and $\mathbf{n_\epsilon\cdot l = 0}$ :

$$\mathbf{l_c^\pm} = \alpha_\mathbf{c}\mathbf{n} \pm \mathbf{i}\beta_\mathbf{c}\mathbf{\hat{n}} \qquad \mathbf{b_0 + b_1\hat{n}\cdot l_c^+} = -\lambda^\mathbf{2}, \quad \mathbf{b_0 + b_1\hat{n}\cdot l_c^-} = -\lambda^\mathbf{2}$$

$$\alpha_\mathbf{c} = -\frac{\mathbf{r_1^2}}{\mathbf{2\sqrt{k^2}}}, \quad \beta_\mathbf{c} = \lambda \qquad \mathbf{b_0} = -\lambda^\mathbf{2}\,, \ \mathbf{b_1 = 0}$$

Calculate $\mathbf{b_2}$ assuming $\mathbf{d_1(l) = d_2(l) = 0}$ and $\mathbf{l\cdot\hat{n} = 0}$ :

$$\mathbf{l_c^\pm} = \alpha_\mathbf{c}\mathbf{n} \pm \mathbf{i}\beta_\mathbf{c}\mathbf{n_\epsilon} \qquad \mathbf{0 = (1 + b_2)}\lambda^\mathbf{2}$$

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$$\mathbf{l_1} = \gamma_\mathbf{1}\mathbf{n} + \gamma_\mathbf{2}\mathbf{\hat{n}}\,, \quad \gamma_\mathbf{1}^\mathbf{2} + \gamma_\mathbf{2}^\mathbf{2} = \mathbf{m_1^2} \qquad \mathbf{a_{1,0} \pm a_{1,2}m_1} = \frac{\mathbf{m_1^2} + \lambda^\mathbf{2}}{\mathbf{r_1^2}} = \frac{\mathbf{r_1^2}}{\mathbf{4k^2}}$$

Choose $\gamma_\mathbf{1} = \mathbf{0}, \gamma_\mathbf{2} = \pm\mathbf{m_1}$

$$\mathbf{a_{1,2} = 0}\,, \mathbf{a_{1,0} = r_1^2/(4k^2)}\,,$$

Choose $\gamma_\mathbf{2} = \mathbf{0}, \gamma_\mathbf{1} = \mathbf{m_1}$

$$\mathbf{a_{1,1} = -(4k^2)^{-1/2}}$$

# OPP reduction *(Ossola, Papadopoulos, Pittau)*

**First we made direct NV reduction of the loop integrand.**

**Next we have pointed out a generic parametric form of the loop integrand and fitted the parameters with the help of on-shell values of the loop momenta solving first double cut and single cut conditions (iteratively).**

The loop integration can be easily carried out:

$$\mathbf{I(k, m_1^2, m_2^2) = \int \frac{d^{2-2\epsilon}}{i(2\pi)^{2-2\epsilon}} \mathcal{I}(k, m_1, m_2) = b_0 \, I_2(k^2, m_1^2, m_2^2) + a_{1,0} \, I_1(m_1^2) + a_{2,0} \, I_1(m_2^2) + \frac{b_2}{4\pi}}$$

**Exercise:**

Calculate the photon self-energy in D=2 QED (Scwinger-model) using NV reduction at the integrand level.

# OPP reduction: general case

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✦ Parametric integral over the loop momentum. Any integrand is decomposed in terms a few known functions.

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  Ordered amplitudes given as sums of diagrams.
  N different l-dependent scalar propagators. Momentum inflow to the loop.

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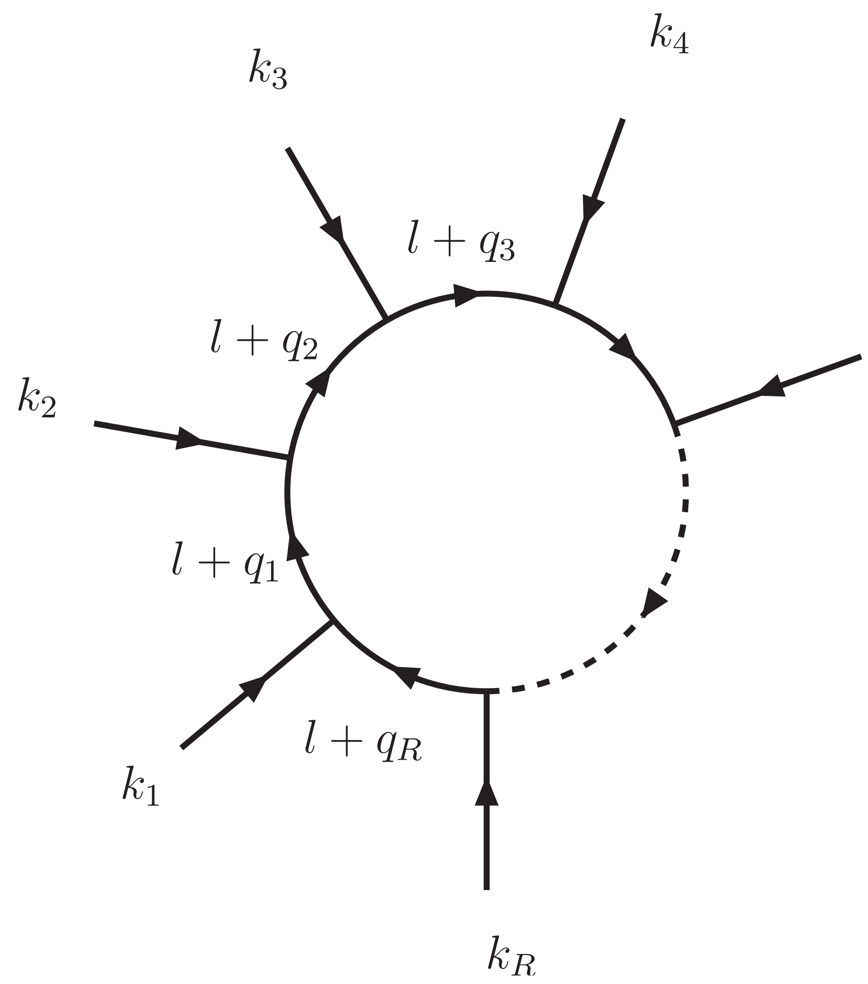

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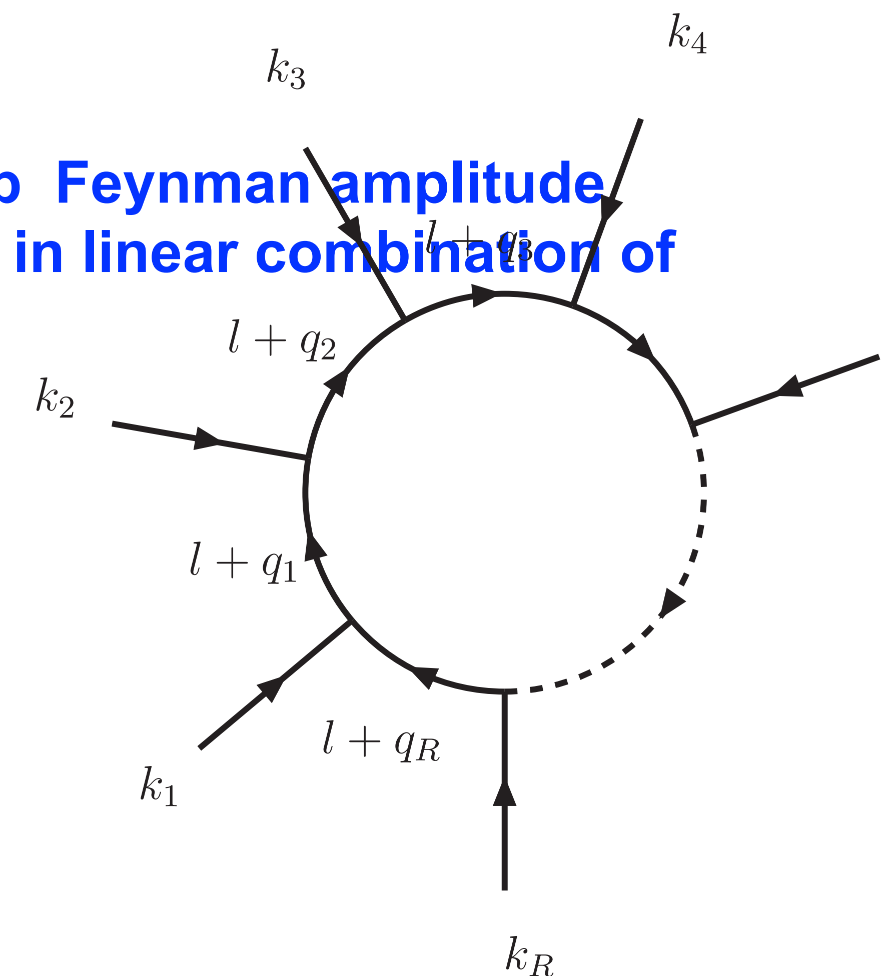
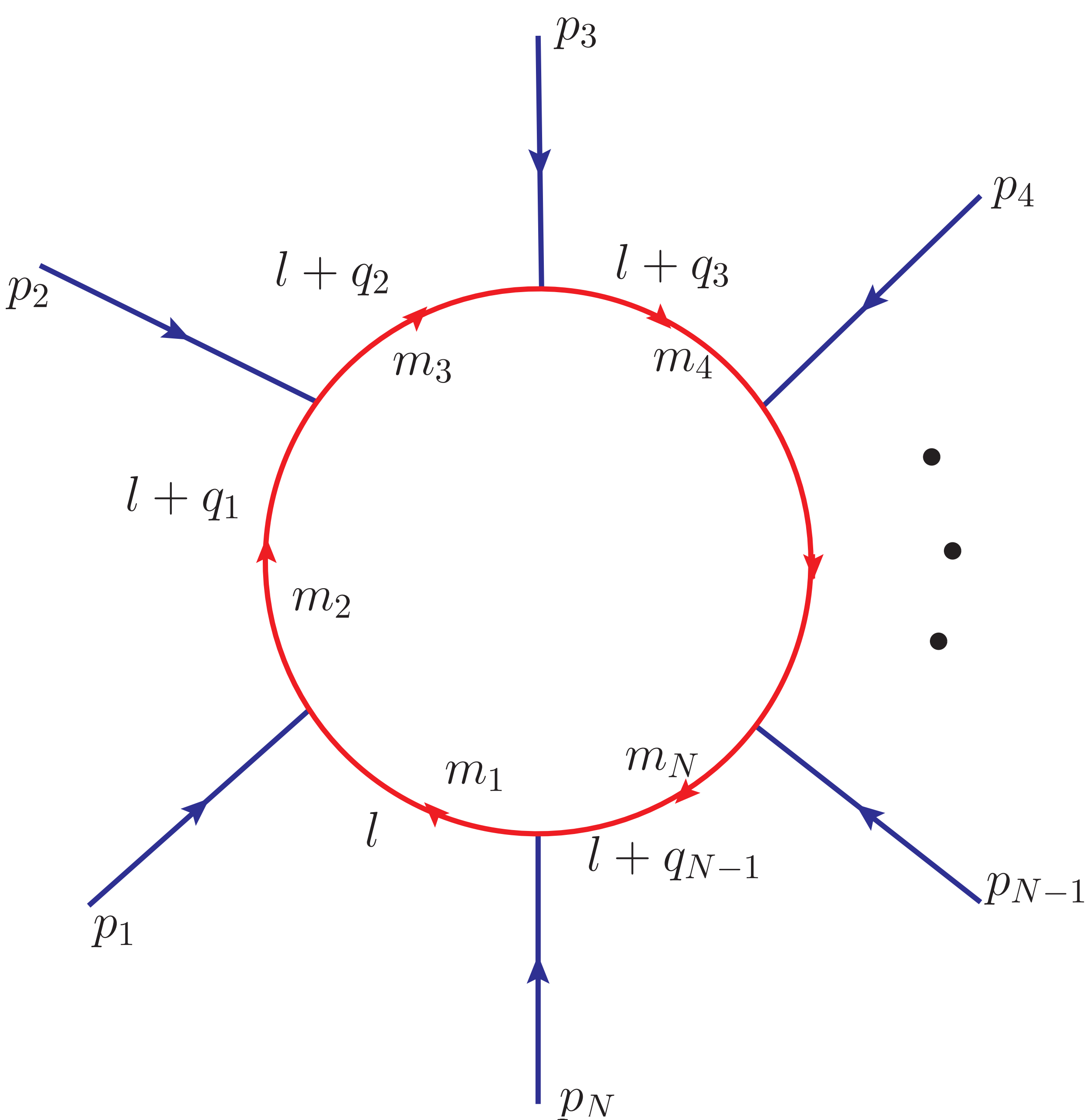
This gives unique prescription of the integrand function as a function of the loop momentum modulo overall shift.

✦ For 4D external kinematics, the integrand of any one-loop Feynman amplitude with arbitrary number of external legs can always be written in linear combination of penta, quadru-,triple-,double- and single-pole terms



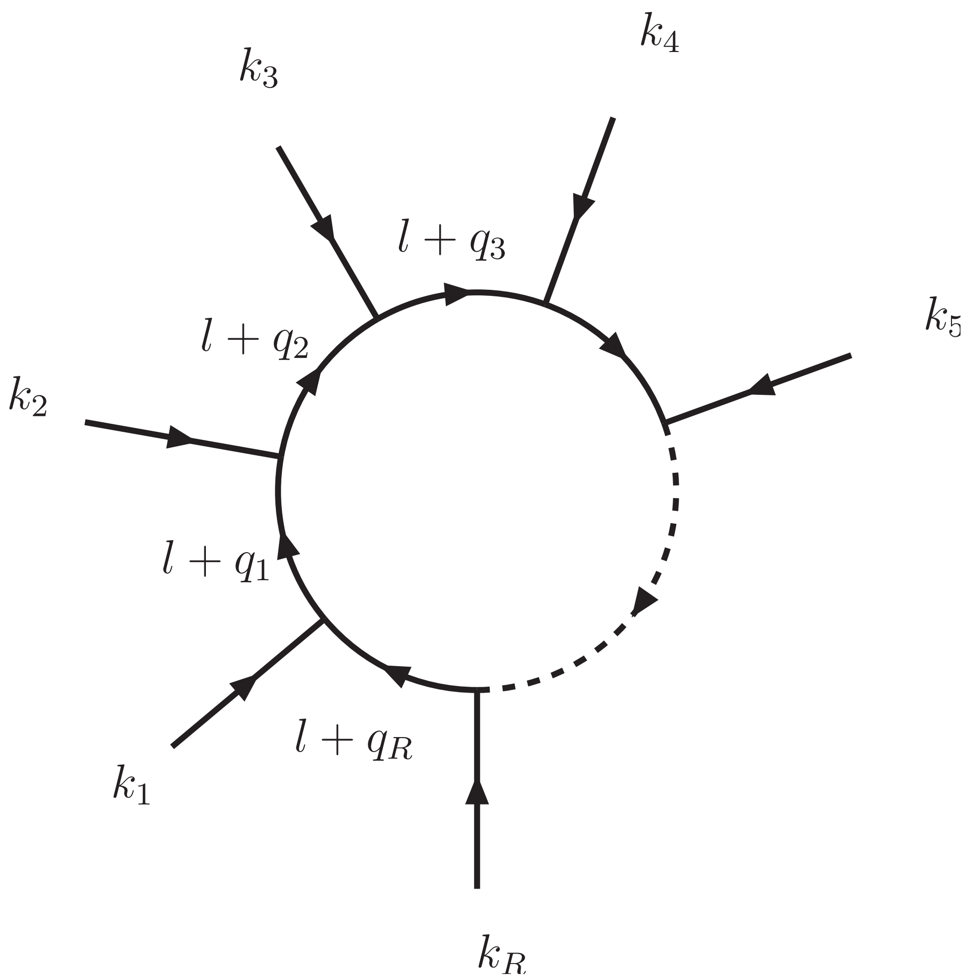
# Ordered amplitudes have well defined integrand

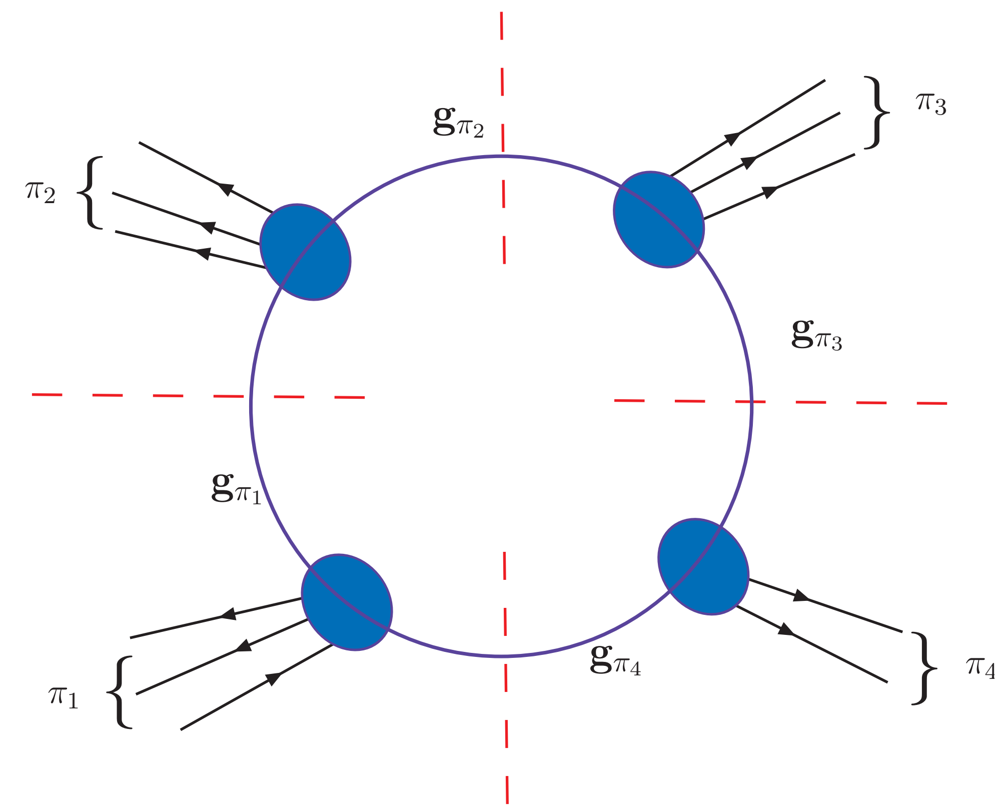
$$\mathcal{I}_N(p_1, p_2, \ldots, p_N, l) = \frac{\mathcal{N}(p_1, p_2, \ldots, p_N; l)}{d_1 d_2 \cdots d_N}$$



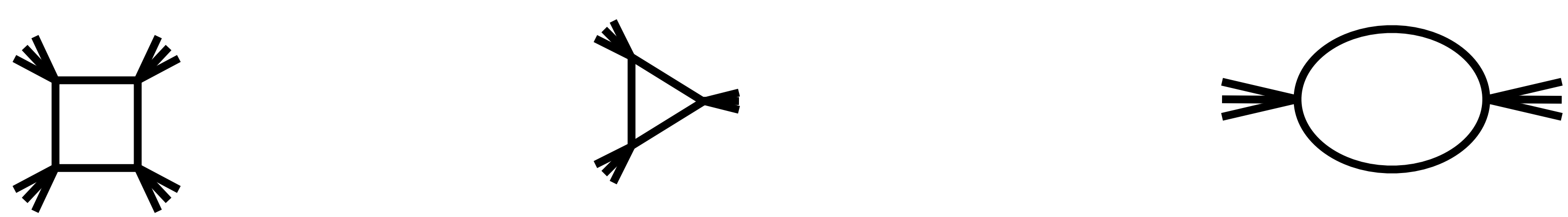

$$\mathcal{I}_N(p_1, p_2, \ldots, p_N, l)$$

## The integrand can be decomposed to pentagon, box, triangle, bubble and tadpole terms




The number of terms with k denominators is $\binom{\mathbf{N}}{\mathbf{k}}$



# The integrand can be decomposed to pentagon, box, triangle, bubble and tadpole terms

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$$
\begin{aligned}
\mathbf{I_N} &= \int \frac{d^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathrm{Num}(\mathbf{l})}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} = \int \frac{d^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathbf{1}}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} \times \Bigg\{ \sum_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \tilde{\mathbf{e}}_{\mathbf{i_1,i_2,i_3,i_4,i_5}}(\mathbf{l}) \prod_{\mathbf{j}\neq[\mathbf{i_1,i_2,i_3,i_4,i_5}]} \mathbf{d_j}(\mathbf{l}) \\
&+ \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}_{\mathbf{i_1,i_2,i_3,i_4}}(\mathbf{l}) \prod_{\mathbf{j}\neq[\mathbf{i_1,i_2,i_3,i_4}]} \mathbf{d_j}(\mathbf{l}) \\
&+ \sum_{\mathbf{i_1,i_2,i_3}} \tilde{\mathbf{c}}_{\mathbf{i_1,i_2,i_3}}(\mathbf{l}) \prod_{\mathbf{j}\neq[\mathbf{i_1,i_2,i_3}]} \mathbf{d_j}(\mathbf{l}) + \sum_{\mathbf{i_1,i_2}} \tilde{\mathbf{b}}_{\mathbf{i_1,i_2}}(\mathbf{l}) \prod_{\mathbf{j}\neq[\mathbf{i_1,i_2}]} \mathbf{d_j}(\mathbf{l}) + \sum_{\mathbf{i_1}} \tilde{\mathbf{a}}_{\mathbf{i_1}}(\mathbf{l}) \prod_{\mathbf{j}\neq\mathbf{i_1}} \mathbf{d_j}(\mathbf{l}) \Bigg\}.
\end{aligned}
$$

$$\mathbf{d_i}(\mathbf{l}) = (\mathbf{l} + \mathbf{q_i})^{\mathbf{2}} - \mathbf{m_i^2}\,, \mathbf{i} = \mathbf{0}, \ldots, \mathbf{4}\,,\ \mathbf{q_0} = \mathbf{0}$$

## The integrand can be decomposed to pentagon, box, triangle, bubble and tadpole terms

$$\mathbf{I_N} = \int \frac{d^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathrm{Num}(\mathbf{l})}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} = \int \frac{d^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathbf{1}}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} \times \Bigg\{ \sum_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \tilde{\mathbf{e}}_{\mathbf{i_1,i_2,i_3,i_4,i_5}}(\mathbf{l}) \prod_{\mathbf{j}\neq[\mathbf{i_1,i_2,i_3,i_4,i_5}]} \mathbf{d_j}(\mathbf{l})$$

$$+ \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}_{\mathbf{i_1,i_2,i_3,i_4}}(\mathbf{l}) \prod_{\mathbf{j}\neq[\mathbf{i_1,i_2,i_3,i_4}]} \mathbf{d_j}(\mathbf{l})$$

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$$\mathbf{d_i}(\mathbf{l}) = (\mathbf{l} + \mathbf{q_i})^{\mathbf{2}} - \mathbf{m_i^2}\,,\ \mathbf{i} = \mathbf{0}, \ldots, \mathbf{4}\,,\ \ \mathbf{q_0} = \mathbf{0}$$

$$\mathrm{Num}(\mathbf{l}) = \mathbf{N_5}(\mathbf{l}) = \prod_{\mathbf{i=1}}^{\mathbf{5}} \mathbf{u_i} \cdot \mathbf{l}, \qquad \mathbf{l}^{\mu} = \sum_{\mathbf{i=1}}^{\mathbf{4}} (\mathbf{l} \cdot \mathbf{q_i}) \mathbf{v_i^{\mu}} + (\mathbf{l} \cdot \mathbf{n}_{\epsilon}) \mathbf{n}_{\epsilon}^{\mu} \quad \mathbf{l} \cdot \mathbf{q_i} = \frac{\mathbf{1}}{\mathbf{2}} \left( \mathbf{d_i} - \mathbf{d_0} - (\mathbf{q_i^2} - \mathbf{m_i^2} + \mathbf{m_0^2}) \right)$$

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$$I_N = \int \frac{d^D l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod_i d_i(l)} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{\prod_i d_i(l)} \times \Bigg\{ \sum_{i_1,i_2,i_3,i_4,i_5} \tilde{e}_{i_1,i_2,i_3,i_4,i_5}(l) \prod_{j \neq [i_1,i_2,i_3,i_4,i_5]} d_j(l)$$

$$+ \sum_{i_1,i_2,i_3,i_4} \tilde{d}_{i_1,i_2,i_3,i_4}(l) \prod_{j \neq [i_1,i_2,i_3,i_4]} d_j(l)$$

$$+ \sum_{i_1,i_2,i_3} \tilde{c}_{i_1,i_2,i_3}(l) \prod_{j \neq [i_1,i_2,i_3]} d_j(l) + \sum_{i_1,i_2} \tilde{b}_{i_1,i_2}(l) \prod_{j \neq [i_1,i_2]} d_j(l) + \sum_{i_1} \tilde{a}_{i_1}(l) \prod_{j \neq i_1} d_j(l) \Bigg\}.$$

$$d_i(l) = (l + q_i)^2 - m_i^2 \,, i = 0, \ldots, 4 \,, \; q_0 = 0$$

$$\mathrm{Num}(l) = N_5(l) = \prod_{i=1}^{5} u_i \cdot l, \qquad l^\mu = \sum_{i=1}^{4} (l \cdot q_i) v_i^\mu + (l \cdot n_\epsilon) n_\epsilon^\mu \quad l \cdot q_i = \frac{1}{2} \left( d_i - d_0 - (q_i^2 - m_i^2 + m_0^2 \right)$$

$$N_5(l) = \left( \prod_i^4 u_i \cdot l \right) (u_5 \cdot l) = \frac{1}{2} \sum_{j=1}^{4} (u_5 \cdot v_j) \left( \prod_i^4 u_i \cdot l \right) (d_j - d_0) \;\; - \frac{1}{2} \sum_{j=1}^{4} (u_5 \cdot v_j) \left( \prod_i^4 u_i \cdot l \right) (q_j^2 - m_j^2 + m_0^2)$$

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$$I_N = \int \frac{d^D l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod_i d_i(l)} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{\prod_i d_i(l)} \times \Bigg\{ \sum_{i_1,i_2,i_3,i_4,i_5} \tilde{e}_{i_1,i_2,i_3,i_4,i_5}(l) \prod_{j\neq[i_1,i_2,i_3,i_4,i_5]} d_j(l)$$

$$+ \sum_{i_1,i_2,i_3,i_4} \tilde{d}_{i_1,i_2,i_3,i_4}(l) \prod_{j\neq[i_1,i_2,i_3,i_4]} d_j(l)$$

$$+ \sum_{i_1,i_2,i_3} \tilde{c}_{i_1,i_2,i_3}(l) \prod_{j\neq[i_1,i_2,i_3]} d_j(l) + \sum_{i_1,i_2} \tilde{b}_{i_1,i_2}(l) \prod_{j\neq[i_1,i_2]} d_j(l) + \sum_{i_1} \tilde{a}_{i_1}(l) \prod_{j\neq i_1} d_j(l) \Bigg\}.$$

$$d_i(l) = (l + q_i)^2 - m_i^2 \,, i = 0, \ldots, 4 \,, \;\; q_0 = 0$$

$$\mathrm{Num}(l) = N_5(l) = \prod_{i=1}^{5} u_i \cdot l, \qquad l^\mu = \sum_{i=1}^{4} (l\cdot q_i) v_i^\mu + (l \cdot n_\epsilon) n_\epsilon^\mu \quad l \cdot q_i = \frac{1}{2}\left(d_i - d_0 - (q_i^2 - m_i^2 + m_0^2)\right)$$

$$N_5(l) = \left(\prod_i^4 u_i \cdot l\right)(u_5 \cdot l) = \frac{1}{2}\sum_{j=1}^{4}(u_5\cdot v_j)\left(\prod_i^4 u_i \cdot l\right)(d_j - d_0) \;\; - \frac{1}{2}\sum_{j=1}^{4}(u_5\cdot v_j)\left(\prod_i^4 u_i \cdot l\right)(q_j^2 - m_j^2 + m_0^2)$$

$$\tilde{e}_{01234}(l) = \tilde{e}^{(0)}_{01234}$$

# Parameter counting

The numerators are simple polynomials of the loop momentum components of the corresponding trivial space.

For a given 5,-4-,3-,2-,1 denominator 1, 5, 10, 10, 5 parameters;

Pentagon (rank five): $\mathbf{D_P = 4\,, D_T = 0 + 1\,, l_T^2} = (\mathbf{ln_\epsilon})^\mathbf{2} = \text{constant terms} + \mathcal{O}(\mathbf{d_i})$

Box (rank four): $\mathbf{D_P = 3\,, D_T = 1 + 1\,, l_T^2} = (\mathbf{ln_4})^\mathbf{2} + (\mathbf{ln_\epsilon})^\mathbf{2} = \text{const} + \mathcal{O}(\mathbf{d_i})$

Triangle (rank three): $\mathbf{D_P = 2\,, D_T = 3 + 1\,, l_T^2} = (\mathbf{ln_3})^\mathbf{2} + (\mathbf{ln_4})^\mathbf{2} + (\mathbf{ln_\epsilon})^\mathbf{2} = \text{const} + \mathcal{O}(\mathbf{d_i})$

Bubble (rank two): $\mathbf{D_P = 1\,, D_T = 3{+}1\,, l_T^2} = (\mathbf{ln_2})^\mathbf{2} + (\mathbf{ln_3})^\mathbf{2} + (\mathbf{ln_4})^\mathbf{2} + (\mathbf{ln_\epsilon})^\mathbf{2} = \text{const} + \mathcal{O}(\mathbf{d_i})$

Tadpole (rank one): $\mathbf{D_P = 0\,, D_T = 4{+}1\,, l_T^2} = (\mathbf{ln_1})^\mathbf{2} + (\mathbf{ln_2})^\mathbf{2} + (\mathbf{ln_3})^\mathbf{2} + (\mathbf{ln_4})^\mathbf{2} + (\mathbf{ln_\epsilon})^\mathbf{2} = \text{const} + \mathcal{O}(\mathbf{d_i})$

# Parameter counting

The numerators are simple polynomials of the loop momentum components of the corresponding trivial space.

For a given 5,-4-,3-,2-,1 denominator 1, 5, 10, 10, 5 parameters; $l_T^2 = l^2 - l_P^2\,, \quad l_P^\mu = (l q_i) v_i^\mu$

Pentagon (rank five): $\mathbf{D_P = 4\,, D_T = 0 + 1\,, l_T^2 = (l n_\epsilon)^2} = \text{constant terms} + \mathcal{O}(\mathbf{d_i})$

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Triangle (rank three): $\mathbf{D_P = 2\,, D_T = 3 + 1\,, l_T^2 = (l n_3)^2 + (l n_4)^2 + (l n_\epsilon)^2} = \text{const} + \mathcal{O}(\mathbf{d_i})$

Bubble (rank two): $\mathbf{D_P = 1\,, D_T = 3 + 1\,, l_T^2 = (l n_2)^2 + (l n_3)^2 + (l n_4)^2 + (l n_\epsilon)^2} = \text{const} + \mathcal{O}(\mathbf{d_i})$

Tadpole (rank one): $\mathbf{D_P = 0\,, D_T = 4 + 1\,, l_T^2 = (l n_1)^2 + (l n_2)^2 + (l n_3)^2 + (l n_4)^2 + (l n_\epsilon)^2} = \text{const} + \mathcal{O}(\mathbf{d_i})$

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$$\tilde{\mathbf{e}}_{\mathbf{01234}}(\mathbf{l}) = \tilde{\mathbf{e}}^{(\mathbf{0})}_{\mathbf{01234}}$$

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$$\tilde{\mathbf{d}}_{\mathbf{0123}}(\mathbf{l}) = \tilde{\mathbf{d}}_{\mathbf{0}} + \tilde{\mathbf{d}}_{\mathbf{1}}(\mathbf{l}\cdot\mathbf{n_4}) + \tilde{\mathbf{d}}_{\mathbf{2}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{2}} + \tilde{\mathbf{d}}_{\mathbf{3}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{2}}(\mathbf{l}\cdot\mathbf{n_4}) + \tilde{\mathbf{d}}_{\mathbf{4}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{4}},$$

# Parameter counting

$$\tilde{\mathbf{e}}_{01234}(\mathbf{l}) = \tilde{\mathbf{e}}^{(0)}_{01234}$$

$$\tilde{\mathbf{d}}_{0123}(\mathbf{l}) = \tilde{\mathbf{d}}_0 + \tilde{\mathbf{d}}_1(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_2(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{d}}_3(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_4(\mathbf{l}\cdot\mathbf{n}_\epsilon)^4,$$

$$\begin{aligned}\tilde{\mathbf{c}}_{012}(\mathbf{l}) = {} & \tilde{\mathbf{c}}_0 + \tilde{\mathbf{c}}_1(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{c}}_2(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{c}}_3((\mathbf{l}\cdot\mathbf{n}_3)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) + \tilde{\mathbf{c}}_4(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{c}}_5(\mathbf{l}\cdot\mathbf{n}_3)^3 \\ & + \tilde{\mathbf{c}}_6(\mathbf{l}\cdot\mathbf{n}_4)^3 + \tilde{\mathbf{c}}_7(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{c}}_8(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{c}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4)\end{aligned}$$

# Parameter counting

$$\tilde{\mathbf{e}}_{01234}(\mathbf{l}) = \tilde{\mathbf{e}}^{(0)}_{01234}$$

$$\tilde{\mathbf{d}}_{0123}(\mathbf{l}) = \tilde{\mathbf{d}}_0 + \tilde{\mathbf{d}}_1(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_2(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{d}}_3(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_4(\mathbf{l}\cdot\mathbf{n}_\epsilon)^4,$$

$$\begin{aligned}\tilde{\mathbf{c}}_{012}(\mathbf{l}) = {} & \tilde{\mathbf{c}}_0+\tilde{\mathbf{c}}_1(\mathbf{l}\cdot\mathbf{n}_3)+\tilde{\mathbf{c}}_2(\mathbf{l}\cdot\mathbf{n}_4)+\tilde{\mathbf{c}}_3((\mathbf{l}\cdot\mathbf{n}_3)^2-(\mathbf{l}\cdot\mathbf{n}_4)^2)+\tilde{\mathbf{c}}_4(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{c}}_5(\mathbf{l}\cdot\mathbf{n}_3)^3 \\ & + \tilde{\mathbf{c}}_6(\mathbf{l}\cdot\mathbf{n}_4)^3 + \tilde{\mathbf{c}}_7(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{c}}_8(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{c}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4)\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{b}}_{01}(\mathbf{l}) = {} & \tilde{\mathbf{b}}_0+\tilde{\mathbf{b}}_1(\mathbf{l}\cdot\mathbf{n}_2)+\tilde{\mathbf{b}}_2(\mathbf{l}\cdot\mathbf{n}_3)+\tilde{\mathbf{b}}_3(\mathbf{l}\cdot\mathbf{n}_4)+\tilde{\mathbf{b}}_4((\mathbf{l}\cdot\mathbf{n}_2)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) + \tilde{\mathbf{b}}_5((\mathbf{l}\cdot\mathbf{n}_3)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) \\ & + \tilde{\mathbf{b}}_6(\mathbf{l}\cdot\mathbf{n}_2)(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{b}}_7(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{b}}_8(\mathbf{l}\cdot\mathbf{n}_2)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{b}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2,\end{aligned}$$

# Parameter counting

$$\tilde{\mathbf{e}}_{01234}(\mathbf{l}) = \tilde{\mathbf{e}}^{(0)}_{01234}$$

$$\tilde{\mathbf{d}}_{0123}(\mathbf{l}) = \tilde{\mathbf{d}}_0 + \tilde{\mathbf{d}}_1(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_2(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{d}}_3(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_4(\mathbf{l}\cdot\mathbf{n}_\epsilon)^4,$$

$$\begin{aligned}\tilde{\mathbf{c}}_{012}(\mathbf{l}) = {} & \tilde{\mathbf{c}}_0 + \tilde{\mathbf{c}}_1(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{c}}_2(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{c}}_3((\mathbf{l}\cdot\mathbf{n}_3)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) + \tilde{\mathbf{c}}_4(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{c}}_5(\mathbf{l}\cdot\mathbf{n}_3)^3 \\ & + \tilde{\mathbf{c}}_6(\mathbf{l}\cdot\mathbf{n}_4)^3 + \tilde{\mathbf{c}}_7(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{c}}_8(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{c}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4)\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{b}}_{01}(\mathbf{l}) = {} & \tilde{\mathbf{b}}_0 + \tilde{\mathbf{b}}_1(\mathbf{l}\cdot\mathbf{n}_2) + \tilde{\mathbf{b}}_2(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{b}}_3(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{b}}_4((\mathbf{l}\cdot\mathbf{n}_2)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) + \tilde{\mathbf{b}}_5((\mathbf{l}\cdot\mathbf{n}_3)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) \\ & + \tilde{\mathbf{b}}_6(\mathbf{l}\cdot\mathbf{n}_2)(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{b}}_7(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{b}}_8(\mathbf{l}\cdot\mathbf{n}_2)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{b}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2,\end{aligned}$$

$$\tilde{\mathbf{a}}_{\mathbf{i}}(\mathbf{l}) = \tilde{\mathbf{a}}_0 + \tilde{\mathbf{a}}_1(\mathbf{l}\cdot\mathbf{n}_1) + \tilde{\mathbf{a}}_2(\mathbf{l}\cdot\mathbf{n}_2) + \tilde{\mathbf{a}}_3(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{a}}_4(\mathbf{l}\cdot\mathbf{n}_4)$$

# Parameter counting

$$\tilde{\mathbf{e}}_{01234}(\mathbf{l}) = \tilde{\mathbf{e}}^{(0)}_{01234}$$

$$\tilde{\mathbf{d}}_{0123}(\mathbf{l}) = \tilde{\mathbf{d}}_0 + \tilde{\mathbf{d}}_1(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_2(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{d}}_3(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_4(\mathbf{l}\cdot\mathbf{n}_\epsilon)^4,$$

$$\begin{aligned}\tilde{\mathbf{c}}_{012}(\mathbf{l}) = {} & \tilde{\mathbf{c}}_0+\tilde{\mathbf{c}}_1(\mathbf{l}\cdot\mathbf{n}_3)+\tilde{\mathbf{c}}_2(\mathbf{l}\cdot\mathbf{n}_4)+\tilde{\mathbf{c}}_3((\mathbf{l}\cdot\mathbf{n}_3)^2-(\mathbf{l}\cdot\mathbf{n}_4)^2)+\tilde{\mathbf{c}}_4(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{c}}_5(\mathbf{l}\cdot\mathbf{n}_3)^3 \\ & + \tilde{\mathbf{c}}_6(\mathbf{l}\cdot\mathbf{n}_4)^3 + \tilde{\mathbf{c}}_7(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{c}}_8(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{c}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4)\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{b}}_{01}(\mathbf{l}) = {} & \tilde{\mathbf{b}}_0+\tilde{\mathbf{b}}_1(\mathbf{l}\cdot\mathbf{n}_2)+\tilde{\mathbf{b}}_2(\mathbf{l}\cdot\mathbf{n}_3)+\tilde{\mathbf{b}}_3(\mathbf{l}\cdot\mathbf{n}_4)+\tilde{\mathbf{b}}_4((\mathbf{l}\cdot\mathbf{n}_2)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) + \tilde{\mathbf{b}}_5((\mathbf{l}\cdot\mathbf{n}_3)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) \\ & + \tilde{\mathbf{b}}_6(\mathbf{l}\cdot\mathbf{n}_2)(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{b}}_7(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{b}}_8(\mathbf{l}\cdot\mathbf{n}_2)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{b}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2,\end{aligned}$$

$$\tilde{\mathbf{a}}_\mathbf{i}(\mathbf{l}) = \tilde{\mathbf{a}}_0 + \tilde{\mathbf{a}}_1(\mathbf{l}\cdot\mathbf{n}_1) + \tilde{\mathbf{a}}_2(\mathbf{l}\cdot\mathbf{n}_2) + \tilde{\mathbf{a}}_3(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{a}}_4(\mathbf{l}\cdot\mathbf{n}_4)$$

The coefficients $\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{e}}_{01234}$ are independent from the loop momenta and in all numerator functions we can replace $(\mathbf{l}\cdot\mathbf{n_i})$ $\mathbf{with}$ $(\mathbf{l_T}\cdot\mathbf{n_i})$

Note that the integration over the transverse space is trivial

$$\int d^{D_1}\mathbf{l}_\perp \delta(\mathbf{l}_\perp^2 - \mu_\mathbf{0}^2)\,(\mathbf{l}_\perp^\mu, \mathbf{l}_\perp^\mu \mathbf{l}_\perp^\nu) = \int d^{D_1}\mathbf{l}_\perp \delta(\mathbf{l}_\perp^2 - \mu_\mathbf{0}^2)\left(\mathbf{0},\ \frac{\mathbf{g}_\perp^{\mu\nu}}{\mathbf{D_1}}\,\mathbf{l}_\perp^2\right), \qquad \mathbf{D_1} = \mathbf{D} - \mathbf{1}$$

Only the constant terms and some of the terms depending on $(\mathbf{l} \cdot \mathbf{n}_\epsilon)^2$ give non-vanishing integrals

Only the constant terms and some of the terms depending on $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{2}}$ give non-vanishing integrals

$$\mathbf{I_N} = \int \frac{\mathrm{d}^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathrm{Num}(\mathbf{l})}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} = \sum_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \tilde{\mathbf{e}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \mathbf{I}_{\mathbf{i_1 i_2 i_3 i_4 i_5}}$$

$$+ \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4}} \mathbf{I}_{\mathbf{i_1 i_2 i_3 i_4}} + \sum_{\mathbf{i_1,i_2,i_3}} \tilde{\mathbf{c}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3}} \mathbf{I}_{\mathbf{i_1 i_2 i_3}} + \sum_{\mathbf{i_1,i_2}} \tilde{\mathbf{b}}^{(\mathbf{0})}_{\mathbf{i_1,i_2}} \mathbf{I}_{\mathbf{i_1 i_2}} + \sum_{\mathbf{i_1}} \tilde{\mathbf{a}}^{(\mathbf{0})}_{\mathbf{i_1}} \mathbf{I}_{\mathbf{i_1}} + \mathcal{R}$$

Only the constant terms and some of the terms depending on $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^\mathbf{2}$ give non-vanishing integrals

$$\mathbf{I_N} = \int \frac{\mathrm{d}^\mathbf{D}\mathbf{l}}{(\mathbf{2}\pi)^\mathbf{D}} \frac{\mathrm{Num}(\mathbf{l})}{\prod_\mathbf{i} \mathbf{d_i}(\mathbf{l})} = \sum_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \tilde{\mathbf{e}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \mathbf{I_{i_1i_2i_3i_4i_5}}$$
$$+ \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4}} \mathbf{I_{i_1i_2i_3i_4}} + \sum_{\mathbf{i_1,i_2,i_3}} \tilde{\mathbf{c}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3}} \mathbf{I_{i_1i_2i_3}} + \sum_{\mathbf{i_1,i_2}} \tilde{\mathbf{b}}^{(\mathbf{0})}_{\mathbf{i_1,i_2}} \mathbf{I_{i_1i_2}} + \sum_{\mathbf{i_1}} \tilde{\mathbf{a}}^{(\mathbf{0})}_{\mathbf{i_1}} \mathbf{I_{i_1}} + \mathcal{R}$$

Non-vanishing master integrals with $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^\mathbf{2}$ factors in the numerator

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = -\frac{D-4}{2} I^{D+2}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^4}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = \frac{(D-2)(D-4)}{4} I^{D+4}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2 i_3},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2}.$$

$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3 i_4} = 0,$$
$$\lim_{D\to 4} \frac{(D-4)(D-2)}{4} I^{(D+4)}_{i_1 i_2 i_3 i_4} = -\frac{1}{3},$$
$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3} = \frac{1}{2},$$
$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2} = -\frac{m_{i_1}^2 + m_{i_2}^2}{2} + \frac{1}{6}(q_{i_1} - q_{i_2})^2$$

Only the constant terms and some of the terms depending on $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{2}}$ give non-vanishing integrals

$$\mathbf{I_N} = \int \frac{\mathrm{d}^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathrm{Num}(\mathbf{l})}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} = \sum_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \tilde{\mathbf{e}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \mathbf{I}_{\mathbf{i_1 i_2 i_3 i_4 i_5}}$$
$$+ \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4}} \mathbf{I}_{\mathbf{i_1 i_2 i_3 i_4}} + \sum_{\mathbf{i_1,i_2,i_3}} \tilde{\mathbf{c}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3}} \mathbf{I}_{\mathbf{i_1 i_2 i_3}} + \sum_{\mathbf{i_1,i_2}} \tilde{\mathbf{b}}^{(\mathbf{0})}_{\mathbf{i_1,i_2}} \mathbf{I}_{\mathbf{i_1 i_2}} + \sum_{\mathbf{i_1}} \tilde{\mathbf{a}}^{(\mathbf{0})}_{\mathbf{i_1}} \mathbf{I}_{\mathbf{i_1}} + \mathcal{R}$$

Non-vanishing master integrals with $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{2}}$ factors in the numerator

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = -\frac{D-4}{2} I^{D+2}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^4}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = \frac{(D-2)(D-4)}{4} I^{D+4}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2 i_3},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2}.$$

$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3 i_4} = 0,$$
$$\lim_{D\to 4} \frac{(D-4)(D-2)}{4} I^{(D+4)}_{i_1 i_2 i_3 i_4} = -\frac{1}{3},$$
$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3} = \frac{1}{2},$$
$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2} = -\frac{m_{i_1}^2 + m_{i_2}^2}{2} + \frac{1}{6}(q_{i_1} - q_{i_2})^2$$

they contribute to the rational terms

Only the constant terms and some of the terms depending on $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2$ give non-vanishing integrals

$$\mathbf{I_N} = \int \frac{d^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathrm{Num}(\mathbf{l})}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} = \sum_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \tilde{\mathbf{e}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \mathbf{I}_{\mathbf{i_1 i_2 i_3 i_4 i_5}}$$
$$+ \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4}} \mathbf{I}_{\mathbf{i_1 i_2 i_3 i_4}} + \sum_{\mathbf{i_1,i_2,i_3}} \tilde{\mathbf{c}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3}} \mathbf{I}_{\mathbf{i_1 i_2 i_3}} + \sum_{\mathbf{i_1,i_2}} \tilde{\mathbf{b}}^{(\mathbf{0})}_{\mathbf{i_1,i_2}} \mathbf{I}_{\mathbf{i_1 i_2}} + \sum_{\mathbf{i_1}} \tilde{\mathbf{a}}^{(\mathbf{0})}_{\mathbf{i_1}} \mathbf{I}_{\mathbf{i_1}} + \mathcal{R}$$

Non-vanishing master integrals with $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2$ factors in the numerator

$$\int \frac{d^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = -\frac{D-4}{2} I^{D+2}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{d^D l}{(i\pi)^{D/2}} \frac{s_e^4}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = \frac{(D-2)(D-4)}{4} I^{D+4}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{d^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2 i_3},$$
$$\int \frac{d^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2}.$$

$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3 i_4} = 0,$$
$$\lim_{D\to 4} \frac{(D-4)(D-2)}{4} I^{(D+4)}_{i_1 i_2 i_3 i_4} = -\frac{1}{3},$$
$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3} = \frac{1}{2},$$
$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2} = -\frac{m_{i_1}^2 + m_{i_2}^2}{2} + \frac{1}{6}(q_{i_1} - q_{i_2})^2$$

they contribute to the rational terms

$$\mathcal{R} = -\sum_{\mathbf{i_1,i_2,i_3,i_4}} \frac{\tilde{\mathbf{d}}^{(4)}_{\mathbf{i_1 i_2 i_3 i_4}}}{\mathbf{6}} + \sum_{\mathbf{i_1,i_2,i_3}} \frac{\tilde{\mathbf{c}}^{(7)}_{\mathbf{i_1 i_2 i_3}}}{\mathbf{2}} + \sum_{\mathbf{i_1,i_2}} \left[\frac{\mathbf{m}^2_{\mathbf{i_1}} + \mathbf{m}^2_{\mathbf{i_2}}}{\mathbf{2}} - \frac{(\mathbf{q_{i_1}} - \mathbf{q_{i_2}})^2}{\mathbf{6}}\right] \tilde{\mathbf{b}}^{(9)}_{\mathbf{i_1 i_2}}.$$

# Example: Projecting out individual (quadrupole) coefficients in D-dimension

$$\mathcal{I}_N(p_1,p_2,\ldots,p_N,l) = \frac{\mathcal{N}(p_1,p_2,\ldots,p_N;l)}{d_1 d_2 \cdots d_N} =$$

$$= \sum_{i_1 \le i_2 \le i_3 \le i_4 \le i_5 \le n} \frac{\bar{e}_{i_1 i_2 i_3 i_4 i_5}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} +$$

$$\sum_{1 \le i_1 < i_2 < i_3 < i_4 \le N} \frac{\bar{d}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{1 \le i_1 < i_2 < i_3 \le N} \frac{\bar{c}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{1 \le i_1 < i_2 \le N} \frac{\bar{b}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{1 \le i_1 \le N} \frac{\bar{a}_{i_1}(l)}{d_{i_1}}$$

**Denote:**

$$\mathbf{RR(l) = Residuum_{0123}(Integrand)}$$
$$\mathbf{- \ pentagon\ contributions\ = \tilde{d}_{0123}(l)}$$

## Projecting out individual quadrupole coefficients in D-dimension:

$$\tilde{\mathbf{d}}_{\mathbf{0123}}(\mathbf{l}) = \tilde{\mathbf{d}}_{\mathbf{0}} + \tilde{\mathbf{d}}_{\mathbf{1}}(\mathbf{l}\cdot\mathbf{n_4}) + \tilde{\mathbf{d}}_{\mathbf{2}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{2}} + \tilde{\mathbf{d}}_{\mathbf{3}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{2}}(\mathbf{l}\cdot\mathbf{n_4}) + \tilde{\mathbf{d}}_{\mathbf{4}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{4}},$$

$$\mathbf{l}^\mu = \mathbf{V}^\mu + \mathbf{l}_\perp(\cos\phi\,\mathbf{n}_\mathbf{4}^\mu + \sin\phi\,\mathbf{n}_\epsilon^\mu), \qquad \mathbf{V}^\mu = -\frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{i}}^{\mathbf{3}} \mathbf{v}_\mathbf{i}^\mu\left(\mathbf{q}_\mathbf{i}^\mathbf{2} - \mathbf{m}_\mathbf{i}^\mathbf{2} + \mathbf{m}_\mathbf{0}^\mathbf{2}\right),$$

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calculate the residuum of the integrand for these values

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$$\tilde{\mathbf{d}}_2 + \tilde{\mathbf{d}}_4\mathbf{l}_\perp^2 = \frac{\mathrm{Num}(\mathbf{l}_\epsilon) - \tilde{\mathbf{d}}_0}{\mathbf{l}_\perp^2}.$$

## Comments on the rational part:

The origin is UV divergent tensor integrals. Reduction requires regularization.
After reduction: D-dimensional finite tensor integrals. In the limiting case D=4 they provide finite constants independent from the kinematics.

It may happen that the numerator has manifest dependence on (D-4) coming from polarization sum. These terms can only contribute if it appears in a term leading to UV divergent integrals. There are only few UV divergent one-loop Feynman diagrams even for large number of external particles (OPP).

Sophisticated recursion relations (BDK) as an option in Black Hat

# Comment on N=4 sYM amplitudes

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- $N = 4$ sYM scattering amplitudes are free from UV divergences.
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The maximum number of loop momentum in the numerator of Feynman-diagrams is reduced by one for N=1 and by four for N=4 .

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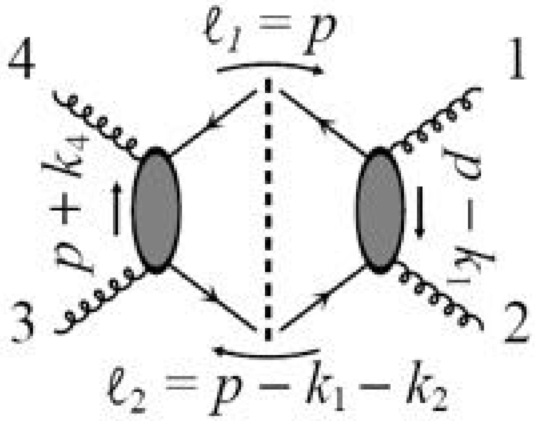
**Excercise (not easy):**

Find proper parametrization for the bubble numerator $\tilde{b}_{01}(l)$
in case of light-like inflow momentum.

# Lecture 3: Unitarity method and amplitudes

# Constraints from Unitarity: $M^\dagger - M = -iM^\dagger M$




$$-i\,\mathrm{Disc}\,A_4(1,2,3,4)\Big|_{s-\mathrm{cut}} = \int \frac{d^4p}{(2\pi)^4}\, 2\pi\delta^{(+)}(\ell_1^2 - m^2)\, 2\pi\delta^{(+)}(\ell_2^2 - m^2) \times A_4^{\mathrm{tree}}(-\ell_1, 1, 2, \ell_2)\, A_4^{\mathrm{tree}}(-\ell_2, 3, 4, \ell_1)\,,$$

Imaginary part of NLO an amplitude is  calculated from tree amplitudes.

- Non-linear relation, iterative in the coupling.
- Iterative in amplitudes. Building blocks are amplitudes and not Feynman diagrams
- Manifestly gauge invariant.

# Unitarity and Cutkosky rule:

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Scattering amplitudes of scalar particles are functions of the external momenta $\mathbf{A_N(\{p_i\})}$

In case of 2 to 2 scattering with equal masses the scattering amplitude $\mathbf{A_4(s,t)}$ depends on Lorenz-invariant Mandelstam-variables s,t , $\mathbf{s = (p_1+p_2)^2, t = (p_1+p_3)^2}$,

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The scattering amplitude as analytic function of s at fixed $\theta_{\mathrm{cms}}$ ?

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$\mathbf{M(s,t)}$ has branch cuts in the complex s-plane at $\mathbf{(nm)^2}$, $\mathbf{(n = 2, 3, \ldots)}$

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Scattering amplitudes of scalar particles are functions of the external momenta $\mathbf{A_N(\{p_i\})}$

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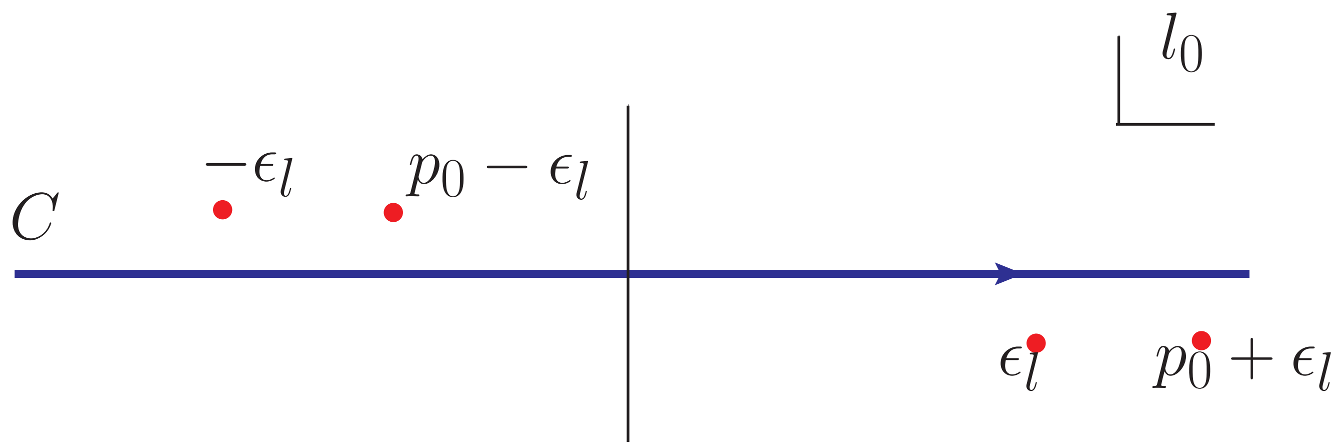


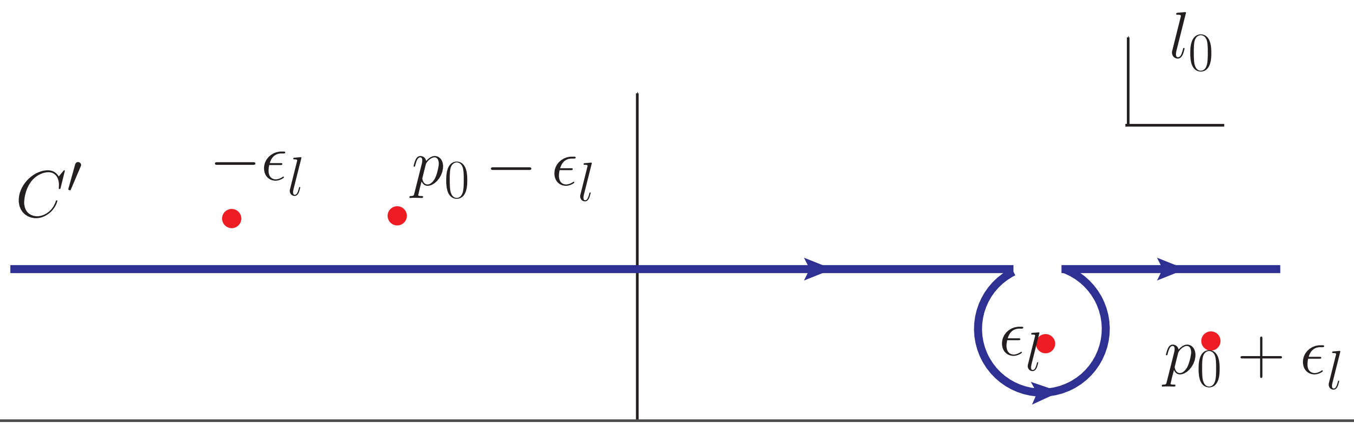

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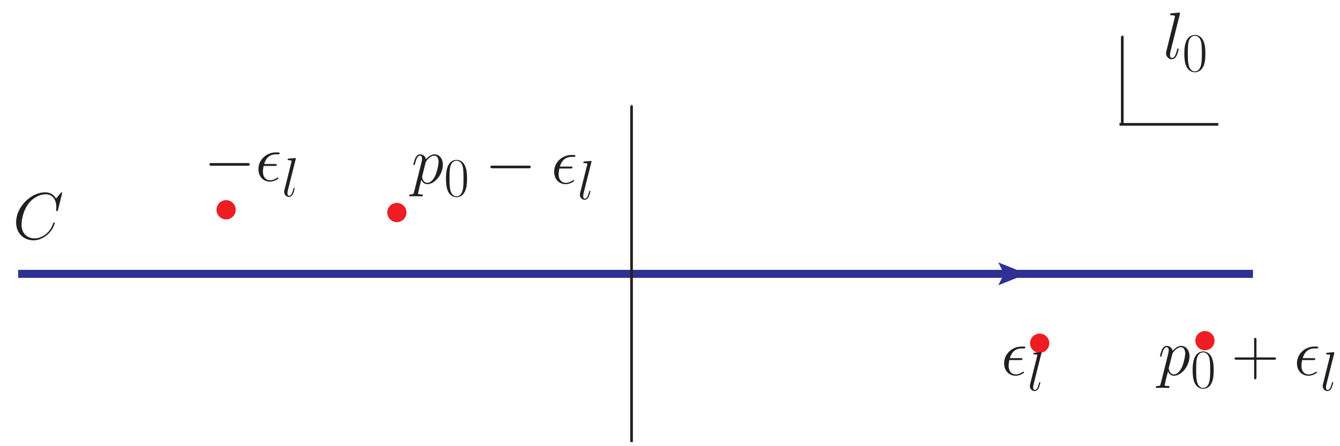


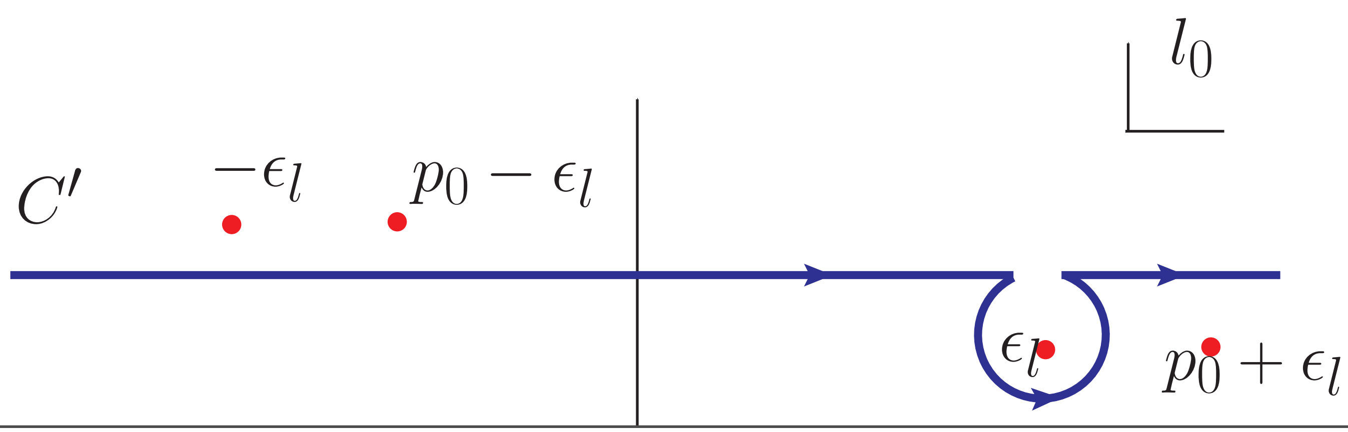

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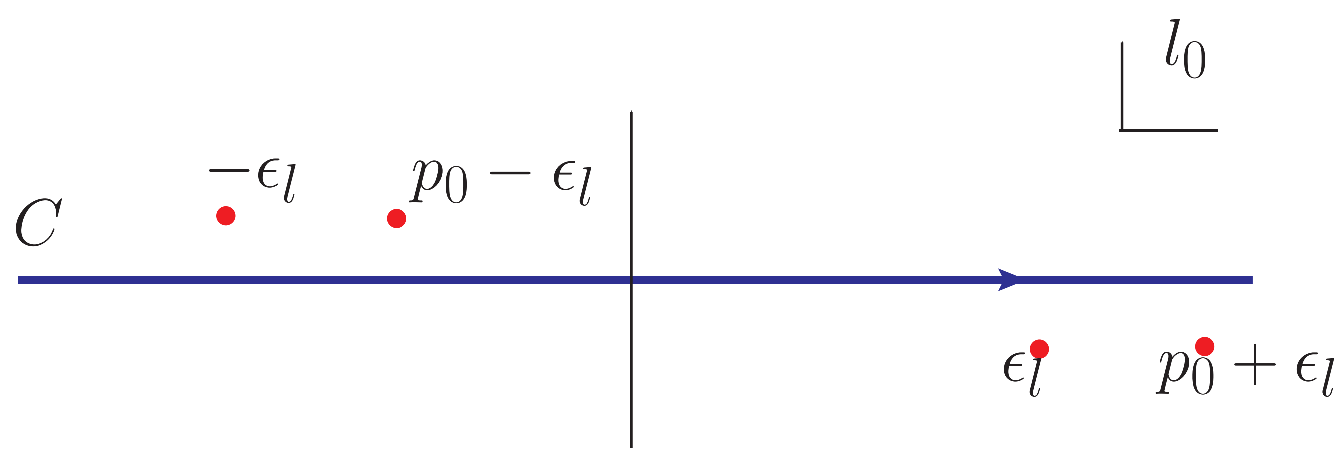
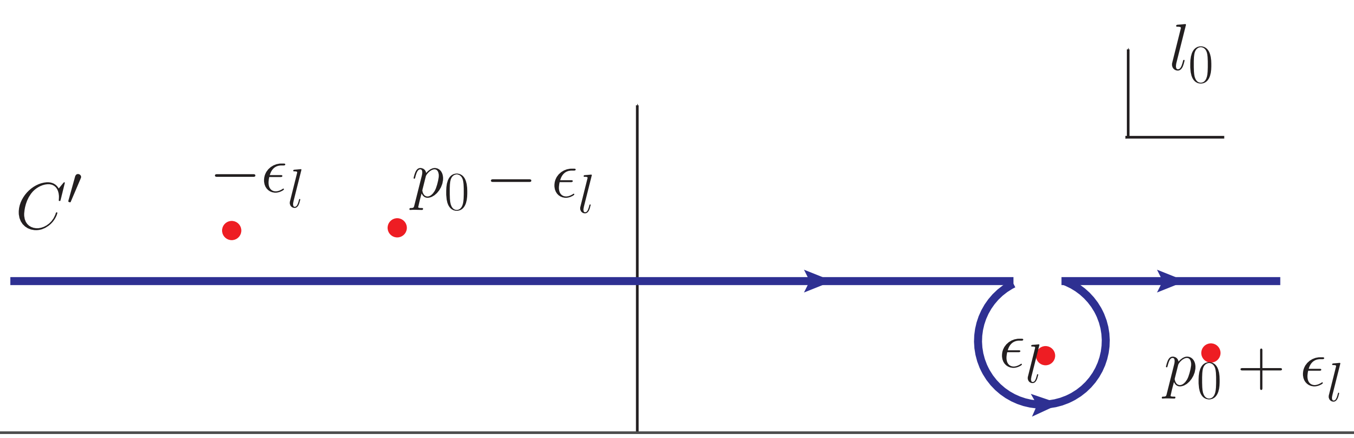
The distance of the poles $a_1, b_2$ can vanish if $p_0 > 2m$ when these poles can pinch the contour.




One can avoid pinching the contour by moving the first pole above the contour.
To compensate the difference for the integral we add an integral over a closed small circle around the first pole

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Excercise: Work out the Landau equations and solve them for the self energy and two different internal mass and find the location of the branch cut.

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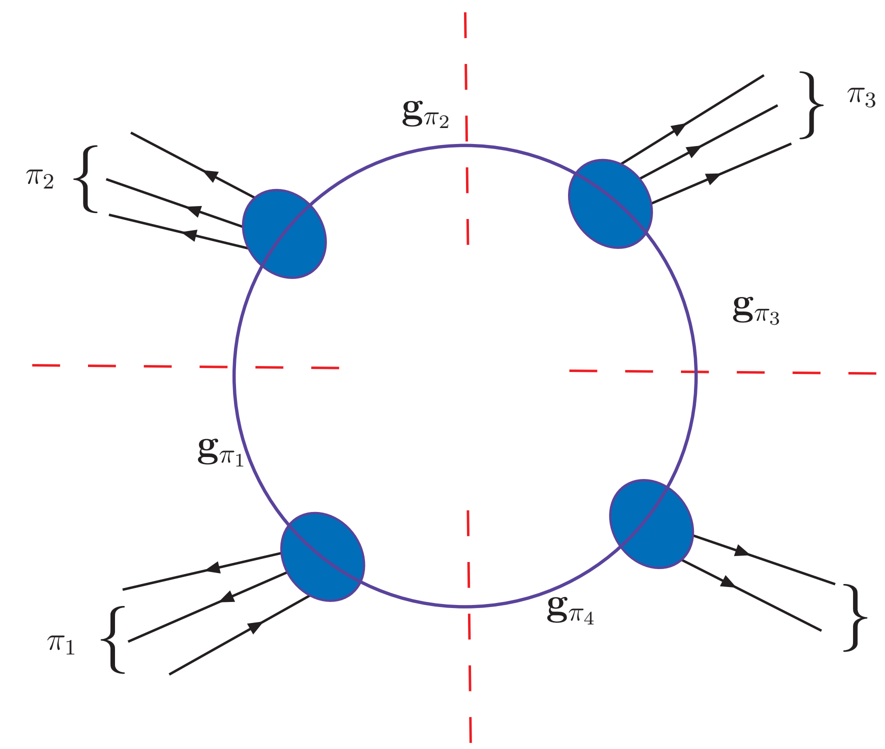

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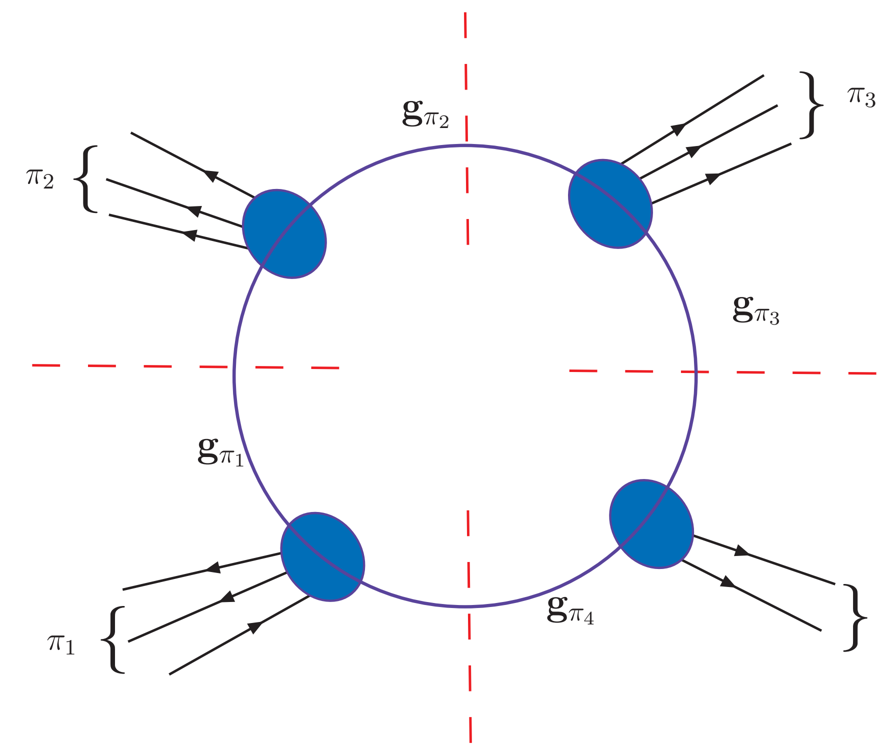

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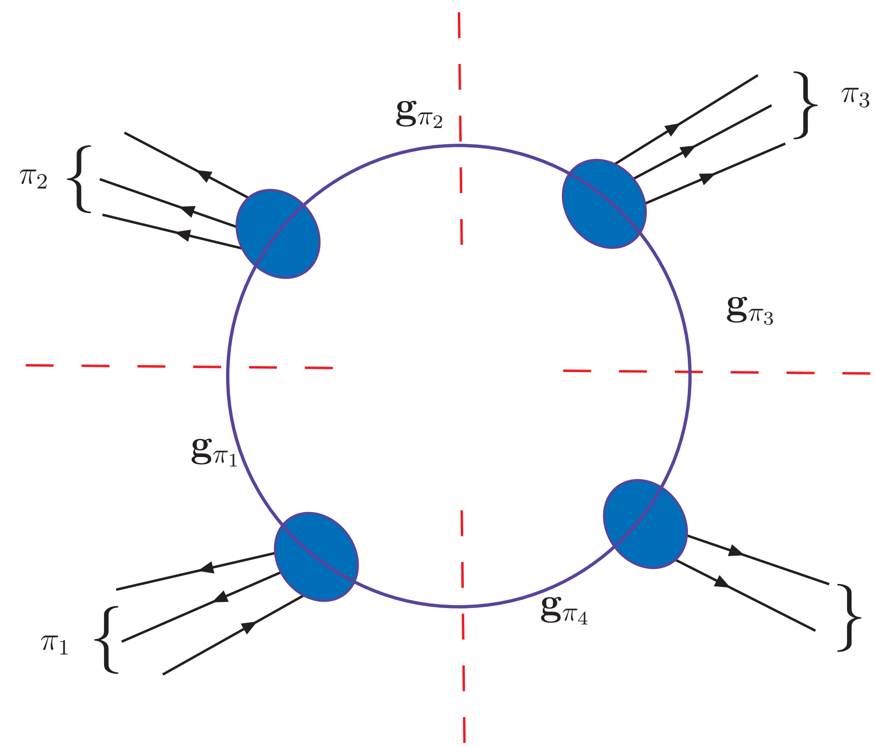


$$-\mathbf{g}_{\mu\nu}^{(\mathbf{D_s})}+\frac{\mathbf{l}_\mu\mathbf{b}_\nu+\mathbf{b}_\mu\mathbf{l}_\nu}{\mathbf{l}\cdot\mathbf{b}}=\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{D_s}-\mathbf{2}}\mathbf{e}_\mu^{(\mathbf{i})}(\mathbf{l})\mathbf{e}_\nu^{(\mathbf{i})}(\mathbf{l}),$$

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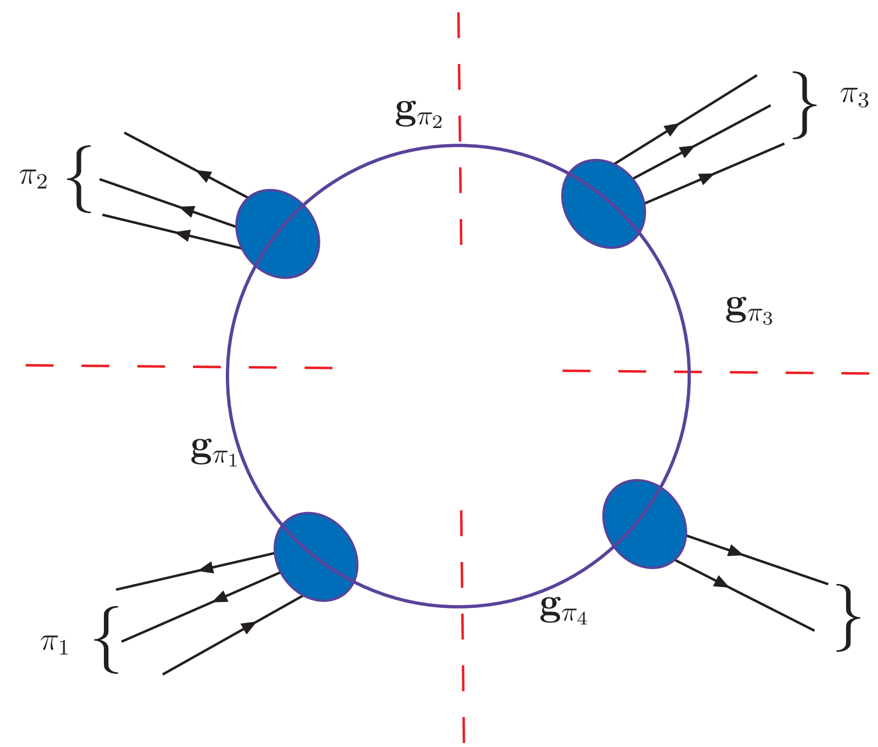


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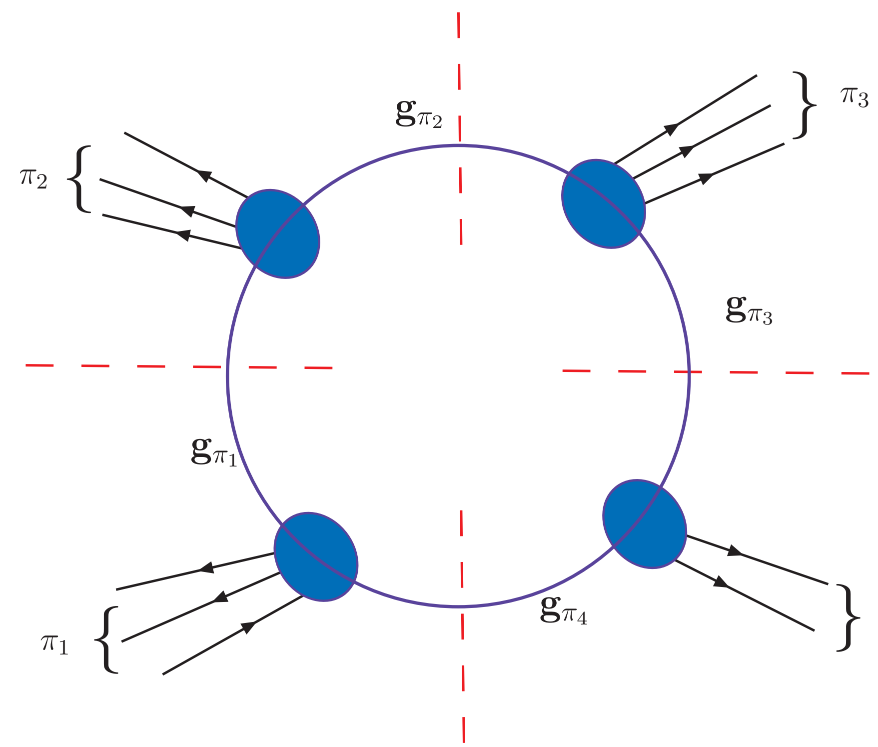


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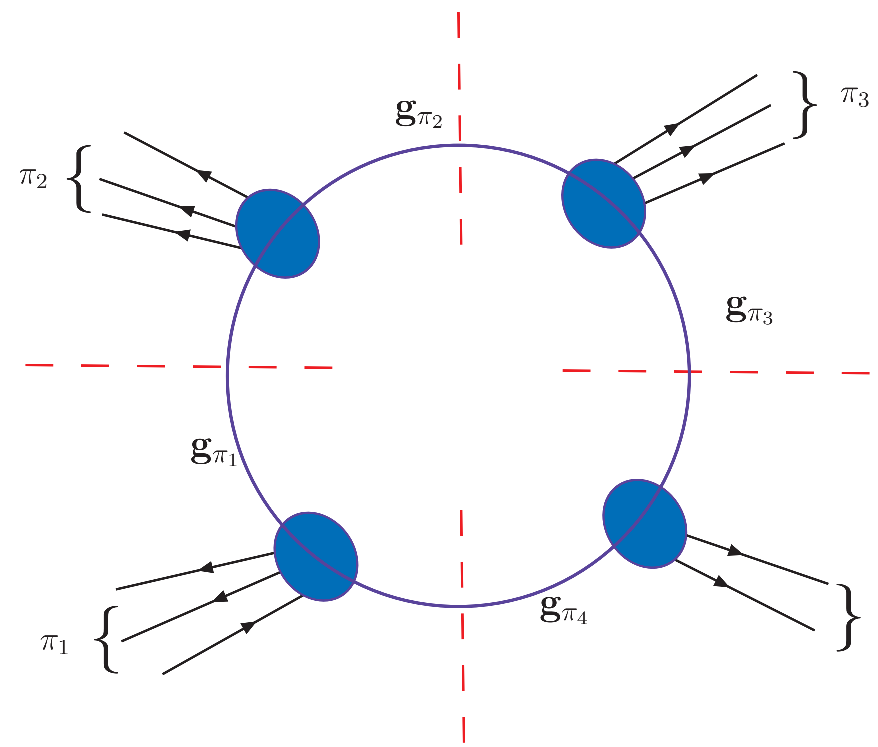


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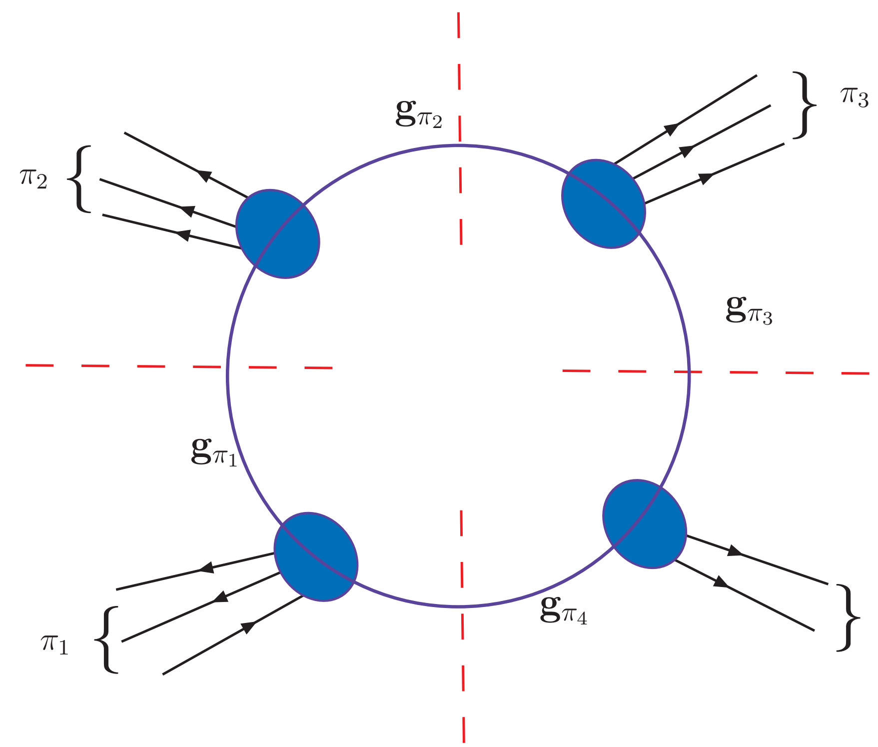


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$$\mathrm{Res}_{\mathbf{l}\to\mathbf{l}*=\mathbf{l}_{\mathbf{n_1 n_2 n_3 n_4}}}\left[\mathcal{A}_{\mathbf{n}}^{1-\mathrm{loop}}\right] = \sum_{\mathbf{n_5}} \frac{\tilde{\mathbf{e}}_{\mathbf{n_1 n_2 n_3 n_4 n_5}}(\mathbf{l}^*)}{\mathbf{d}_{\mathbf{n_5}}(\mathbf{l}^*)} + \tilde{\mathbf{d}}_{\mathbf{n_1 n_2 n_3 n_4}}(\mathbf{l}^*)$$

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# The parameters are fixed by linear algebraic equations

$$\text{Res}_{ij\cdots k}\left[F(l)\right] \equiv \left[d_i(l)d_j(l)\cdots d_k(l)F(l)\right]\Big|_{l=l_{ij\cdots k}} .$$

$$\bar{d}_{ijkl}(l) = \text{Res}_{ijkl}\left(\mathcal{A}_N(l)\right)$$ $d_i=d_j=d_k=d_l=0$ two solutions

$$\bar{c}_{ijk}(l) = \text{Res}_{ijk}\left(\mathcal{A}_N(l) - \sum_{l\neq i,j,k}\frac{\bar{d}_{ijkl}(l)}{d_id_jd_kd_l}\right)$$ $d_i=d_j=d_k=0$ infinite # of solutions

$$\bar{b}_{ij}(l) = \text{Res}_{ij}\left(\mathcal{A}_N(l) - \sum_{k\neq i,j}\frac{\bar{c}_{ijk}(l)}{d_id_jd_k} - \frac{1}{2!}\sum_{k,l\neq i,j}\frac{\bar{d}_{ijkl}(l)}{d_id_jd_kd_l}\right)$$ $d_i=d_j=0$ infinite # of solutions

**unitarity: the residues factorize into the products of tree amplitudes**

**we fully reconstruct the integrand in terms of product of tree amplitudes**

**no Feynman diagrams**

## Unitarity method: one-loop amplitudes from tree amplitudes + scalar integral functions

- Decompose the amplitude in terms of basic set of scalar integral functions and read out the coefficients using unitarity cuts ('98) (BDK)

- Consider the integrand, the amplitude is parametric integral over the loop momentum OPP('06) ( EGK ('07))

- Use generalized cuts, read out the coefficients in terms of tree amplitudes at cut-momenta (complex) BCF/BDK('05)

- Rational part is obtained by carrying out the algorithm in two different integer D>4 dimensions GKM (08) (see also Badger,BDK, OPP)

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Bern, Dixon, Kosower (BDK) ; Ellis, Giele, ZK, Melnikov (EGKM); Britto, Cachazo, Feng (BCF)

# Managing color of amplitudes

Unitarity: instead of Feynman diagrams amplitudes
Tree and one loop Feynman diagrams: color and space time part

$$\mathcal{A}_{\mathbf{n}}^{\mathrm{tree}}(\{\mathbf{k}\},\{\mathbf{h}\},\{\mathbf{a}\}),\ \mathcal{A}_{\mathbf{n}}^{1-\mathrm{loop}}(\{\mathbf{k}\},\{\mathbf{h}\},\{\mathbf{a}\})$$

➡ Color decomposition of amplitudes with the help of a basis color space (T-based, F-based, color flow based}

$$[T^a, T^b] = -F^a_{bc} T^c\,, \qquad \mathrm{Tr}(T^a T^b) = \delta_{ab}.$$

$$[F^a, F^b] = -F^a_{bc} F^c\,, \quad F^a_{bc} = -i\sqrt{2} f^{abc}\,, \quad \mathrm{Tr}(F^a F^b) = 2N_c \delta_{ab}.$$

$$(F^{a_1})_{a_2 a_3} = -\frac{1}{2N_c}\mathrm{Tr}\left([F^{a_1}, F^{a_2}]F^{a_3}\right) = -\mathrm{Tr}\left([T^{a_1}, T^{a_2}]T^{a_3}\right).$$

$$(T^a)_{i_1}^{\bar{j}_1}(T^a)_{i_2}^{\bar{j}_2} = (\delta)_{i_1}^{\bar{j}_2}(\delta)_{i_2}^{\bar{j}_1} - \frac{1}{N_c}(\delta)_{i_1}^{\bar{j}_1}(\delta)_{i_2}^{\bar{j}_2}$$

$$\begin{aligned}(F^{a_2} F^{a_3} \ldots F^{a_{(n-2)}} F^{a_{(n-1)}})_{a_1 a_n} &= \frac{1}{2N_c}\mathrm{Tr}\left([[\ldots[[F^{a_1}, F^{a_2}], F^{a_3}], \ldots, F^{a_{n-2}}]\,[F^{a_{n-1}}, F^{a_n}]\right)\\ &= \mathrm{Tr}\left([[\ldots[[T^{a_1}, T^{a_2}], T^{a_3}], \ldots, T^{a_{n-2}}]\,[T^{a_{n-1}}, T^{a_n}]\right).\end{aligned}$$

## Color ordered n-gluon tree sub-amplitudes

$$\mathcal{A}_{\mathbf{n}}^{\text{tree}} = \frac{\mathbf{g_s^{n-2}}}{\mathbf{2N_c}} \sum_{\sigma \in \mathbf{S_n/Z_n}} \mathrm{Tr}\left(\mathbf{F}^{\mathbf{a}_{\sigma(1)}} \mathbf{F}^{\mathbf{a}_{\sigma(2)}} \mathbf{F}^{\mathbf{a}_{\sigma(3)}} \ldots \mathbf{F}^{\mathbf{a}_{\sigma(\mathbf{n})}}\right) \mathbf{A}_{\mathbf{n},\sigma}^{\text{tree}},$$

(n-1)! color ordered sub-amplitudes

$$\mathcal{A}_{\mathbf{n}}^{\text{tree}} = \mathbf{g_s^{n-2}} \sum_{\sigma \in \mathbf{S_n/Z_n}} \mathrm{Tr}\left(\mathbf{T}^{\mathbf{a}_{\sigma(1)}} \mathbf{T}^{\mathbf{a}_{\sigma(2)}} \mathbf{T}^{\mathbf{a}_{\sigma(3)}} \ldots \mathbf{T}^{\mathbf{a}_{\sigma(\mathbf{n})}}\right) \mathbf{A}_{\mathbf{n},\sigma}^{\text{tree}}$$

$$\mathbf{A}_{\mathbf{n},\sigma}^{\text{tree}} = \mathbf{m_n}(\mathbf{g}_{\sigma(\mathbf{1})}, \mathbf{g}_{\sigma(\mathbf{2})}, \mathbf{g}_{\sigma(\mathbf{3})}, \cdots, \mathbf{g}_{\sigma(\mathbf{n})})$$

Some properties of sub-amplitudes:

$$\mathbf{m_n}(\mathbf{g_1}, \mathbf{g_2}, \mathbf{g_3}, \ldots, \mathbf{g_n}) = \mathbf{m_n}(\mathbf{g_2}, \mathbf{g_3}, \ldots, \mathbf{g_n}, \mathbf{g_1})$$ (cyclic identity)

$$\mathbf{m_n}(\mathbf{g_1}, \mathbf{g_2}, \mathbf{g_3}, \ldots, \mathbf{g_{n-1}}, \mathbf{g_n}) = (-\mathbf{1})^{\mathbf{n}} \mathbf{m_n}(\mathbf{g_n}, \mathbf{g_{n-1}}, \ldots, \mathbf{g_2}, \mathbf{g_1}),$$ (reflection identity)

$$\mathbf{m_n}(\mathbf{1}, \underline{\mathbf{2}, \ldots \mathbf{n_1}}, \overline{\mathbf{n_1}+\mathbf{1}, \ldots, \mathbf{n}}) \equiv \sum_{\sigma(\mathbf{n})} \mathbf{m_n}(\mathbf{g_1}, \mathbf{g}_{\sigma(\mathbf{2})}, \mathbf{g}_{\sigma(\mathbf{3})}, \cdots, \mathbf{g}_{\sigma(\mathbf{n})}) = \mathbf{0}$$ (Abelian identity)

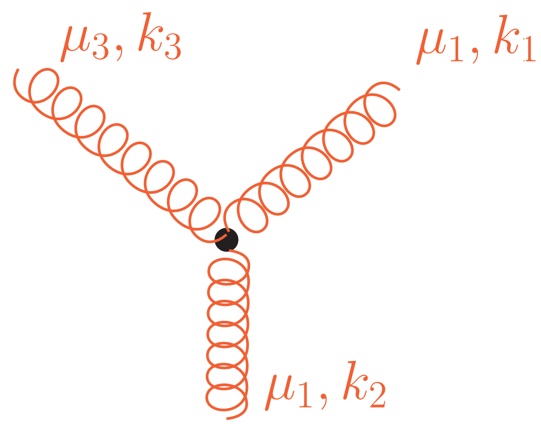
~ Kleiss-Kuijf relations

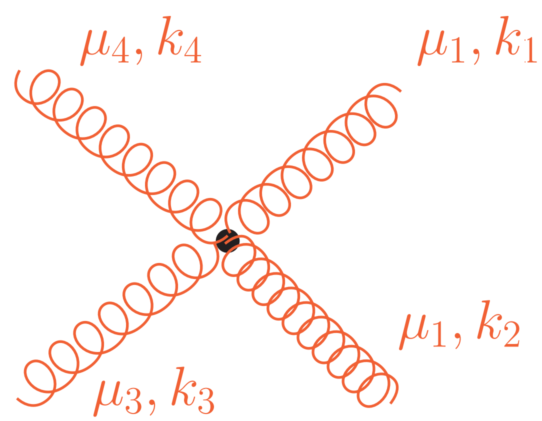
$$\mathcal{A}_{\mathbf{n}}^{\text{tree}}(\mathbf{1},\mathbf{2},\mathbf{3},\ldots,\mathbf{n}) = \mathbf{g_s^{n-2}} \sum_{\sigma=\mathcal{P}(\mathbf{2},\mathbf{3},\ldots,\mathbf{n-1})} (\mathbf{F}^{\mathbf{a}_{\sigma(2)}} \ldots \mathbf{F}^{\mathbf{a}_{\sigma(\mathbf{n-1})}})_{\mathbf{a_1 a_n}} \mathbf{m_n}(\mathbf{g_1}, \mathbf{g}_{\sigma(\mathbf{2})}, \mathbf{g}_{\sigma(\mathbf{3})}, \cdots, \mathbf{g}_{\sigma(\mathbf{n-1})}, \mathbf{g_n}).$$

(n-2)! color ordered sub-amplitudes (see also BCJ relations)

Unitary color basis: on each pole of the tree amplitude the color factor of a given colorless amplitude also factorizes three gluon vertex

$$i\frac{g}{\sqrt{2}}\left((k_1-k_2)_{\mu_3}g_{\mu_1\mu_2}+(k_2-k_3)_{\mu_1}g_{\mu_2\mu_3}+(k_3-k_1)_{\mu_2}g_{\mu_3\mu_1}\right.$$



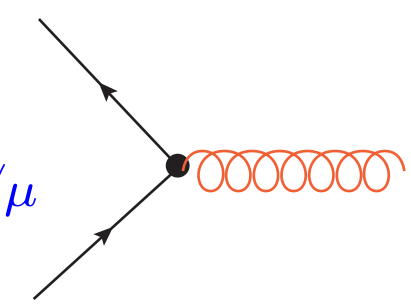




four gluon vertex

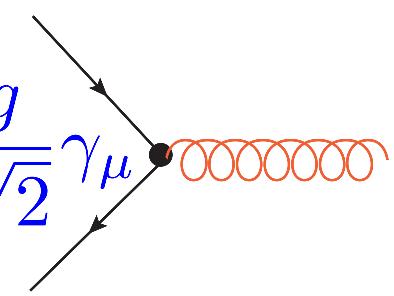
$$i\frac{g^2}{2}(2g_{\mu_1\mu_3}g_{\mu_2\mu_4}-g_{\mu_1\mu_4}g_{\mu_2\mu_3}-g_{\mu_1\mu_2}g_{\mu_3\mu_4})$$

gluon-quark-antiquark vertex

$$i\frac{g}{\sqrt{2}}\gamma_\mu$$



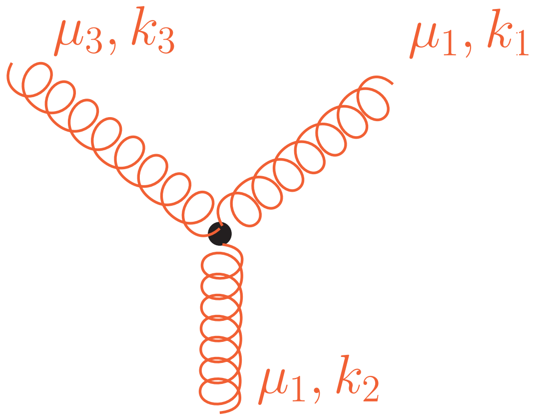
$$-\ i\frac{g}{\sqrt{2}}\gamma_\mu$$



sign depends on orientation

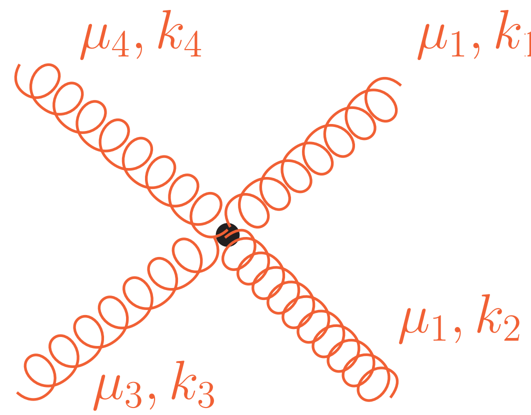
# Color stripped Feynman rules for color ordered sub-amplitudes

three gluon vertex



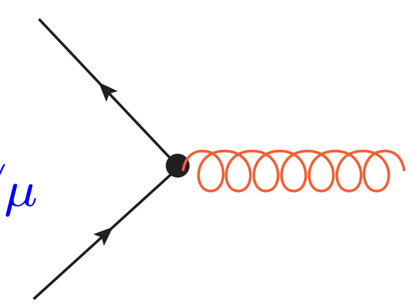
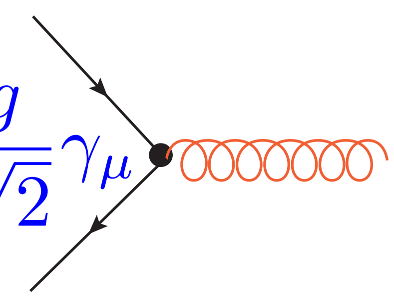

$$i\frac{g}{\sqrt{2}}\left((k_1-k_2)_{\mu_3}g_{\mu_1\mu_2}+(k_2-k_3)_{\mu_1}g_{\mu_2\mu_3}+(k_3-k_1)_{\mu_2}g_{\mu_3\mu_1}\right.$$

four gluon vertex




$$i\frac{g^2}{2}(2g_{\mu_1\mu_3}g_{\mu_2\mu_4}-g_{\mu_1\mu_4}g_{\mu_2\mu_3}-g_{\mu_1\mu_2}g_{\mu_3\mu_4})$$

gluon-quark-antiquark vertex

$i\frac{g}{\sqrt{2}}\gamma_\mu$  - $i\frac{g}{\sqrt{2}}\gamma_\mu$ 

sign depends on orientation

# Color ordered n-gluon one-loop sub-amplitudes

$$\mathcal{A}_{\mathbf{n}}^{1-\mathrm{loop}} = \mathbf{g_s^n c_\Gamma} \sum_{\sigma \in \mathbf{S_{n-1}}} \mathbf{A_n^{(1)}}(\mathbf{g_1}, \mathbf{g}_{\sigma(\mathbf{2})}, \ldots, \mathbf{g}_{\sigma(\mathbf{n})}),$$

$$\mathbf{c_\Gamma} = \frac{\mathbf{1}}{(4\pi)^{\mathbf{2}-\epsilon}} \frac{\mathbf{\Gamma}(\mathbf{1}+\epsilon)\mathbf{\Gamma}^{\mathbf{2}}(\mathbf{1}-\epsilon)}{\mathbf{\Gamma}(\mathbf{1}-\mathbf{2}\epsilon)}$$

Consider a double cut between external line k and k+1 and n and 1

$$\begin{aligned}
\mathrm{Im}_{(\mathbf{k},\mathbf{n})}\left[\mathbf{A_n^{(1)}}(\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_n})\right] &= \{\mathbf{a_1 a_2} \cdots \mathbf{a_k}\}_{\mathbf{vu}}\{\mathbf{a_{k+1}} \cdots \mathbf{a_n}\}_{\mathbf{uv}} \\
&\quad \times \mathbf{m_{k+2}}(\mathbf{g_v}, \mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_k}, \mathbf{g_u})\mathbf{m_{n-k+2}}(\mathbf{g_u}, \mathbf{g_{k+1}}, \ldots, \mathbf{g_n}, \mathbf{g_v}) \\
&= \{\mathbf{a_1 a_2} \cdots \mathbf{a_n}\}\, \mathrm{Im}_{(\mathbf{k},\mathbf{n})}\left[\mathbf{m_n^{(1)}}(\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_n})\right]
\end{aligned}$$

$$\mathcal{A}_{\mathbf{n}}^{1-\mathrm{loop}} = \mathbf{g_s^n c_\Gamma} \sum_{\mathcal{P}(\mathbf{2},\cdots,\mathbf{n})/\mathcal{R}} \mathrm{Tr}(\mathbf{F^{a_1}}, \ldots, \mathbf{F^{a_n}})\mathbf{m_n^{(1)}}(\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_n})$$

one loop color order sub-amplitude

A reflection transformation is factored out. The cyclic property and reflection symmetry remain valid. The number of independent one-loop amplitudes is (n-1)!/2.

# Color ordered n-gluon one-loop sub-amplitudes for large $\mathrm{N_c}$

Decomposition in T-basis:

$$\mathcal{A}_{\mathbf{n}}^{1-\mathrm{loop}} = \mathbf{g_s^n N_c c_\Gamma} \sum_{\mathcal{P}(\mathbf{2},\cdots,\mathbf{n})/\mathcal{R}} \mathrm{Tr}(\mathbf{T^{a_1}},\ldots,\mathbf{T^{a_n}})\mathbf{m}^{(\mathbf{1})}_{\mathbf{1,n}}(\mathbf{g_1},\mathbf{g_2},\ldots,\mathbf{g_n})$$

$$+\mathbf{g_s^n c_\Gamma} \sum_{\mathbf{r=2}}^{[\mathbf{n}/\mathbf{2}]+\mathbf{1}} \left( \sum_{\mathcal{P}(\mathbf{2},\cdots,\mathbf{n})/\mathcal{Z}_{\mathbf{r-1}}\times\mathcal{Z}_{\mathbf{n-r+1}}} \mathrm{Tr}(\mathbf{T^{a_1}},\ldots,\mathbf{T^{a_{r-1}}})\,\mathrm{Tr}(\mathbf{T^{a_r}},\ldots,\mathbf{T^{a_n}})\,\mathbf{m}^{(\mathbf{1})}_{\mathbf{2,n}}(\mathbf{g_1},\mathbf{g_2},\ldots,\mathbf{g_n}) \right)$$

- ➡ The single-trace color structures have an explicit factor of Nc out.
- ➡ They  dominate in the large Nc limit.
- ➡ The planar L-loop color decomposition formula  remains the same .
- ➡ The decomposition remains the same also for the N=4sYM theory.
- ➡ T-based color decomposition is preferred.

Two particle unitarity gives color decomposition of a quark-loop
to n-gluon amplitudes

$$\mathcal{A}_{\mathbf{n;n_f}}^{1-\mathrm{loop}} = \mathbf{g_s^n c_\Gamma n_f} \sum_{\sigma\in\mathbf{S_{n-1}}} \mathrm{Tr}(\mathbf{T^{a_1}T^{a_{\sigma(2)}}}\ldots\mathbf{T^{a_{\sigma n}}})\,\mathbf{m}^{(\mathbf{1})}_{\mathbf{n;n_f}}\left(\mathbf{g_1},\mathbf{g}_{\sigma(\mathbf{2})},\ldots,\mathbf{g}_{\sigma(\mathbf{n})}\right),$$

# $\mathbf{\bar{q}q + (n-2)g}$ amplitudes and fully ordered primitive amplitudes

Color factors of Feynman diagrams $\quad (\mathbf{T^{b_1} \ldots T^{b_k} \ldots})_{\mathbf{j\bar{\imath}}} \times (\mathbf{F^{a_1} \ldots F^{a_r}})_{\mathbf{b_1 a_{r+1}}} \cdots (\mathbf{F^{a_p} \ldots F^{a_t-1}})_{\mathbf{b_k a_t}} \cdots$

$$\mathcal{A}_{\mathbf{n}}^{\mathrm{tree}}(\mathbf{\bar{q}_1, q_2, g_3, \ldots, g_n}) = \mathbf{g_s^{n-2}} \sum_{\sigma \in \mathbf{S_{n-2}}} (\mathbf{T^{a_{\sigma(3)}} T^{a_{\sigma(4)}} \ldots T^{a_{\sigma(n)}}})_{\mathbf{i_2 \bar{\imath}_1}} \; \mathbf{m_n(\bar{q}_1, q_2, g_{\sigma(3)}, \ldots, g_{\sigma(n)})}.$$

(n-2)! colorless color ordered tree sub-amplitudes

the quark labels do not participate in the permutation sum

Decomposition in mixed basis such that the quark is also in the permutation sum

$$\mathcal{A}_{\mathbf{n}}^{\mathrm{tree}}(\mathbf{\bar{q}_1, q_2, g_3, \ldots, g_n}) = \mathbf{g_s^{n-2}}(-\mathbf{1})^{\mathbf{n}} \sum_{\mathbf{k=3}}^{\mathbf{n}} \sum_{\mathcal{P}(\mathbf{4,\ldots,n})} (\mathbf{T^y\, T^{a_{k+1}} .. T^{a_n}})_{\mathbf{i_2 \bar{\imath}_1}} \mathrm{Tr}\, (\mathbf{F^{a_4} \ldots F^{a_k}})_{\mathbf{a_3 y}}$$
$$\times \mathbf{\tilde{m}_n(\bar{q}_1, g_n, \ldots, g_{k+1}, q_2, g_k, \ldots, g_3)}$$

Color ordered tree “left primitive amplitudes”

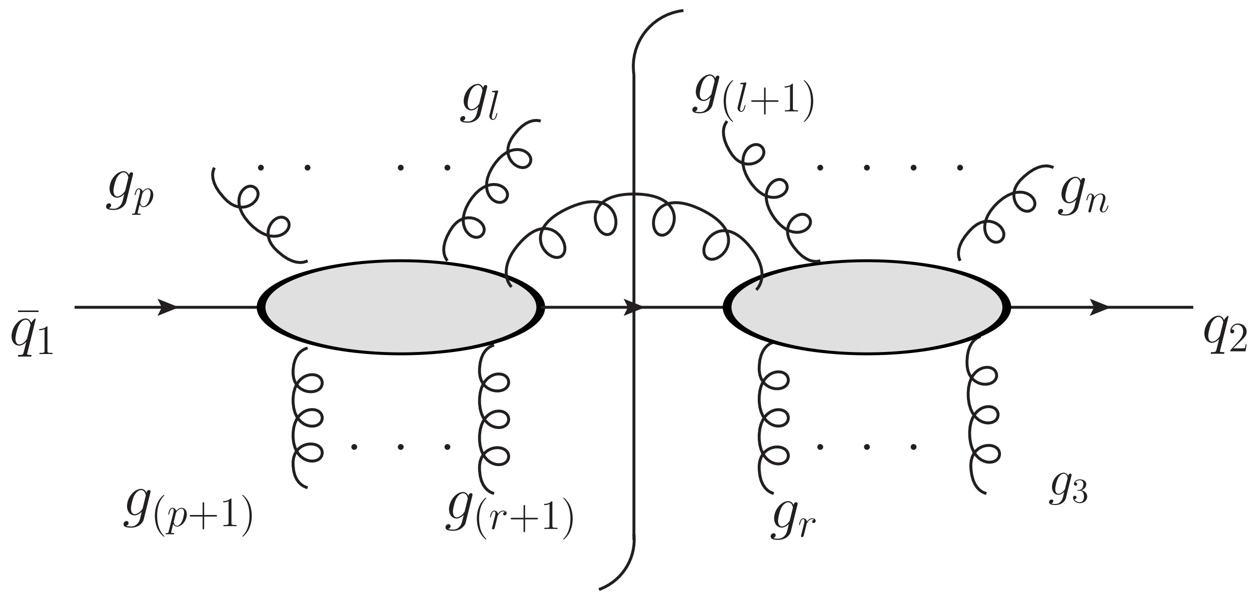
$$\mathbf{\tilde{m}_n(\bar{q}_1, g_n, .., g_{(k+1)}, q_2, g_k, .., g_3)} = (-\mathbf{1})^{\mathbf{k}} \mathbf{m_n(\bar{q}_1, q_2, \underline{k, \ldots, 3}, \overline{(k+1), .., n})}$$

**Excercise**: derive this relation using commutator identities

When anti-quark is in the permutation sum: “right primitive amplitude”

This mixed basis is “unitary”

# $\bar{\mathbf{q}}\mathbf{q} + (\mathbf{n}-\mathbf{2})\mathbf{g}$ **colorless one-loop fully ordered left primitive amplitude**




$$\mathcal{A}_{\mathbf{n}}^{1-\mathrm{loop}}(\bar{\mathbf{q}}_1, \mathbf{q}_2, \mathbf{g}_3, .., \mathbf{g}_\mathbf{n}) = \mathbf{g}_\mathbf{s}^\mathbf{n} \mathbf{c}_\Gamma \sum_{\mathbf{p}=2}^{\mathbf{n}} \sum_{\sigma \in \mathbf{S}_{\mathbf{n}-2}} (\mathbf{T}^{\mathbf{x}_2} \mathbf{T}^{\mathbf{a}_{\sigma_3}} .. \mathbf{T}^{\mathbf{a}_{\sigma_\mathbf{p}}} \mathbf{T}^{\mathbf{x}_1})_{\mathbf{i}_2 \bar{\imath}_1} (\mathbf{F}^{\mathbf{a}_{\sigma_{\mathbf{p}+1}}} .. \mathbf{F}^{\mathbf{a}_{\sigma_\mathbf{n}}})_{\mathbf{x}_1 \mathbf{x}_2}$$
$$\times (-\mathbf{1})^\mathbf{n} \tilde{\mathbf{m}}_\mathbf{n}^{(1)}(\bar{\mathbf{q}}_1, \mathbf{g}_{\sigma(\mathbf{p})}, .., \mathbf{g}_{\sigma(\mathbf{3})}, \mathbf{q}_2, \mathbf{g}_{\sigma(\mathbf{n})}, .., \mathbf{g}_{\sigma(\mathbf{p}+1)})$$

Color ordered amplitudes for n-gluons and primitive amplitudes
for $\bar{\mathbf{q}}\mathbf{q} + (\mathbf{n}-\mathbf{2})\mathbf{g}$ can be calculated using colorless Feynman rules
Berends-Giele recursion relations

Comment: Leading color is good approximations for gluons only

# Amplitudes with multiple quarks

Colorless ordered amplitudes, primitive amplitudes, parent diagrams

$$\mathcal{B}^{\rm tree}(\bar{\mathbf{q}}_1,\mathbf{q}_2,\bar{\mathbf{Q}}_3,\mathbf{Q}_4,\mathbf{g}_5) = \mathbf{g}_s^3\Big[(\mathbf{T}^{\mathbf{a}_5})_{\mathbf{i}_4\bar{\imath}_1}\delta_{\mathbf{i}_2\bar{\imath}_3}\mathbf{B}^{\rm tree}_{5;1} + \frac{\mathbf{1}}{\mathbf{N_c}}(\mathbf{T}^{\mathbf{a}_5})_{\mathbf{i}_2\bar{\imath}_1}\delta_{\mathbf{i}_4\bar{\imath}_3}\mathbf{B}^{\rm tree}_{5;2}$$

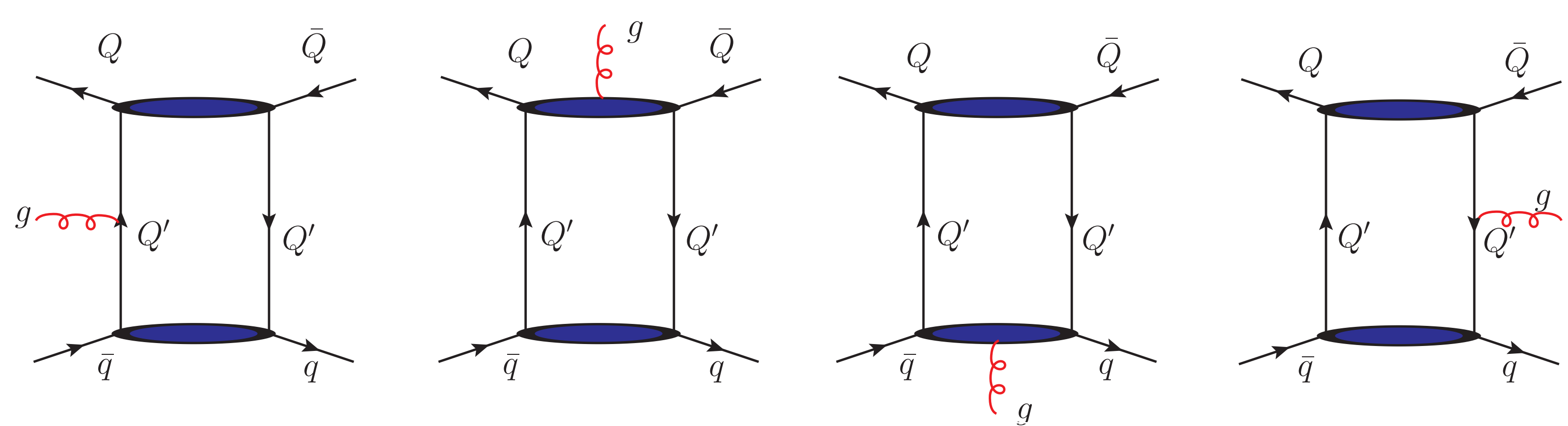
$$+(\mathbf{T}^{\mathbf{a}_5})_{\mathbf{i}_2\bar{\imath}_3}\delta_{\mathbf{i}_4\bar{\imath}_1}\mathbf{B}^{\rm tree}_{5;3} + \frac{\mathbf{1}}{\mathbf{N_c}}(\mathbf{T}^{\mathbf{a}_5})_{\mathbf{i}_4\bar{\imath}_3}\delta_{\mathbf{i}_2\bar{\imath}_1}\mathbf{B}^{\rm tree}_{5;4}\Big],$$

$$\mathcal{B}^{1-\rm loop}(\bar{\mathbf{q}}_1,\mathbf{q}_2,\bar{\mathbf{Q}}_3,\mathbf{Q}_4,\mathbf{g}_5) = \mathbf{g}_s^5\Big[\mathbf{N_c}(\mathbf{T}^{\mathbf{a}_5})_{\mathbf{i}_4\bar{\imath}_1}\delta_{\mathbf{i}_2\bar{\imath}_3}\mathbf{B}_{5;1} + (\mathbf{T}^{\mathbf{a}_5})_{\mathbf{i}_2\bar{\imath}_1}\delta_{\mathbf{i}_4\bar{\imath}_3}\mathbf{B}_{5;2} + \mathbf{N_c}(\mathbf{T}^{\mathbf{a}_5})_{\mathbf{i}_2\bar{\imath}_3}\delta_{\mathbf{i}_4\bar{\imath}_1}\mathbf{B}_{5;3}$$

$$+ (\mathbf{T}^{\mathbf{a}_5})_{\mathbf{i}_4\bar{\imath}_3}\delta_{\mathbf{i}_2\bar{\imath}_1}\mathbf{B}_{5;4}\Big]$$

$$\mathbf{B}_{5;\mathbf{i}} = \mathbf{B}^{[1]}_{5;\mathbf{i}} + \frac{\mathbf{n_f}}{\mathbf{N_c}}\mathbf{B}^{[1/2]}_{5;\mathbf{i}}, \qquad \mathbf{i} = \mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},$$

$$\mathbf{B}^{[1/2]}_{5;1} = -\mathbf{A}^{[1/2]}_{\mathbf{L}}(\bar{\mathbf{q}}_1,\mathbf{g}_5,\mathbf{Q}_4,\bar{\mathbf{Q}}_3,\mathbf{q}_2), \quad \mathbf{B}^{[1/2]}_{5;2} = -\mathbf{A}^{[1/2]}_{\mathbf{L}}(\bar{\mathbf{q}}_1,\mathbf{Q}_4,\bar{\mathbf{Q}}_3,\mathbf{q}_2,\mathbf{g}_5),$$
$$\mathbf{B}^{[1/2]}_{5;3} = -\mathbf{A}^{[1/2]}_{\mathbf{L}}(\bar{\mathbf{q}}_1,\mathbf{Q}_4,\bar{\mathbf{Q}}_3,\mathbf{g}_5,\mathbf{q}_2), \quad \mathbf{B}^{[1/2]}_{5;4} = -\mathbf{A}^{[1/2]}_{\mathbf{L}}(\bar{\mathbf{q}}_1,\mathbf{Q}_4,\mathbf{g}_5,\bar{\mathbf{Q}}_3,\mathbf{q}_2).$$



## Amplitudes with multiple quarks

Colorless ordered amplitudes, primitive amplitudes, parent diagrams

More and more complicated.....

**End of color management**

# Singular behavior of one loop primitive amplitudes

**n-gluon:**

$$\mathbf{m_n^{(1)}(g_1, g_2, ...g_n) = -\sum_{i=1}^{n} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{11}{3n} + L_{i,i+1}\right)\right] m_n^{(0)}(g_1, g_2, ...g_n),}$$

$$\mathbf{L_{kn}} = \ln\left(\mu^2/(-\mathbf{s_{kn}} - \mathbf{i0})\right)$$

$\mathbf{\bar{q}q + (n-2)g}$ :

$$\mathbf{\tilde{m}_n^{(1)}(\bar{q}_n, g_{k+1}, ..., g_{n-1}, q_2, g_3, ...g_k) =}$$
$$\mathbf{\tilde{m}_n(\bar{q}_n, g_{k+1}, ..., g_{n-1}, q_2, g_3, ...g_k)\left[-\frac{k}{\epsilon^2} - \frac{1}{\epsilon}\left(\frac{3}{2} + \sum_{i=1}^{k-1} L_{i\,i+1} + L_{kn}\right)\right]}$$

Color is eliminated. Important for testing the calculations.
Primitive tree amplitude is calculated with Berends-Giele recursion relations based on color stripped Feynman rules or BCFW recursion relations

# Parma International School of Theoretical Physics

## September 3 - September 8, 2012

## Parma, Italy

***Scattering Amplitudes in QCD, Supersymmetric Gauge Theories and Supergravity***

# OUTLINE

Lecture 1: QCD and review its main features.

Lecture 2: One loop tensor integrals and their reduction

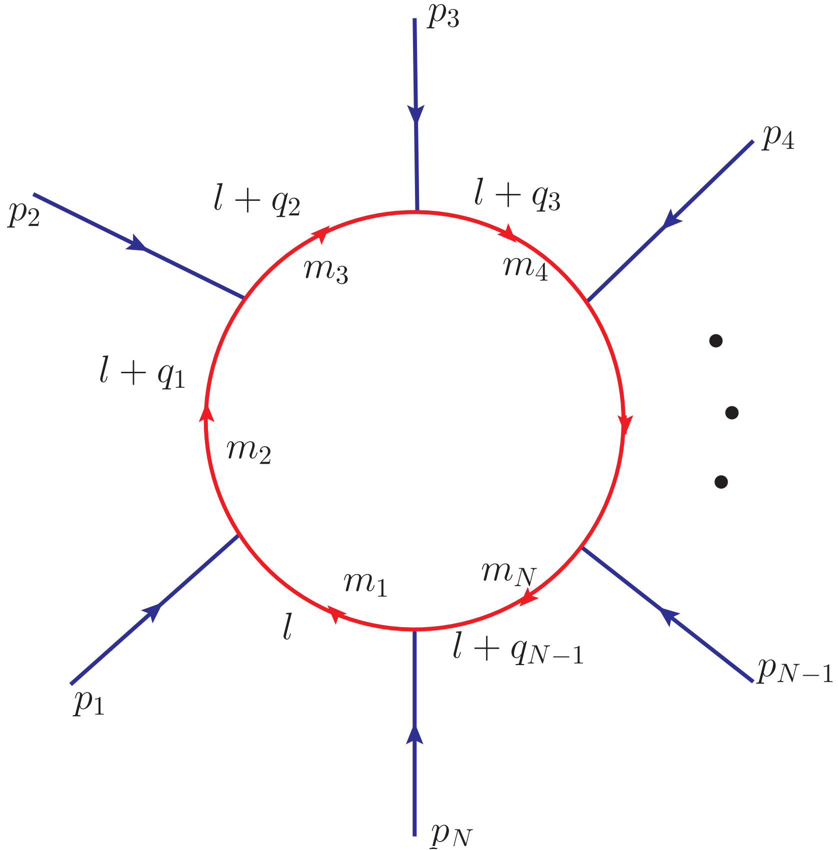
Lecture 3: Unitarity method and amplitudes

Lecture 4: Analytic and numerical computations

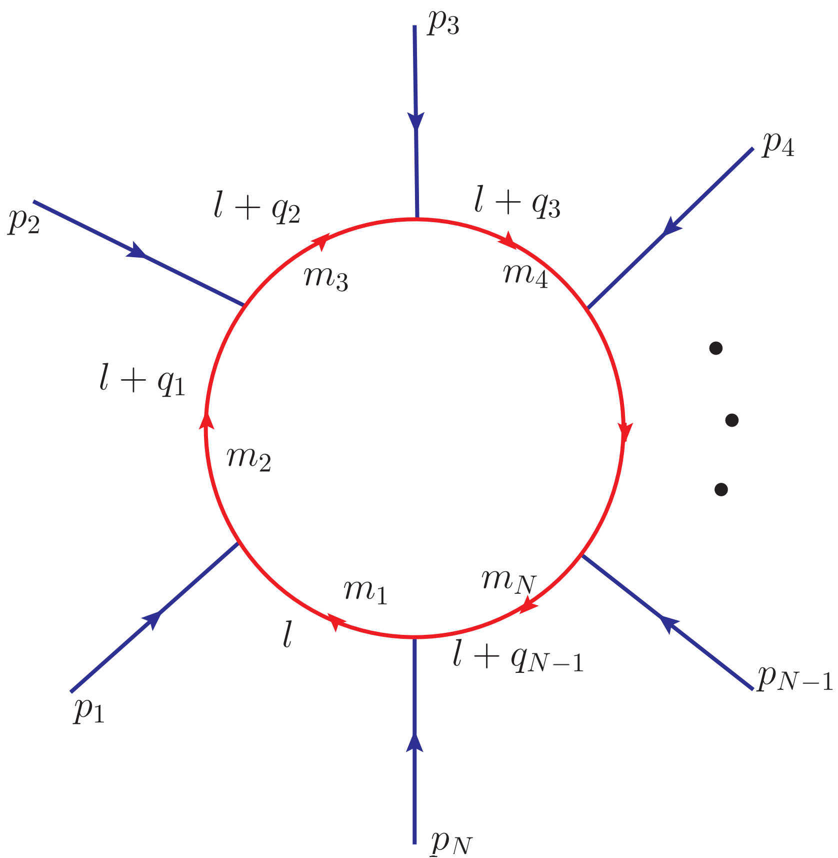
Lecture 5: Different implementations, Outlook

**One-loop calculations in quantum field theory: from Feynman diagrams to unitarity cuts.**
R.Keith Ellis,,Zoltan Kunszt, Kirill Melnikov, Giulia Zanderighi,
**to appear in Phys. Rep. arXiv:1105.4319** [hep-ph]

# One loop calculation with traditional methods

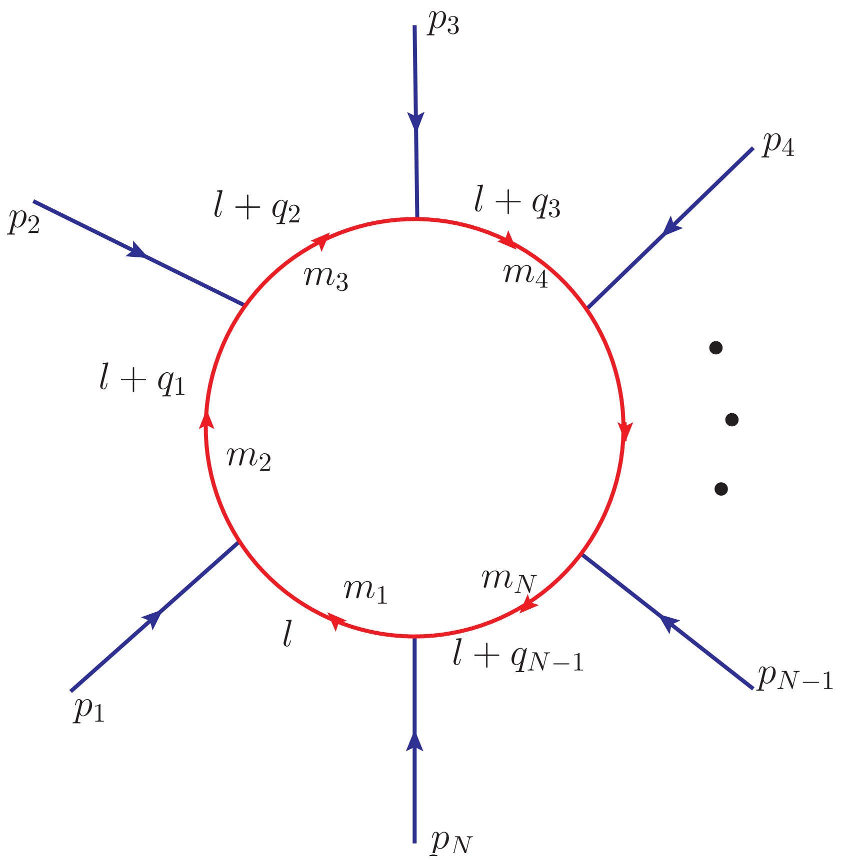


# One loop calculation with traditional methods




$\mathcal{N}(\mathbf{l})$ =1 one loop scalar, otherwise one loop tensor integral

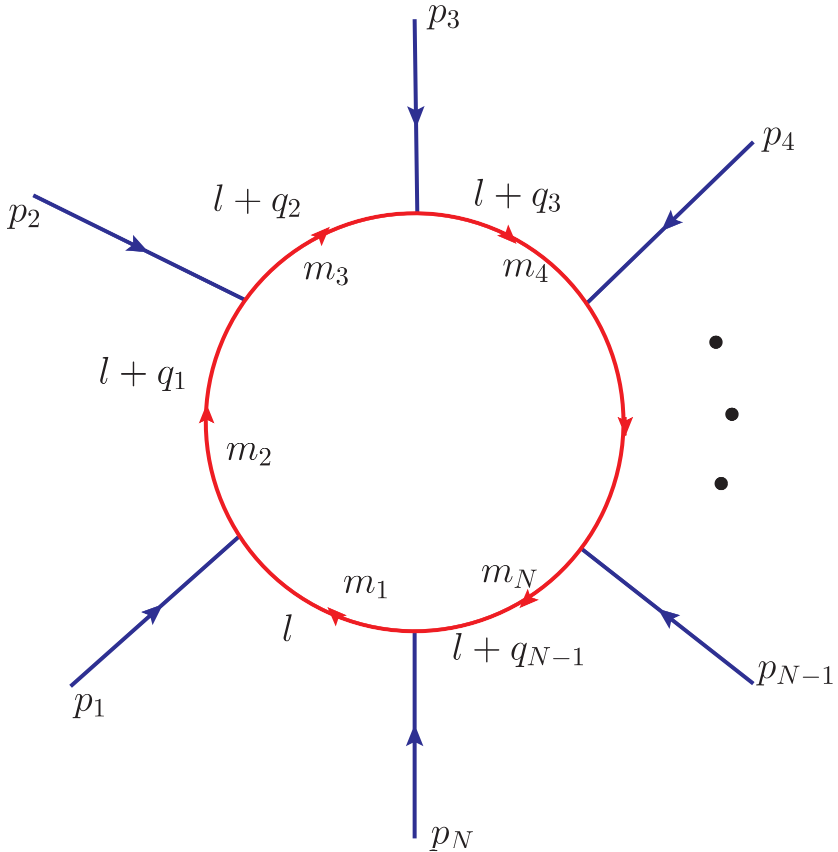
# One loop calculation with traditional methods




$\mathcal{N}(\mathbf{l})$ =1 one loop scalar, otherwise one loop tensor integral

$\mathcal{N}(\mathbf{l})$ is polynomial of degree r in l $\mathbf{r} \leq \mathbf{N}$

# One loop calculation with traditional methods

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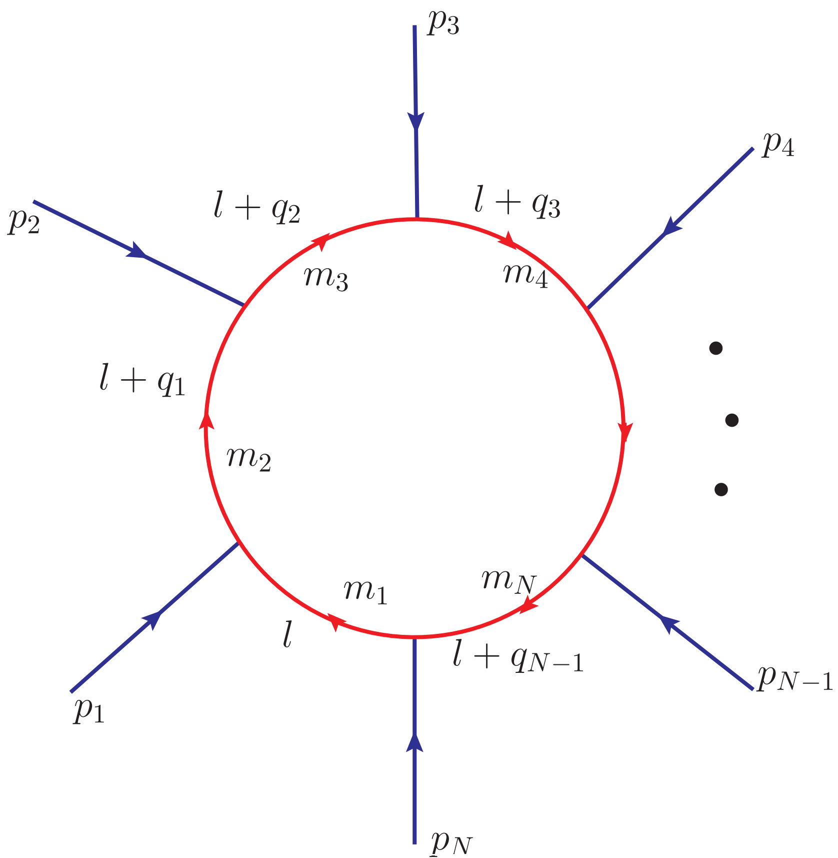


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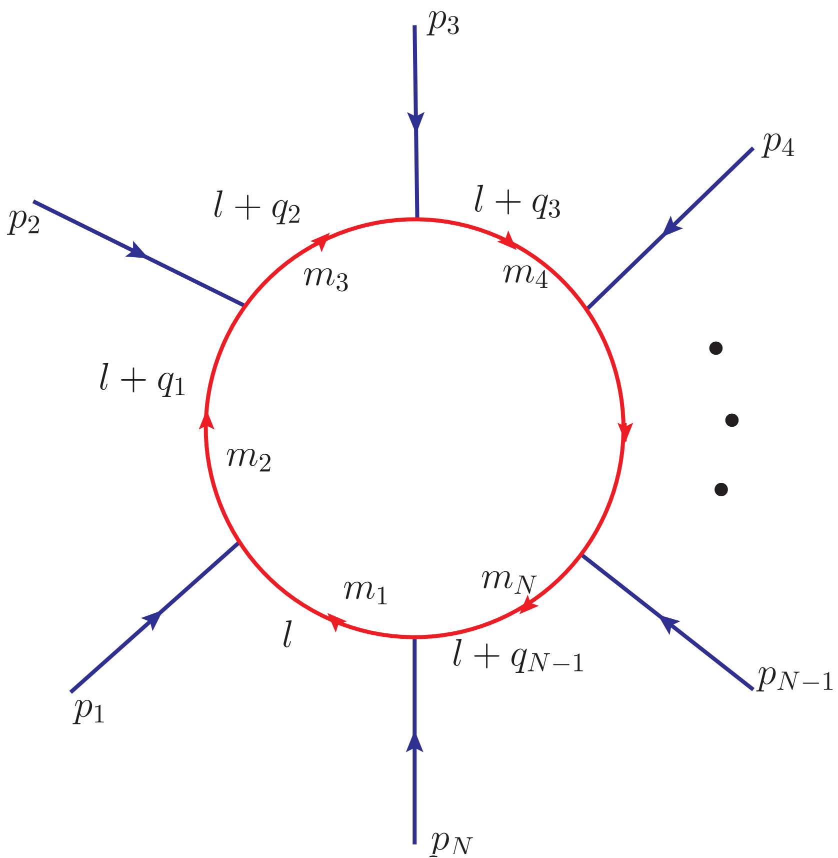


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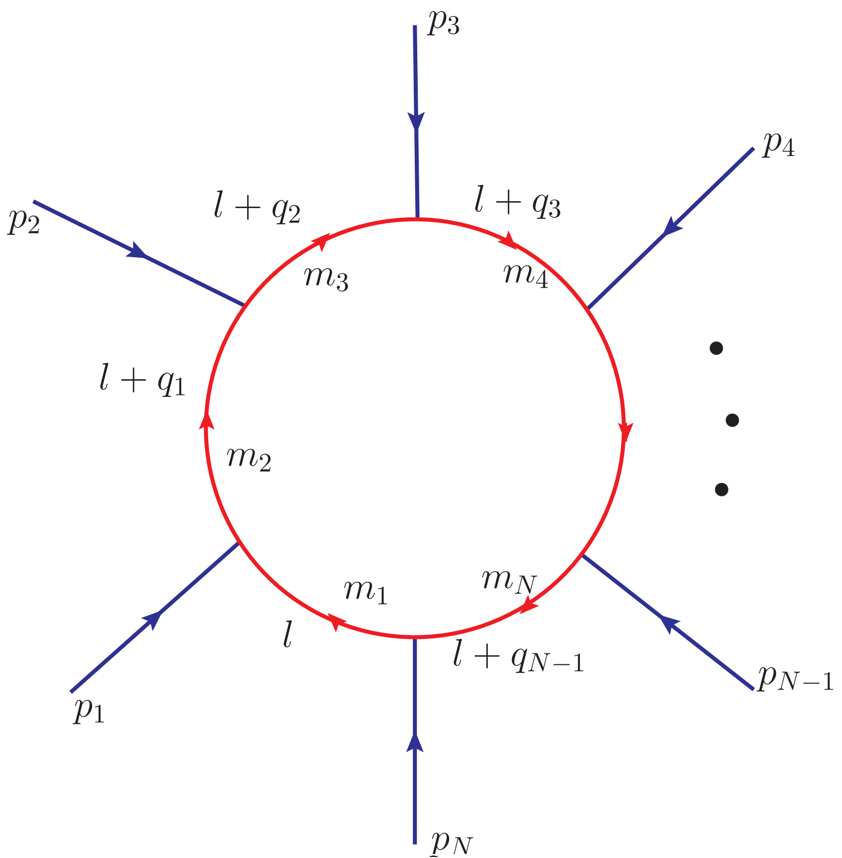


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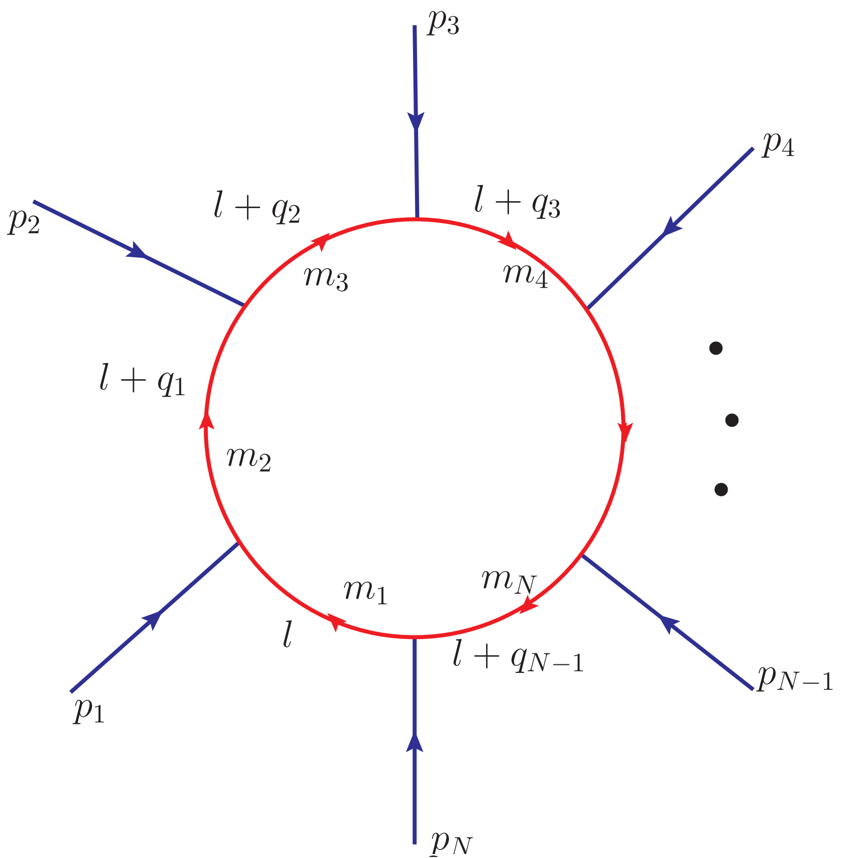


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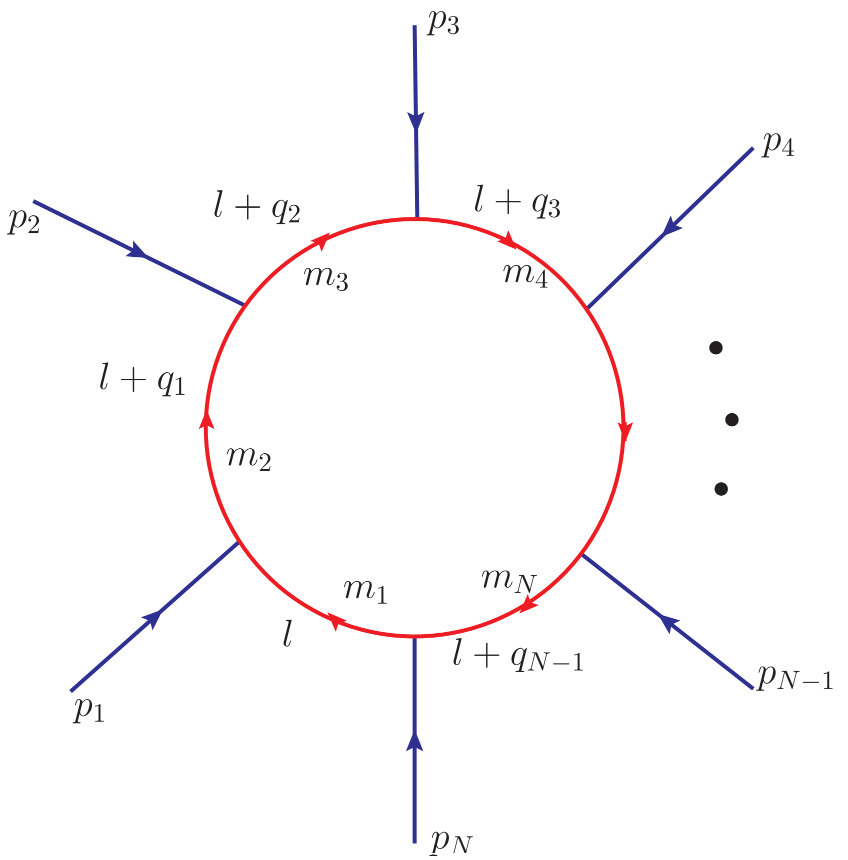


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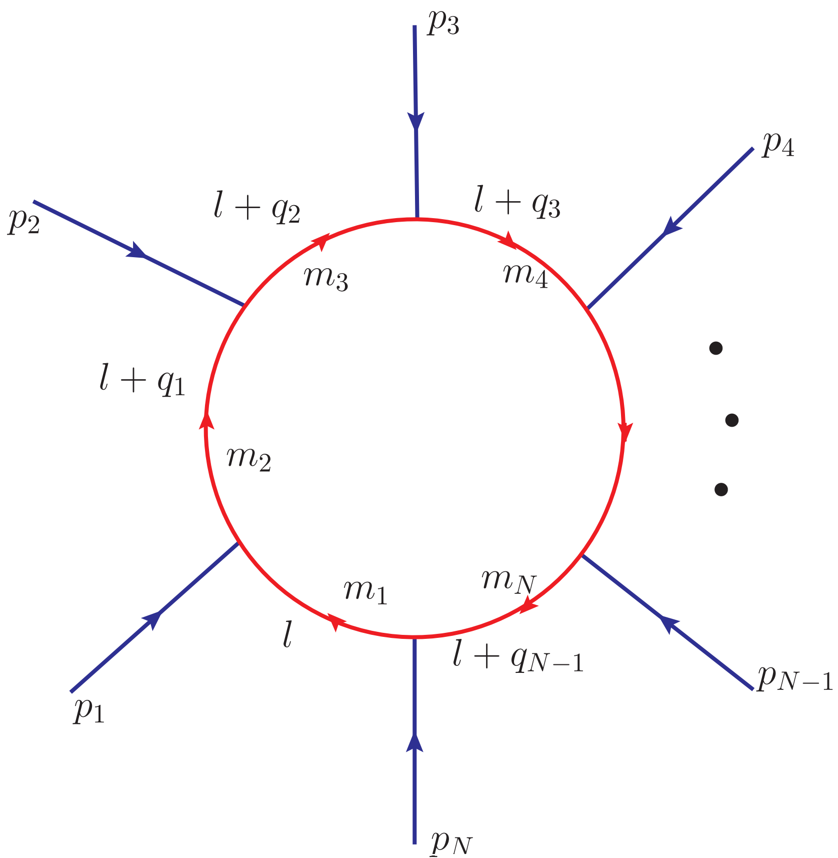


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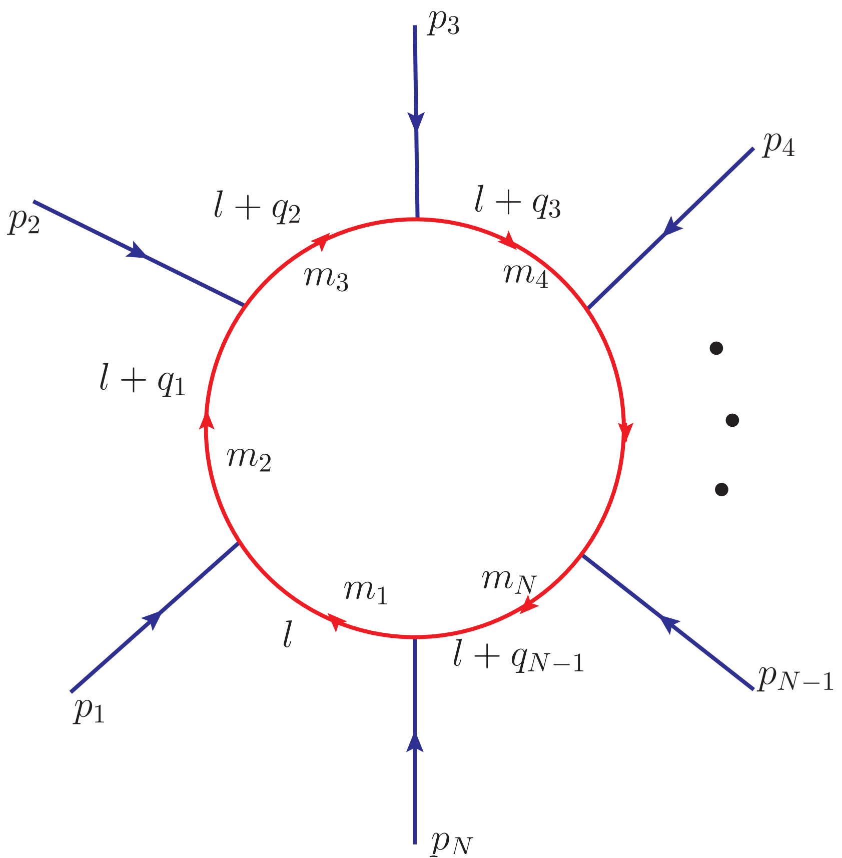


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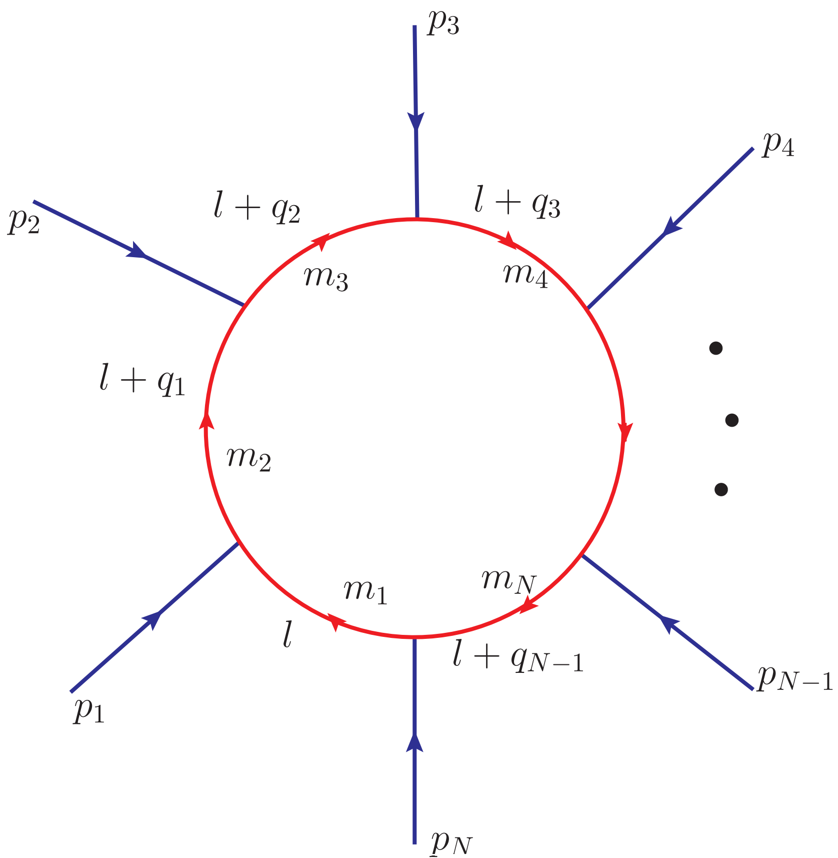


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Finite result for $\quad \mathbf{D \neq 4 \, , m = 4 \, , s = 2}$

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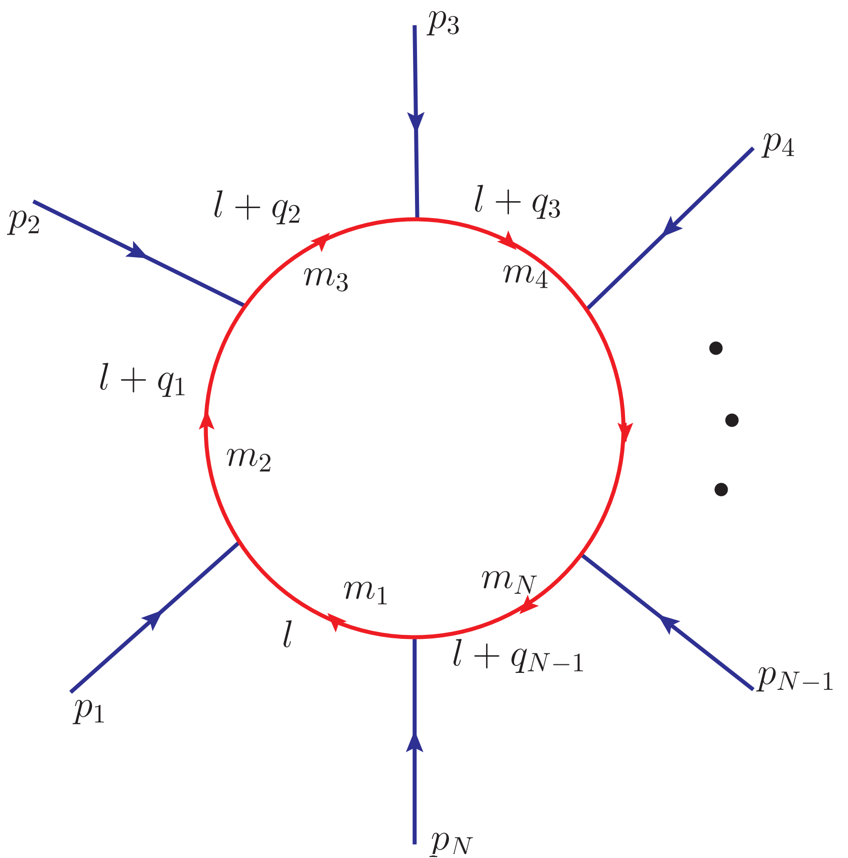


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IR divergent if sufficient number of propagators can go to mass-shell.
Soft and collinear singularities.

# Integral basis in the limit $\mathrm{D} \rightarrow 4$

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Any one-loop integral can be written as linear combination of scalar one-loop integrals and rational terms:

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## Integral basis in the limit $D \to 4$

Any one-loop integral can be written as linear combination of scalar one-loop integrals and rational terms:

$$\mathbf{I_N} = \sum_{\mathbf{j}} \mathbf{c_{4;j} I_{4;j} + c_{3;j} I_{3;j} + c_{2;j} I_{2;j} + c_{1;j} I_{1;j}} + \mathcal{R} + \mathcal{O}(\mathbf{D}-\mathbf{4}).$$

Only one-,two-,three-,four-point scalar integrals occur.

**The problem of the analytic calculation of one-loop scalar integrals** $\mathbf{I_{r;j}}, \quad r = 1, \ldots, 4$
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The problem of one-loop calculation is reduced to the determination of the coefficient $\mathbf{c_{n;j}}$ and $\mathcal{R}$

$$\mathbf{I_1(m_1^2)} = \frac{\mu^{\mathbf{4-D}}}{\mathbf{i}\pi^{\frac{\mathbf{D}}{\mathbf{2}}} \mathbf{r_\Gamma}} \int \frac{\mathbf{d^D l}}{\mathbf{d_1}},$$

$$\mathbf{d_i} = (\mathbf{l} + \mathbf{q_{i-1}})^{\mathbf{2}} - \mathbf{m_i^2} + \mathbf{i}\varepsilon \qquad \mathbf{s_{ij}} = (\mathbf{p_i} + \mathbf{p_j})^{\mathbf{2}},$$

$$\mathbf{q_n} \equiv \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{p_i}, \quad \mathbf{q_0} = \mathbf{0}$$

$$\mathbf{I_2(p_1^2; m_1^2, m_2^2)} = \frac{\mu^{\mathbf{4-D}}}{\mathbf{i}\pi^{\frac{\mathbf{D}}{\mathbf{2}}} \mathbf{r_\Gamma}} \int \frac{\mathbf{d^D l}}{\mathbf{d_1 d_2}},$$

$$\mathbf{r_\Gamma} = \mathbf{\Gamma^2}(\mathbf{1}-\epsilon)\mathbf{\Gamma}(\mathbf{1}+\epsilon)/\mathbf{\Gamma}(\mathbf{1}-\mathbf{2}\epsilon)$$

$$\mathbf{I_3(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2)} = \frac{\mu^{\mathbf{4-D}}}{\mathbf{i}\pi^{\frac{\mathbf{D}}{\mathbf{2}}} \mathbf{r_\Gamma}} \int \frac{\mathbf{d^D l}}{\mathbf{d_1 d_2 d_3}},$$

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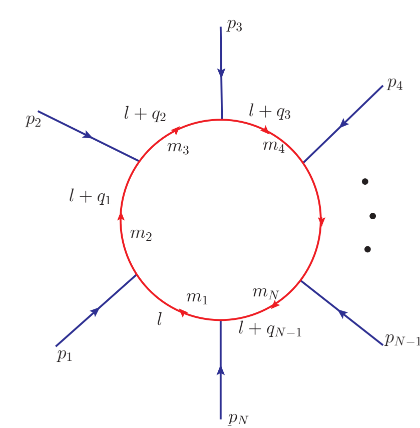


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Non-vanishing master integrals with $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2$ factors in the numerator

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = -\frac{D-4}{2} I^{D+2}_{i_1 i_2 i_3 i_4},$$
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$$\lim_{D\to4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3 i_4} = 0,$$
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Examples:

$$\mathbf{C^{\mu} = p_1^{\mu} C_1 + p_2^{\mu} C_2 \,, \quad C^{\mu\nu} = g^{\mu\nu} C_{00} + \sum_{i,j=1}^{2} p_i^{\mu} p_j^{\nu} C_{ij} \,, \quad C_{12} = C_{21}}$$

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rank 1 and 2 tensor three point integrals

Reduction of $\mathbf{C}^{\mu} = \mathbf{p}_{\mathbf{1}}^{\mu}\mathbf{C}_{\mathbf{1}} + \mathbf{p}_{\mathbf{2}}^{\mu}\mathbf{C}_{\mathbf{2}}\,,$

Reduction of $\mathbf{C^\mu = p_1^\mu C_1 + p_2^\mu C_2}\,,$

$$\mathbf{G_2}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \langle l\cdot p_1\rangle \\ \langle l\cdot p_2\rangle \end{pmatrix} = \begin{pmatrix} R_1^{[c]} \\ R_2^{[c]} \end{pmatrix},$$

trace with $\mathbf{p_1}$ and $\mathbf{p_2}$

$$\langle \mathbf{l\cdot p_j}\rangle = \int \frac{\mathbf{d^D l}}{\mathbf{i}\pi^{\mathbf{D/2}}}\,\frac{\mathbf{l\cdot p_j}}{\mathbf{d_1 d_2 d_3}},\quad \text{for } \mathbf{j=1,2}$$

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cancel terms in numerator with denominator factors

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$$\mathbf{p_1}_{\,\mu}\mathbf{C}^{\mu\nu} = \mathbf{p}_1^\nu(\mathbf{p_1.p_1 C_{11} + p_1.p_2 C_{12} + C_{00}}) + \mathbf{p}_2^\nu(\mathbf{p_1.p_1 C_{12} + p_1.p_2 C_{22}}),$$
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trace with $\mathbf{p_1}$ and $\mathbf{p_2}$

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trace with $\mathbf{p_1}$ and $\mathbf{p_2}$

$$\mathbf{C}_\mu^\nu = \mathbf{B_0(2,3) + m_1^2 C_0(1,2,3) = D C_{00} + p_1^2 C_{11} + 2p_1.p_2 C_{12} + p_2^2 C_{22}}$$

trace with $\mathbf{g}^{\mu\nu}$

## Reduction chains for Passarino-Veltman procedure.

$$
\begin{array}{lcl}
\mathbf{D_{ijkl}} & \to & D_{00ij}, D_{ijk}, C_{ijk}, C_{ij}, C_i, C_0 \\
\mathbf{D_{00ij}} & \to & D_{ijk}, D_{ij}, C_{ij}, C_i \\
\mathbf{D_{0000}} & \to & D_{00i}, D_{00}, C_{00} \\
\mathbf{D_{ijk}} & \to & D_{00i}, D_{ij}, C_{ij}, C_i \\
\mathbf{D_{00i}} & \to & D_{ij}, D_i, C_i, C_0 \\
\mathbf{D_{ij}} & \to & D_{00}, D_i, C_i, C_0 \\
\mathbf{D_{00}} & \to & D_i, D_0, C_0 \\
\mathbf{D_{i}} & \to & D_0, C_0 \\
\hline
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## Reduction chains for Passarino-Veltman procedure.

For more then 4 particles it is cumbersome

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For more then 4 particles it is cumbersome

- ✦ The number of Feynman diagrams grows dramatically with the number of external particle (factorial).
- ✦ The number of terms generated during the reduction of tensor integrals grows rapidly with the number of external particles and with the rank of the integral.
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Well established technique, applied in many cases.

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For 5-,6-, and more particles external momenta not linearly independent (additional input), Denner, Dittmaier 2006

# Singular regions in PV reduction:

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Example: rank 1 triangle

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Exercise: Investigate the collinear behaviour of reduction using form factors with
using $\mathbf{p_1^\mu}$ and $\mathbf{\tilde{p}_2^\mu = p_2^\mu - \frac{p_1 \cdot p_2}{p_1^2} p_1^\mu}$

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$$\mathbf{v}_{\mathbf{1}}^{\mu} = \frac{\epsilon_{\mathbf{q_1 q_2}} \epsilon^{\mu \mathbf{q_2}}}{\epsilon_{\mathbf{q_1 q_2}} \epsilon^{\mathbf{q_1 q_2}}}, \quad \mathbf{v}_{\mathbf{2}}^{\mu} = \frac{\epsilon_{\mathbf{q_1 q_2}} \epsilon^{\mathbf{q_1} \mu}}{\epsilon_{\mathbf{q_1 q_2}} \epsilon^{\mathbf{q_1 q_2}}}, \quad \epsilon^{\mu_1 \mu_2} \epsilon_{\nu_1 \nu_2} = \delta^{\mu_1}_{\nu_1} \delta^{\mu_2}_{\nu_2} - \delta^{\mu_1}_{\nu_2} \delta^{\mu_2}_{\nu_1} = \det |\delta^{\mu}_{\nu}| \equiv \delta^{\mu_1 \mu_2}_{\nu_1 \nu_2},$$

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In Kronecker-deltas we can have Lorenz-vector indices in any space and Shouten-identies also valid in any space

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# Physical and transverse space

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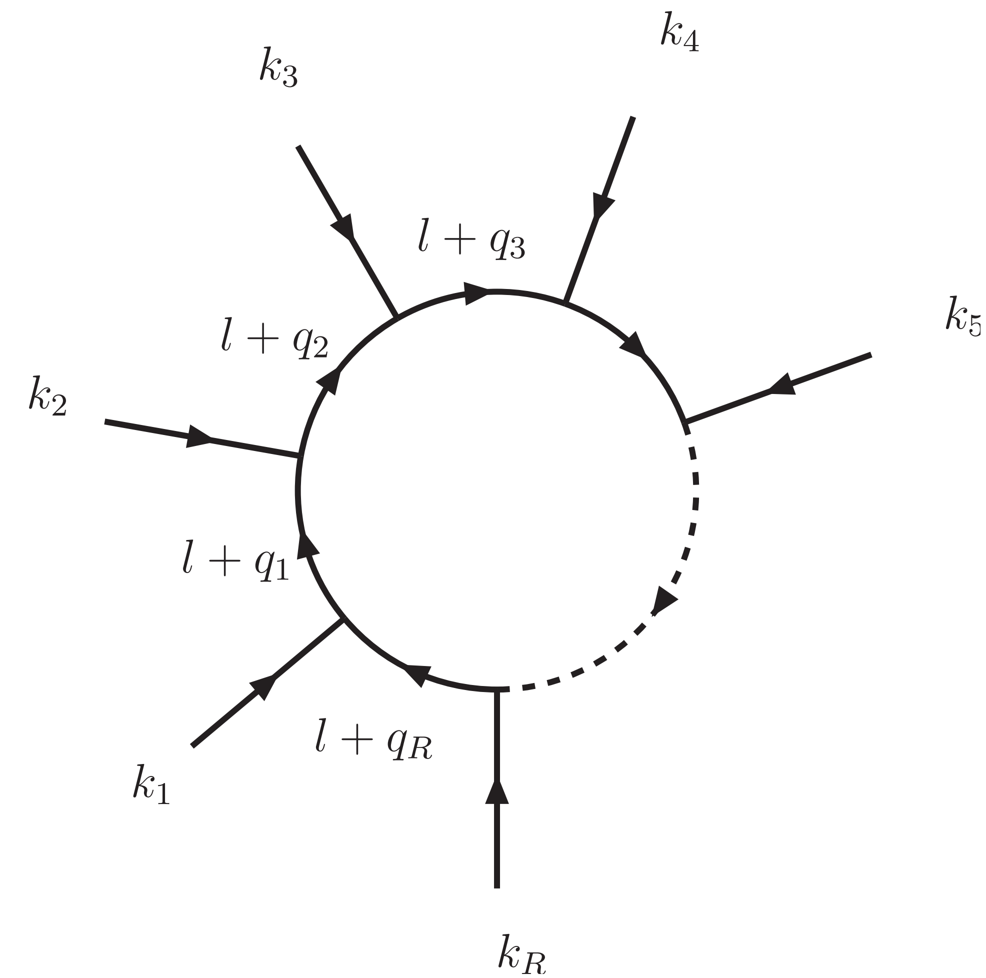
QFT in D-dimension, N-particle scattering amplitude, consider one one-loop Feynman-diagram with R loop-momentum dependent propagator. The integrand is a rational function of the loop momentum

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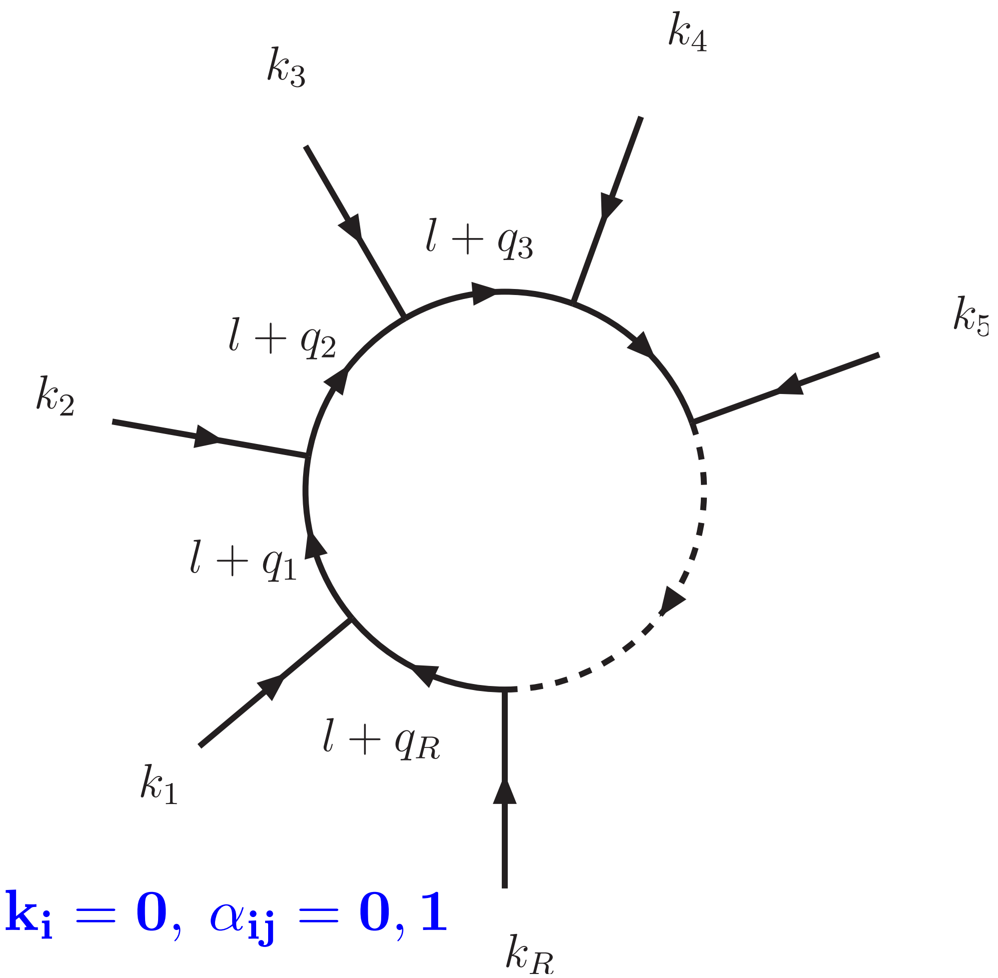


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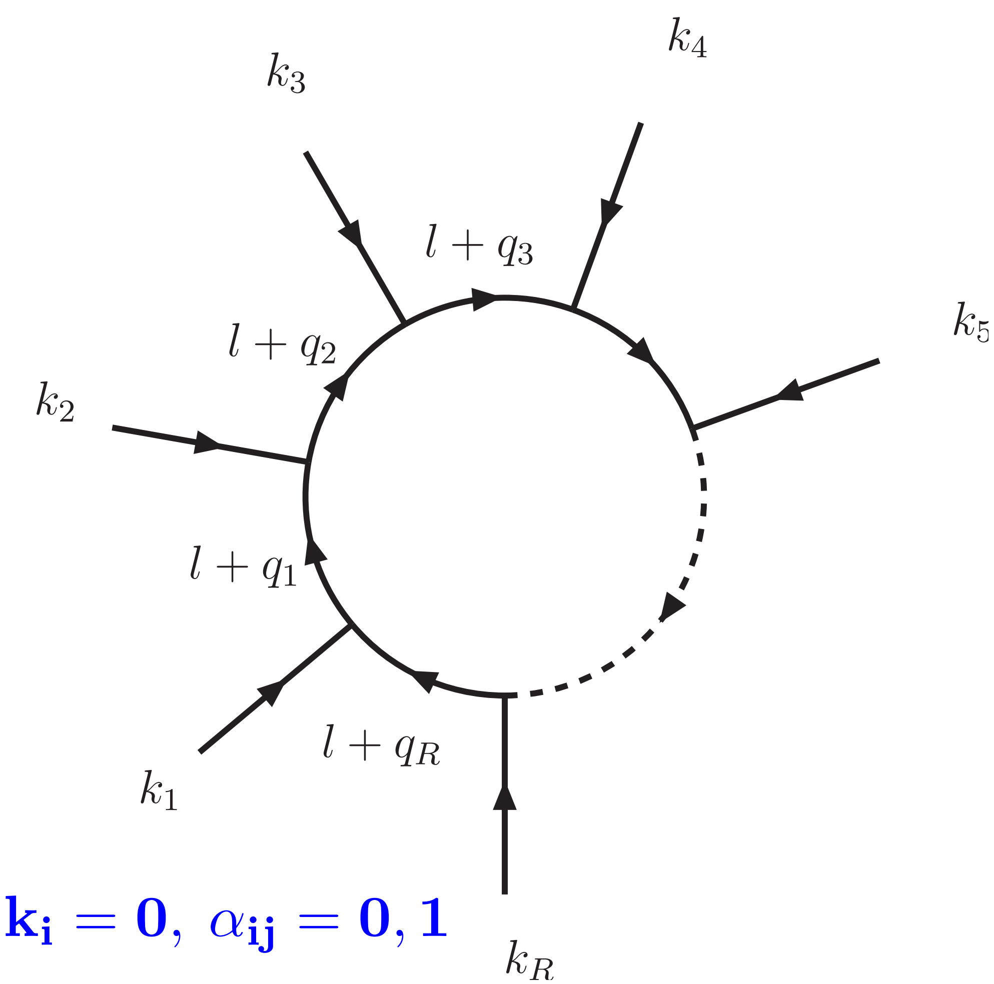


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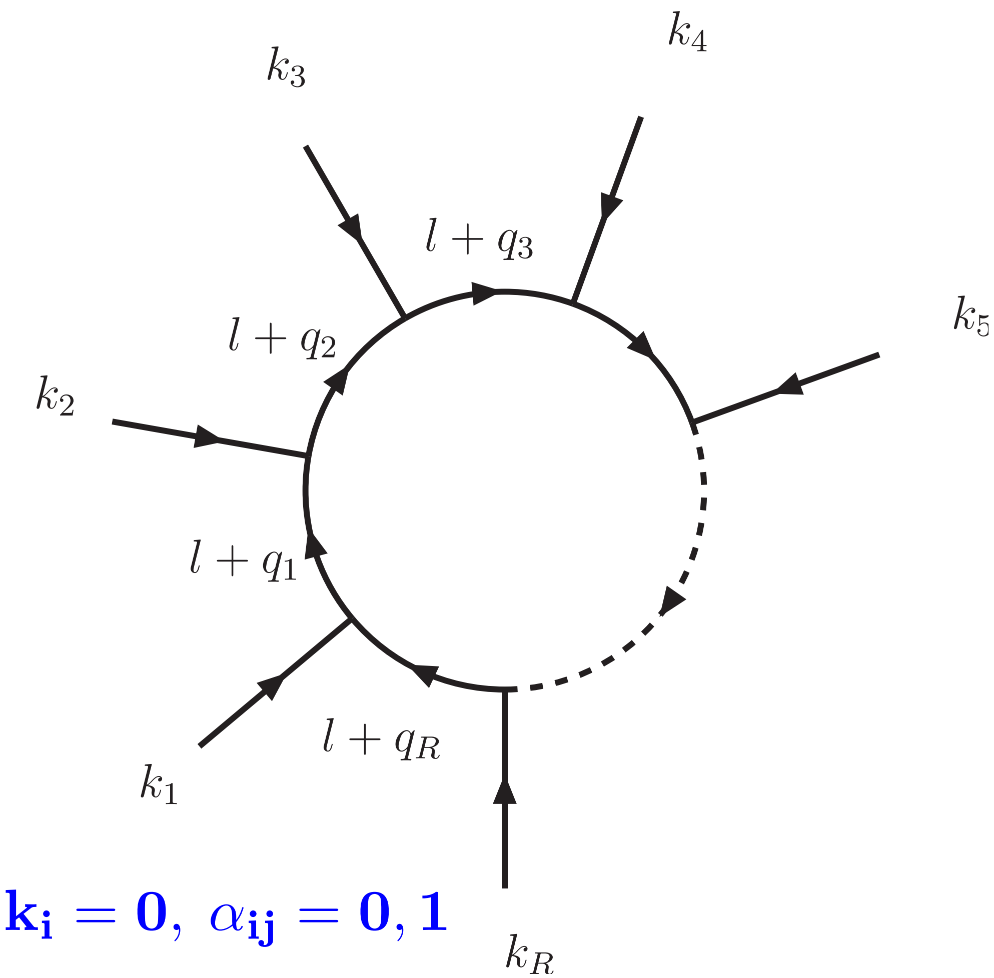


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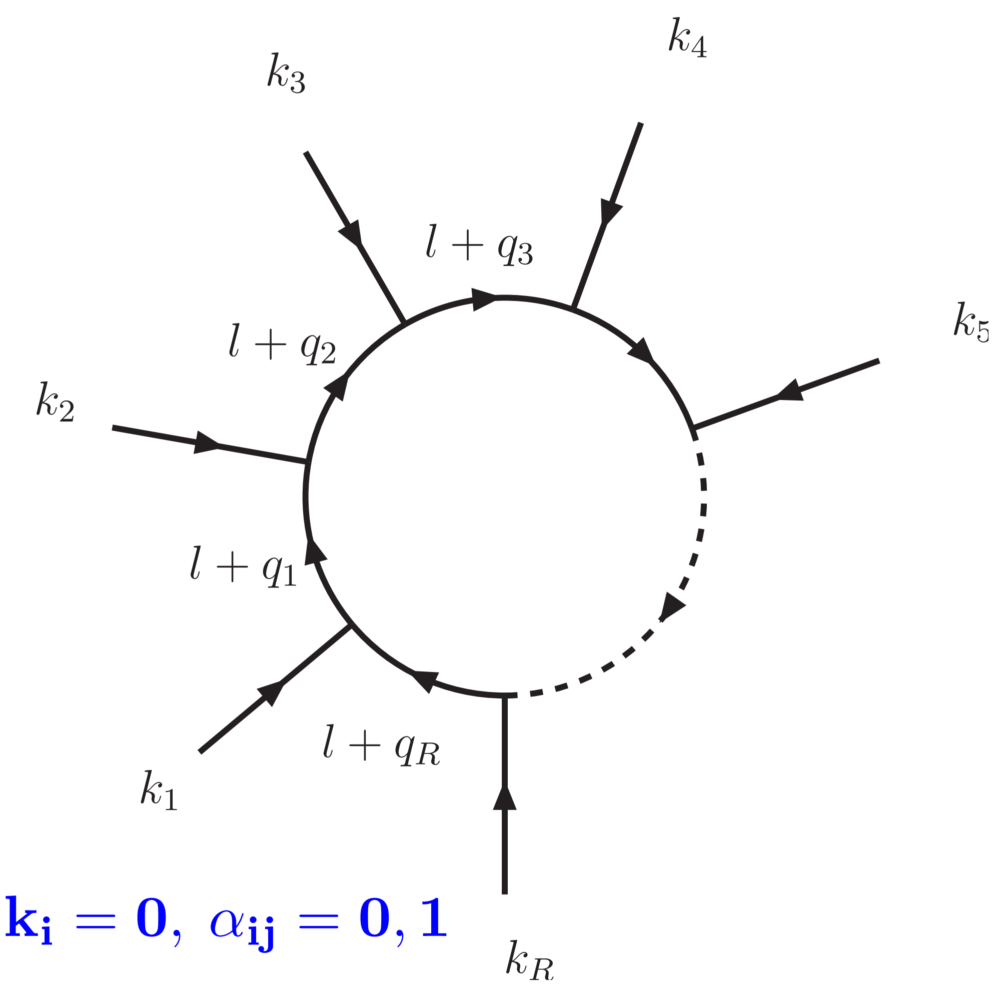


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If R>D, the transverse space is zero dimensional.

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$$\mathbf{g}^{\mu\nu} = \sum_{\mathbf{i=1}}^{\mathbf{D_P}} \mathbf{k_i}^{\mu} \mathbf{v_i}^{\nu} + \sum_{\mathbf{i=1}}^{\mathbf{D_T}} \mathbf{n_i}^{\mu} \mathbf{n_i}^{\nu}$$

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The dual coordinates of the loop momentum vector provide us the reduction at the integrand level:

$$\mathbf{l}\cdot\mathbf{k}_i = \frac{1}{2}\left[\mathbf{d}_i - \mathbf{d}_{i-1} - (\mathbf{q}_i^2 - \mathbf{m}_i^2) + (\mathbf{q}_{i-1}^2 - \mathbf{m}_{i-1}^2)\right]$$

$$\mathbf{l}^\mu = \mathbf{V}_R^\mu + \frac{1}{2}\sum_{i=1}^{D_P} (\mathbf{d}_i - \mathbf{d}_{i-1})\, \mathbf{v}_i^\mu + \sum_{i=1}^{D_T} (\mathbf{l}\cdot\mathbf{n}_i)\, \mathbf{n}_i^\mu \,,$$

$$\mathbf{V}_R^\mu = -\frac{1}{2}\sum_{i=1}^{D_P} \Big( (\mathbf{q}_i^2 - \mathbf{m}_i^2) - (\mathbf{q}_{i-1}^2 - \mathbf{m}_{i-1}^2) \Big)\, \mathbf{v}_i^\mu$$

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$$\mathbf{l}\cdot\mathbf{q_i} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{d_i} - \mathbf{d_0} - \mathbf{r_i})\,, \quad \mathbf{r_i} = \mathbf{q_i^2} - \mathbf{m_i^2} + \mathbf{m_0^2}\,, \quad \mathbf{i} = \mathbf{1}, \mathbf{2}\,,$$

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$$2(\mathbf{d_0} + \mathbf{m_0^2}) = \sum_{\mathbf{i}=1}^{2}(\mathbf{l}\cdot\mathbf{v_i})(\mathbf{d_i} - \mathbf{d_0}) - \sum_{\mathbf{i}=1}^{2}(\mathbf{l}\cdot\mathbf{v_i})\mathbf{r_i}$$

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$$\frac{\mathbf{1}}{\mathbf{d_0 d_1 d_2}} = \frac{\mathbf{1}}{(\mathbf{4m_0^2} - \mathbf{w^2})}\left\{\frac{\mathbf{2(l\cdot v_1) - (w\cdot v_1)}}{\mathbf{d_0 d_2}} + \frac{\mathbf{2(l\cdot v_2) - (w\cdot v_2)}}{\mathbf{d_0 d_1}} - \frac{\mathbf{4 + (2l - w)\cdot(v_1+v_2)}}{\mathbf{d_1 d_2}}\right\}.$$

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In the right hand side the terms proportional to $l_\mu$ depend only on $l_T^\mu$ and integrate to zero.

The inflow momenta for the three bubble denominators are different

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**Exercise:**

Any N-point scalar one-loop integrand function, for N > D, where D is the dimensionality of space-time, can be written as a linear combination of the D-point scalar and vector integrand functions.

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$$\mathbf{l}^2_{(\mathbf{2})} = \mathbf{d_1} + \mathbf{m_1^2} - \mu^2, \quad \mathbf{2\,l \cdot k} = \mathbf{d_2} - \mathbf{d_1} - \mathbf{r_1^2},$$

$$\mu^2 = (\mathbf{n}_\epsilon \cdot \mathbf{l})^2$$

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The integral is UV divergent. We regularize it dimensionally $\mathbf{D = 2 - 2\epsilon}$

$$\mathbf{n}^\mu = \frac{\mathbf{k}^\mu}{\sqrt{\mathbf{k^2}}}, \quad \mathbf{n^2} = \mathbf{1},\ \mathbf{n}^\mu \mathbf{n}^\nu + \hat{\mathbf{n}}^\mu \hat{\mathbf{n}}^\nu = \mathbf{g}^{\mu\nu}_{(\mathbf{2})},$$

$$(\hat{\mathbf{n}} \cdot \mathbf{l})^2 = \mathbf{l}^2_{(\mathbf{2})} - (\mathbf{n} \cdot \mathbf{l})^2 = \mathbf{l}^2_{(\mathbf{2})} - \frac{(\mathbf{l} \cdot \mathbf{k})^2}{\mathbf{k^2}}$$

$$\mathbf{l}^\mu = (\mathbf{l} \cdot \mathbf{n})\mathbf{n}^\mu + (\mathbf{l} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}^\mu + \mathbf{n}_\epsilon^\mu (\mathbf{l} \cdot \mathbf{n}_\epsilon)$$

$$\mathbf{l}^2_{(\mathbf{2})} = \mathbf{d_1} + \mathbf{m_1^2} - \mu^2, \quad \mathbf{2\,l \cdot k} = \mathbf{d_2} - \mathbf{d_1} - \mathbf{r_1^2},$$

$$\mu^2 = (\mathbf{n}_\epsilon \cdot \mathbf{l})^2$$

Result of the reduction:

$$\frac{(\hat{\mathbf{n}} \cdot \mathbf{l})^2}{\mathbf{d_1 d_2}} = -\frac{(\lambda^2 + \mu^2)}{\mathbf{d_1 d_2}} + \frac{\mathbf{1}}{\mathbf{4k^2}} \left[ \frac{\mathbf{r_1^2} - \mathbf{2\,l \cdot k}}{\mathbf{d_1}} + \frac{\mathbf{r_2^2} + \mathbf{2\,l \cdot k} + \mathbf{2k^2}}{\mathbf{d_2}} \right].$$

$$\mathbf{r_1^2} = \mathbf{k^2} + \mathbf{m_1^2} - \mathbf{m_2^2}, \quad \mathbf{r_2^2} = \mathbf{k^2} + \mathbf{m_2^2} - \mathbf{m_1^2},$$

$$\lambda^2 = \frac{\mathbf{k^4} - \mathbf{2k^2}(\mathbf{m_1^2} + \mathbf{m_2^2}) + (\mathbf{m_1^2} - \mathbf{m_2^2})^2}{\mathbf{4k^2}}$$

Example 2.: **Reduction of a rank-two two point function in** $\mathbf{D = 2 - 2\epsilon}$

$$\mathcal{I}(\mathbf{k}, \mathbf{m_1}, \mathbf{m_2}) = \frac{(\hat{\mathbf{n}} \cdot \mathbf{l})^2}{\mathbf{d_1 d_2}}, \qquad \mathbf{d_1} = \mathbf{l^2} - \mathbf{m_1^2}, \; \mathbf{d_2} = (\mathbf{l} + \mathbf{k})^2 - \mathbf{m_2^2}, \; \hat{\mathbf{n}} \cdot \mathbf{k} = \mathbf{0}, \mathbf{k^2} \neq \mathbf{0}, \; \text{and } \hat{\mathbf{n}}^2 = \mathbf{1}$$

Our aim is to reduce it using VN method.

The integral is UV divergent. We regularize it dimensionally $\mathbf{D = 2 - 2\epsilon}$

$$\mathbf{n}^\mu = \frac{\mathbf{k}^\mu}{\sqrt{\mathbf{k^2}}}, \quad \mathbf{n^2} = \mathbf{1}, \; \mathbf{n}^\mu \mathbf{n}^\nu + \hat{\mathbf{n}}^\mu \hat{\mathbf{n}}^\nu = \mathbf{g}^{\mu\nu}_{(\mathbf{2})},$$

$$(\hat{\mathbf{n}} \cdot \mathbf{l})^2 = \mathbf{l}^2_{(\mathbf{2})} - (\mathbf{n} \cdot \mathbf{l})^2 = \mathbf{l}^2_{(\mathbf{2})} - \frac{(\mathbf{l} \cdot \mathbf{k})^2}{\mathbf{k^2}}$$

$$\mathbf{l}^\mu = (\mathbf{l} \cdot \mathbf{n})\mathbf{n}^\mu + (\mathbf{l} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}^\mu + \mathbf{n}^\mu_\epsilon (\mathbf{l} \cdot \mathbf{n}_\epsilon)$$

$$\mathbf{l}^2_{(\mathbf{2})} = \mathbf{d_1} + \mathbf{m_1^2} - \mu^2, \quad \mathbf{2\,l \cdot k} = \mathbf{d_2} - \mathbf{d_1} - \mathbf{r_1^2},$$

$$\mu^2 = (\mathbf{n}_\epsilon \cdot \mathbf{l})^2$$

Result of the reduction:

$$\frac{(\hat{\mathbf{n}} \cdot \mathbf{l})^2}{\mathbf{d_1 d_2}} = -\frac{(\lambda^2 + \mu^2)}{\mathbf{d_1 d_2}} + \frac{\mathbf{1}}{\mathbf{4k^2}} \left[ \frac{\mathbf{r_1^2} - \mathbf{2\,l \cdot k}}{\mathbf{d_1}} + \frac{\mathbf{r_2^2} + \mathbf{2\,l \cdot k} + \mathbf{2k^2}}{\mathbf{d_2}} \right].$$

$$\mathbf{r_1^2} = \mathbf{k^2} + \mathbf{m_1^2} - \mathbf{m_2^2}, \quad \mathbf{r_2^2} = \mathbf{k^2} + \mathbf{m_2^2} - \mathbf{m_1^2},$$

$$\lambda^2 = \frac{\mathbf{k^4} - \mathbf{2k^2}(\mathbf{m_1^2} + \mathbf{m_2^2}) + (\mathbf{m_1^2} - \mathbf{m_2^2})^2}{\mathbf{4k^2}}$$

It follows the parametrization:

$$\frac{(\hat{\mathbf{n}} \cdot \mathbf{l})^2}{\mathbf{d_1 d_2}} = \frac{\mathbf{b_0} + \mathbf{b_1}(\hat{\mathbf{n}} \cdot \mathbf{l}) + \mathbf{b_2}(\mathbf{n}_\epsilon \cdot \mathbf{l})^2}{\mathbf{d_1 d_2}} + \frac{\mathbf{a_{1,0}} + \mathbf{a_{1,1}}(\mathbf{n} \cdot \mathbf{l}) + \mathbf{a_{1,2}}(\hat{\mathbf{n}} \cdot \mathbf{l})}{\mathbf{d_1}} + \frac{\mathbf{a_{2,0}} + \mathbf{a_{2,1}}(\mathbf{n} \cdot \mathbf{l}) + \mathbf{a_{2,2}}(\hat{\mathbf{n}} \cdot \mathbf{l})}{\mathbf{d_2}}.$$

# Connection to “unitarity” :  evaluate coefficients with “on-shell” loop momenta

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$$(\mathbf{\hat{n}\cdot l})^2 = \left[\mathbf{b_0} + \mathbf{b_1}(\mathbf{\hat{n}\cdot l}) + \mathbf{b_2}(\mathbf{n}_\epsilon\cdot\mathbf{l})^2\right] + \left[\mathbf{a_{1,0}} + \mathbf{a_{1,1}}\cancel{(\mathbf{n\cdot l})} + \mathbf{a_{1,2}}(\mathbf{\hat{n}\cdot l})\right]\mathbf{d_2} \quad + \left[\mathbf{a_{2,0}} + \mathbf{a_{2,1}}\cancel{(\mathbf{n\cdot l})} + \mathbf{a_{2,2}}(\mathbf{\hat{n}\cdot l})\right]\mathbf{d_1}$$

## Connection to “unitarity” : evaluate coefficients with “on-shell” loop momenta

$$(\mathbf{\hat{n}\cdot l})^{\mathbf{2}} = \left[\mathbf{b_0 + b_1(\hat{n}\cdot l) + b_2(n_\epsilon\cdot l)^2}\right] + \left[\mathbf{a_{1,0} + a_{1,1}}\cancel{\mathbf{(n\cdot l)}} + \mathbf{a_{1,2}(\hat{n}\cdot l)}\right]\mathbf{d_2} \quad + \left[\mathbf{a_{2,0} + a_{2,1}}\cancel{\mathbf{(n\cdot l)}} + \mathbf{a_{2,2}(\hat{n}\cdot l)}\right]\mathbf{d_1}$$

Calculate $\mathbf{b_0, b_1}$ assuming $\mathbf{d_1(l) = d_2(l) = 0}$ and $\mathbf{n_\epsilon \cdot l = 0}$ :

$$\mathbf{l_c^\pm} = \alpha_\mathbf{c}\mathbf{n} \pm \mathbf{i}\beta_\mathbf{c}\mathbf{\hat{n}}$$

$$\alpha_\mathbf{c} = -\frac{\mathbf{r_1^2}}{\mathbf{2\sqrt{k^2}}}, \quad \beta_\mathbf{c} = \lambda$$

$$\mathbf{b_0 + b_1\hat{n}\cdot l_c^+} = -\lambda^\mathbf{2}, \quad \mathbf{b_0 + b_1\hat{n}\cdot l_c^-} = -\lambda^\mathbf{2}$$

$$\mathbf{b_0} = -\lambda^\mathbf{2}, \ \mathbf{b_1 = 0}$$

# Connection to "unitarity" : evaluate coefficients with "on-shell" loop momenta

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Calculate $\mathbf{b_0}, \mathbf{b_1}$ assuming $\mathbf{d_1}(\mathbf{l}) = \mathbf{d_2}(\mathbf{l}) = \mathbf{0}$ and $\mathbf{n}_\epsilon\cdot\mathbf{l} = \mathbf{0}$ :

$$\mathbf{l_c^\pm} = \alpha_\mathbf{c}\mathbf{n} \pm \mathbf{i}\beta_\mathbf{c}\hat{\mathbf{n}}$$

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$$\mathbf{b_0} = -\lambda^\mathbf{2}, \ \mathbf{b_1} = \mathbf{0}$$

Calculate $\mathbf{b_2}$ assuming $\mathbf{d_1}(\mathbf{l}) = \mathbf{d_2}(\mathbf{l}) = \mathbf{0}$ and $\mathbf{l}\cdot\hat{\mathbf{n}} = \mathbf{0}$ :

## Connection to "unitarity" : evaluate coefficients with "on-shell" loop momenta

$$(\mathbf{\hat{n}\cdot l})^2 = \left[\mathbf{b_0 + b_1(\hat{n}\cdot l) + b_2(n_\epsilon\cdot l)^2}\right] + \left[\mathbf{a_{1,0} + a_{1,1}\cancel{(n\cdot l)} + a_{1,2}(\hat{n}\cdot l)}\right]\mathbf{d_2} \quad + \left[\mathbf{a_{2,0} + a_{2,1}\cancel{(n\cdot l)} + a_{2,2}(\hat{n}\cdot l)}\right]\mathbf{d_1}$$

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$$\mathbf{0 = (1 + b_2)}\lambda^\mathbf{2}$$

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## Connection to “unitarity” : evaluate coefficients with “on-shell” loop momenta

$$(\hat{\mathbf{n}}\cdot\mathbf{l})^2 = \left[\mathbf{b_0} + \mathbf{b_1}(\hat{\mathbf{n}}\cdot\mathbf{l}) + \mathbf{b_2}(\mathbf{n}_\epsilon\cdot\mathbf{l})^2\right] + \left[\mathbf{a_{1,0}} + \mathbf{a_{1,1}}\cancel{(\mathbf{n}\cdot\mathbf{l})} + \mathbf{a_{1,2}}(\hat{\mathbf{n}}\cdot\mathbf{l})\right]\mathbf{d_2} \quad + \left[\mathbf{a_{2,0}} + \mathbf{a_{2,1}}\cancel{(\mathbf{n}\cdot\mathbf{l})} + \mathbf{a_{2,2}}(\hat{\mathbf{n}}\cdot\mathbf{l})\right]\mathbf{d_1}$$

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$$\mathbf{0} = (\mathbf{1} + \mathbf{b_2})\lambda^2$$

$$\mathbf{b_2} = -\mathbf{1}$$

Calculate tadpole coefficients $\mathbf{a_{1,0}}.\mathbf{a_{1,1}}, \mathbf{a_{1,2}}$ assuming $\mathbf{d_1}(\mathbf{l}) = \mathbf{0}$ :

## Connection to "unitarity" : evaluate coefficients with "on-shell" loop momenta

$$(\hat{n}\cdot l)^2 = \left[b_0 + b_1(\hat{n}\cdot l) + b_2(n_\epsilon\cdot l)^2\right] + \left[a_{1,0} + \cancel{a_{1,1}(n\cdot l)} + a_{1,2}(\hat{n}\cdot l)\right] d_2 \quad + \left[a_{2,0} + \cancel{a_{2,1}(n\cdot l)} + a_{2,2}(\hat{n}\cdot l)\right] d_1$$

Calculate $b_0, b_1$ assuming $d_1(l) = d_2(l) = 0$ and $n_\epsilon \cdot l = 0$ :

$$l_c^\pm = \alpha_c n \pm i\beta_c \hat{n}$$

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$$l_c^{\pm} = \alpha_c n \pm i\beta_c \hat{n}$$

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$$l_1 = \gamma_1 n + \gamma_2 \hat{n}\,, \quad \gamma_1^2 + \gamma_2^2 = m_1^2$$

Choose $\gamma_1 = 0, \gamma_2 = \pm m_1$

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# Connection to “unitarity” : evaluate coefficients with “on-shell” loop momenta

$$(\mathbf{\hat{n}\cdot l})^2 = \left[\mathbf{b_0 + b_1(\hat{n}\cdot l) + b_2(n_\epsilon\cdot l)^2}\right] + [\mathbf{a_{1,0} + a_{1,1}}\cancel{\mathbf{(n\cdot l)}} + \mathbf{a_{1,2}(\hat{n}\cdot l)]\, d_2} \quad + [\mathbf{a_{2,0} + a_{2,1}}\cancel{\mathbf{(n\cdot l)}} + \mathbf{a_{2,2}(\hat{n}\cdot l)]\, d_1}$$

Calculate $\mathbf{b_0, b_1}$ assuming $\mathbf{d_1(l) = d_2(l) = 0}$ and $\mathbf{n_\epsilon \cdot l = 0}$ :

$$\mathbf{l_c^\pm} = \alpha_\mathbf{c}\mathbf{n} \pm \mathbf{i}\beta_\mathbf{c}\mathbf{\hat{n}}$$

$$\mathbf{b_0 + b_1\hat{n}\cdot l_c^+} = -\lambda^\mathbf{2}, \quad \mathbf{b_0 + b_1\hat{n}\cdot l_c^-} = -\lambda^\mathbf{2}$$

$$\alpha_\mathbf{c} = -\frac{\mathbf{r_1^2}}{\mathbf{2\sqrt{k^2}}}, \quad \beta_\mathbf{c} = \lambda$$

$$\mathbf{b_0} = -\lambda^\mathbf{2}, \ \mathbf{b_1 = 0}$$

Calculate $\mathbf{b_2}$ assuming $\mathbf{d_1(l) = d_2(l) = 0}$ and $\mathbf{l\cdot\hat{n} = 0}$ :

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Calculate tadpole coefficients $\mathbf{a_{1,0}.a_{1,1}, a_{1,2}}$ assuming $\mathbf{d_1(l) = 0}$ :

$$\frac{(\mathbf{\hat{n}\cdot l_1})^\mathbf{2}}{\mathbf{d_2(l_1)}} - \frac{\mathbf{b_0 + b_1(\hat{n}\cdot l) + b_2(n_\epsilon\cdot l_1)^2}}{\mathbf{d_2(l_1)}} = \mathbf{a_{1,0} + a_{1,1}(n\cdot l) + a_{1,2}(\hat{n}\cdot l)} + \cancel{\frac{\mathbf{a_{2,0} + a_{2,1}(n\cdot l) + a_{2,2}(\hat{n}\cdot l)}}{\mathbf{d_2(l_1)}}\mathbf{d_1(l_1)}}$$

$$\mathbf{l_1} = \gamma_\mathbf{1}\mathbf{n} + \gamma_\mathbf{2}\mathbf{\hat{n}}, \quad \gamma_\mathbf{1}^\mathbf{2} + \gamma_\mathbf{2}^\mathbf{2} = \mathbf{m_1^2}$$

$$\mathbf{a_{1,0} \pm a_{1,2}m_1} = \frac{\mathbf{m_1^2} + \lambda^\mathbf{2}}{\mathbf{r_1^2}} = \frac{\mathbf{r_1^2}}{\mathbf{4k^2}}$$

Choose $\gamma_\mathbf{1} = \mathbf{0}, \gamma_\mathbf{2} = \pm\mathbf{m_1}$

$$\mathbf{a_{1,2} = 0\,,\, a_{1,0} = r_1^2/(4k^2)\,,}$$

Choose $\gamma_\mathbf{2} = \mathbf{0}, \gamma_\mathbf{1} = \mathbf{m_1}$

$$\mathbf{a_{1,1} = -(4k^2)^{-1/2}}$$

# OPP reduction *(Ossola, Papadopoulos, Pittau)*

**First we made direct NV reduction of the loop integrand.**

**Next we have pointed out a generic parametric form of the loop integrand and fitted the parameters with the help of on-shell values of the loop momenta solving first double cut and single cut conditions (iteratively).**

The loop integration can be easily carried out:

$$\mathbf{I(k, m_1^2, m_2^2)} = \int \frac{\mathbf{d^{2-2\epsilon}}}{\mathbf{i(2\pi)^{2-2\epsilon}}} \mathcal{I}(\mathbf{k, m_1, m_2}) = \mathbf{b_0\, I_2(k^2, m_1^2, m_2^2) + a_{1,0}\, I_1(m_1^2) + a_{2,0}\, I_1(m_2^2)} + \frac{\mathbf{b_2}}{\mathbf{4\pi}}$$

**Exercise:**

Calculate the photon self-energy in D=2 QED (Scwinger-model) using NV reduction at the integrand level.

# OPP reduction: general case

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N different l-dependent scalar propagators. Momentum inflow to the loop.

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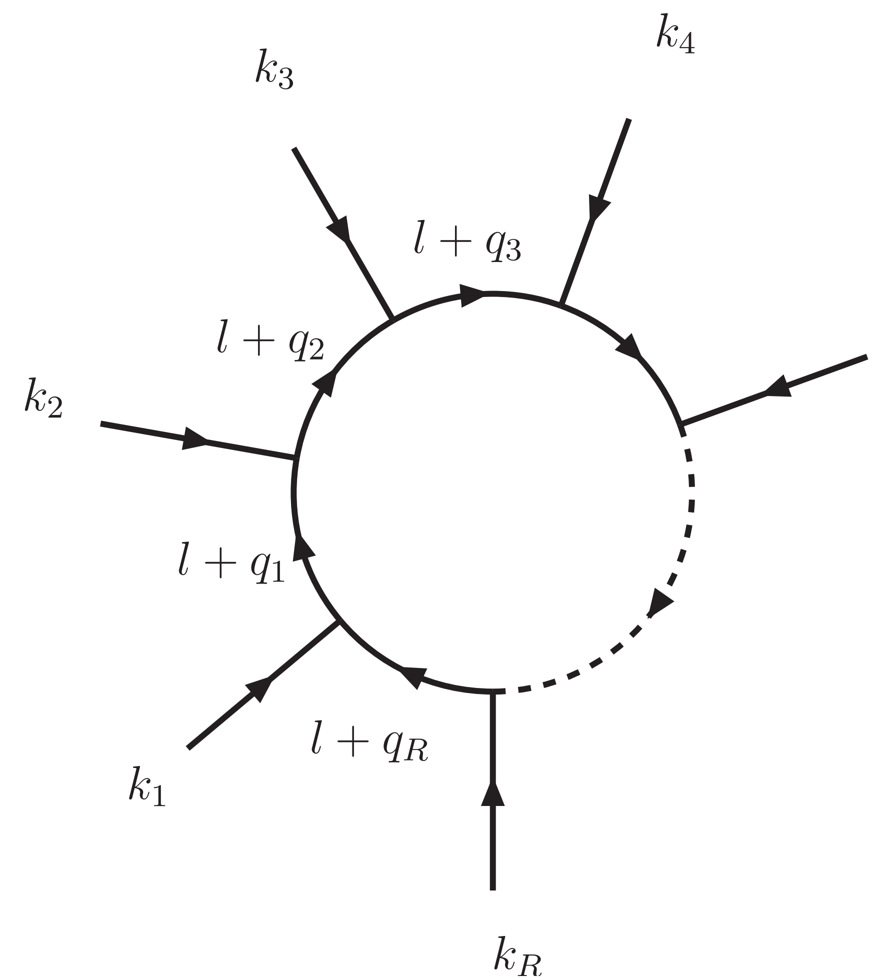

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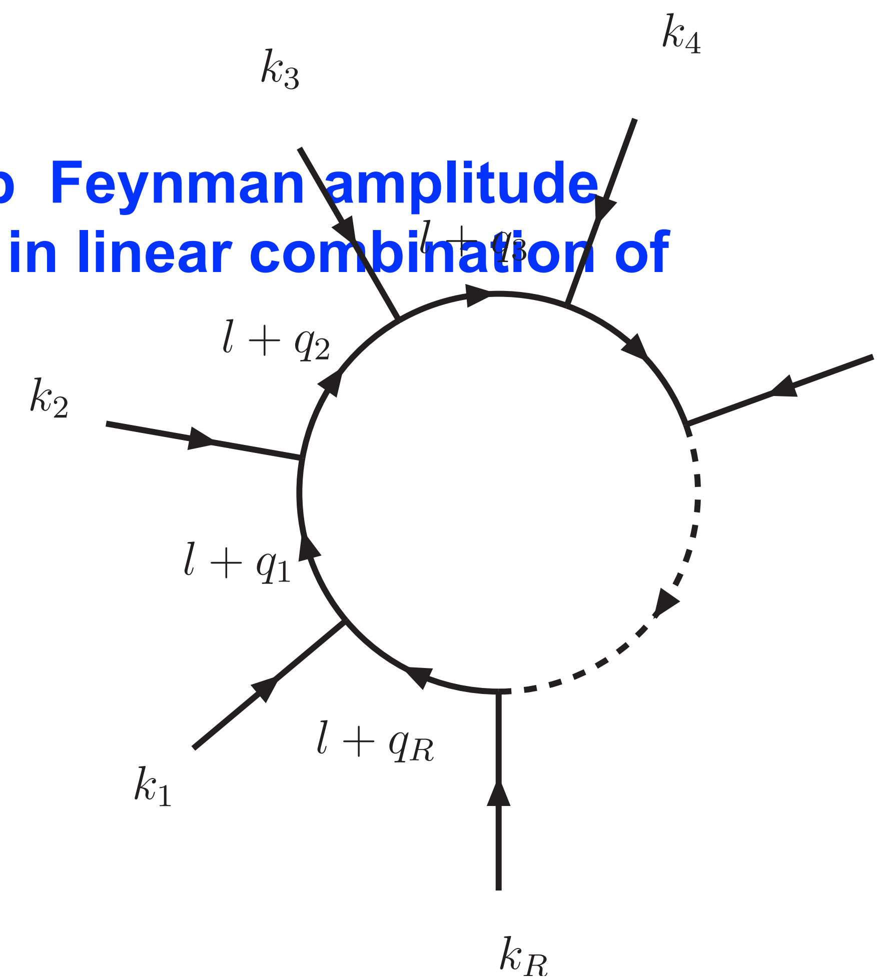
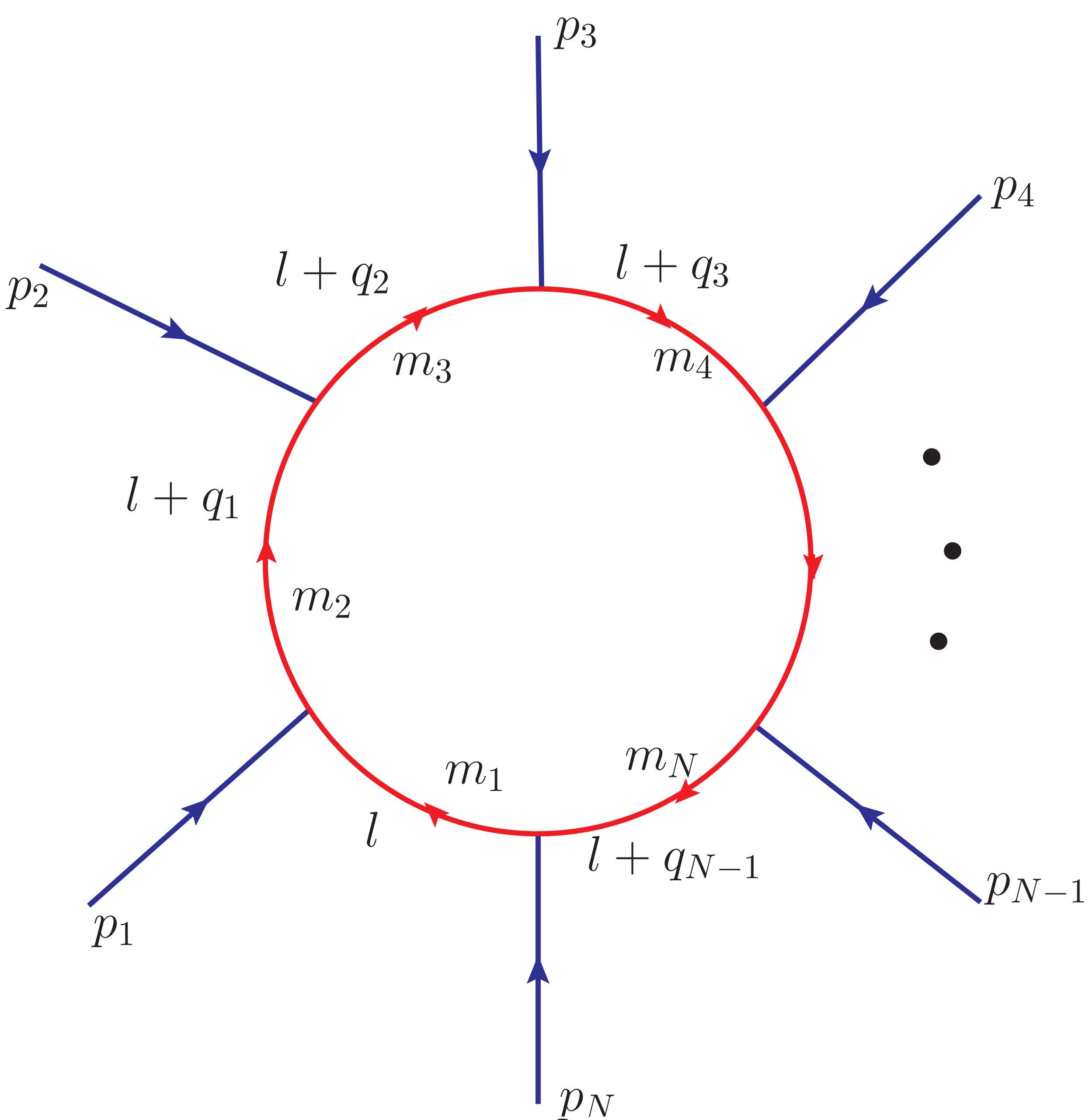
  This gives unique prescription of the integrand function as a function of the loop momentum modulo overall shift.

- For 4D external kinematics, the integrand of any one-loop Feynman amplitude with arbitrary number of external legs can always be written in linear combination of penta, quadru-,triple-,double- and single-pole terms



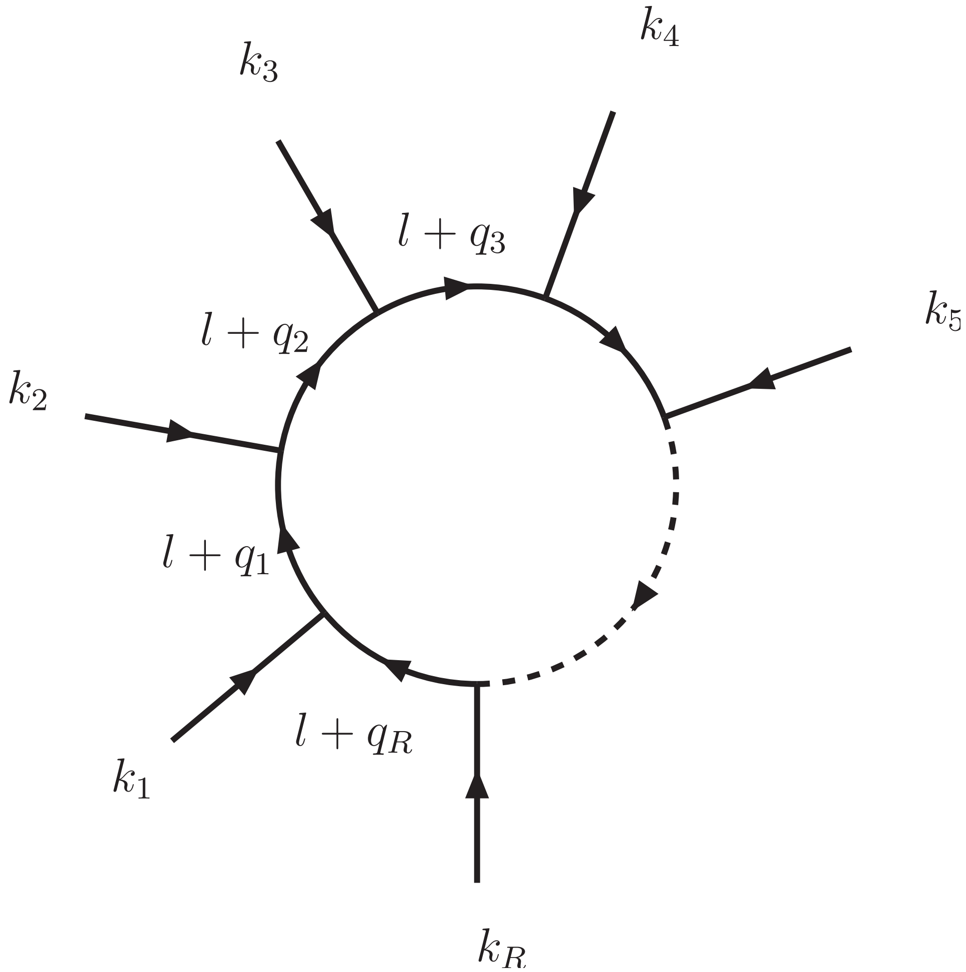
# Ordered amplitudes have well defined integrand

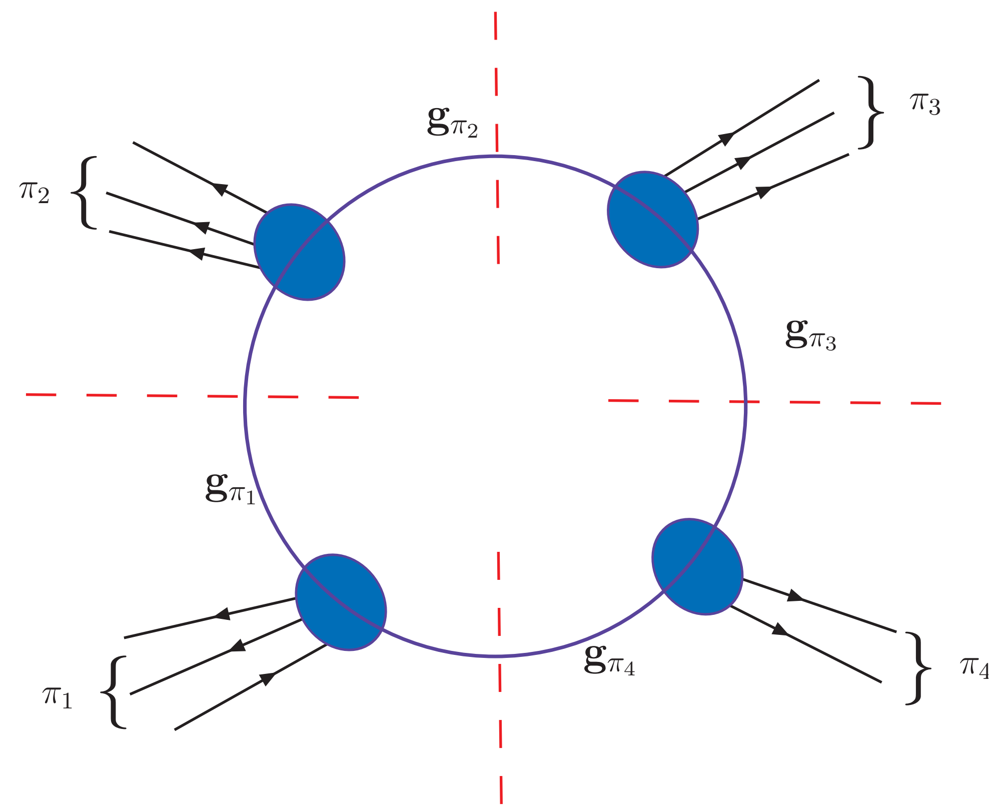
$$\mathcal{I}_N(p_1, p_2, \ldots, p_N, l) = \frac{\mathcal{N}(p_1, p_2, \ldots, p_N; l)}{d_1 d_2 \cdots d_N}$$



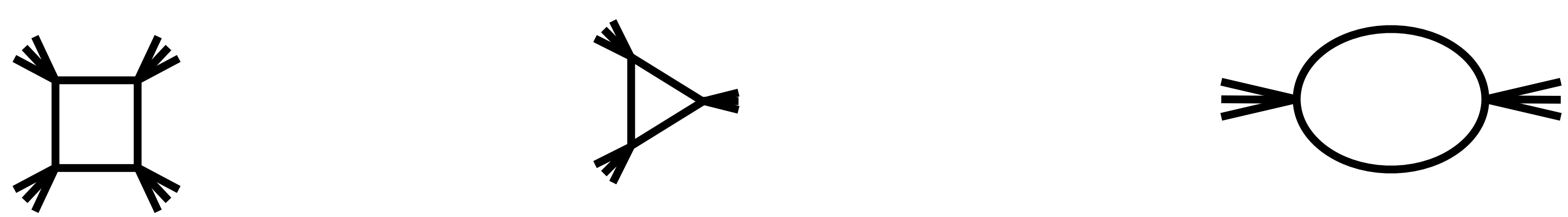

$\mathcal{I}_N(p_1, p_2, \ldots, p_N, l)$

## The integrand can be decomposed to pentagon, box, triangle, bubble and tadpole terms




The number of terms with k denominators is $\binom{\mathbf{N}}{\mathbf{k}}$



# The integrand can be decomposed to pentagon, box, triangle, bubble and tadpole terms

**The integrand can be decomposed to pentagon, box, triangle, bubble and tadpole terms**

$$\mathbf{I_N} = \int \frac{\mathrm{d}^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathrm{Num}(\mathbf{l})}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} = \int \frac{\mathrm{d}^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathbf{1}}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} \times \Bigg\{ \sum_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \tilde{\mathbf{e}}_{\mathbf{i_1,i_2,i_3,i_4,i_5}}(\mathbf{l}) \prod_{\mathbf{j}\neq[\mathbf{i_1,i_2,i_3,i_4,i_5}]} \mathbf{d_j}(\mathbf{l})$$

$$+ \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}_{\mathbf{i_1,i_2,i_3,i_4}}(\mathbf{l}) \prod_{\mathbf{j}\neq[\mathbf{i_1,i_2,i_3,i_4}]} \mathbf{d_j}(\mathbf{l})$$

$$+ \sum_{\mathbf{i_1,i_2,i_3}} \tilde{\mathbf{c}}_{\mathbf{i_1,i_2,i_3}}(\mathbf{l}) \prod_{\mathbf{j}\neq[\mathbf{i_1,i_2,i_3}]} \mathbf{d_j}(\mathbf{l}) + \sum_{\mathbf{i_1,i_2}} \tilde{\mathbf{b}}_{\mathbf{i_1,i_2}}(\mathbf{l}) \prod_{\mathbf{j}\neq[\mathbf{i_1,i_2}]} \mathbf{d_j}(\mathbf{l}) + \sum_{\mathbf{i_1}} \tilde{\mathbf{a}}_{\mathbf{i_1}}(\mathbf{l}) \prod_{\mathbf{j}\neq\mathbf{i_1}} \mathbf{d_j}(\mathbf{l}) \Bigg\}.$$

$$\mathbf{d_i}(\mathbf{l}) = (\mathbf{l} + \mathbf{q_i})^{\mathbf{2}} - \mathbf{m_i^2}\,, \mathbf{i} = \mathbf{0}, \ldots, \mathbf{4}\,,\ \mathbf{q_0} = \mathbf{0}$$

## The integrand can be decomposed to pentagon, box, triangle, bubble and tadpole terms

$$\mathbf{I_N} = \int \frac{d^D l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod_i d_i(l)} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{\prod_i d_i(l)} \times \Bigg\{ \sum_{i_1,i_2,i_3,i_4,i_5} \tilde{e}_{i_1,i_2,i_3,i_4,i_5}(l) \prod_{j\neq[i_1,i_2,i_3,i_4,i_5]} d_j(l)$$

$$+ \sum_{i_1,i_2,i_3,i_4} \tilde{d}_{i_1,i_2,i_3,i_4}(l) \prod_{j\neq[i_1,i_2,i_3,i_4]} d_j(l)$$

$$+ \sum_{i_1,i_2,i_3} \tilde{c}_{i_1,i_2,i_3}(l) \prod_{j\neq[i_1,i_2,i_3]} d_j(l) + \sum_{i_1,i_2} \tilde{b}_{i_1,i_2}(l) \prod_{j\neq[i_1,i_2]} d_j(l) + \sum_{i_1} \tilde{a}_{i_1}(l) \prod_{j\neq i_1} d_j(l) \Bigg\}.$$

$$d_i(l) = (l + q_i)^2 - m_i^2 \, , i = 0, \ldots, 4 \, , \; q_0 = 0$$

$$\mathrm{Num}(l) = N_5(l) = \prod_{i=1}^{5} u_i \cdot l, \qquad l^\mu = \sum_{i=1}^{4} (l\cdot q_i) v_i^\mu + (l \cdot n_\epsilon) n_\epsilon^\mu \quad l \cdot q_i = \frac{1}{2}\left(d_i - d_0 - (q_i^2 - m_i^2 + m_0^2)\right)$$

## The integrand can be decomposed to pentagon, box, triangle, bubble and tadpole terms

$$\mathbf{I_N} = \int \frac{d^D l}{(2\pi)^D} \frac{\mathrm{Num}(\mathbf{l})}{\prod_i d_i(\mathbf{l})} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{\prod_i d_i(\mathbf{l})} \times \left\{ \sum_{i_1,i_2,i_3,i_4,i_5} \tilde{e}_{i_1,i_2,i_3,i_4,i_5}(\mathbf{l}) \prod_{j\neq[i_1,i_2,i_3,i_4,i_5]} d_j(\mathbf{l}) \right.$$

$$+ \sum_{i_1,i_2,i_3,i_4} \tilde{d}_{i_1,i_2,i_3,i_4}(\mathbf{l}) \prod_{j\neq[i_1,i_2,i_3,i_4]} d_j(\mathbf{l})$$

$$\left. + \sum_{i_1,i_2,i_3} \tilde{c}_{i_1,i_2,i_3}(\mathbf{l}) \prod_{j\neq[i_1,i_2,i_3]} d_j(\mathbf{l}) + \sum_{i_1,i_2} \tilde{b}_{i_1,i_2}(\mathbf{l}) \prod_{j\neq[i_1,i_2]} d_j(\mathbf{l}) + \sum_{i_1} \tilde{a}_{i_1}(\mathbf{l}) \prod_{j\neq i_1} d_j(\mathbf{l}) \right\}.$$

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$$N_5(l) = \left(\prod_i^4 u_i \cdot l\right)(u_5 \cdot l) = \frac{1}{2}\sum_{j=1}^{4}(u_5 \cdot v_j)\left(\prod_i^4 u_i \cdot l\right)(d_j - d_0) \ - \frac{1}{2}\sum_{j=1}^{4}(u_5 \cdot v_j)\left(\prod_i^4 u_i \cdot l\right)(q_j^2 - m_j^2 + m_0^2)$$

## The integrand can be decomposed to pentagon, box, triangle, bubble and tadpole terms

$$\mathbf{I_N} = \int \frac{d^D l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod_i d_i(l)} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{\prod_i d_i(l)} \times \Big\{ \sum_{i_1,i_2,i_3,i_4,i_5} \tilde{e}_{i_1,i_2,i_3,i_4,i_5}(l) \prod_{j\neq[i_1,i_2,i_3,i_4,i_5]} d_j(l)$$

$$+ \sum_{i_1,i_2,i_3,i_4} \tilde{d}_{i_1,i_2,i_3,i_4}(l) \prod_{j\neq[i_1,i_2,i_3,i_4]} d_j(l)$$

$$+ \sum_{i_1,i_2,i_3} \tilde{c}_{i_1,i_2,i_3}(l) \prod_{j\neq[i_1,i_2,i_3]} d_j(l) + \sum_{i_1,i_2} \tilde{b}_{i_1,i_2}(l) \prod_{j\neq[i_1,i_2]} d_j(l) + \sum_{i_1} \tilde{a}_{i_1}(l) \prod_{j\neq i_1} d_j(l) \Big\}.$$

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$$N_5(l) = \left(\prod_i^4 u_i \cdot l\right)(u_5 \cdot l) = \frac{1}{2}\sum_{j=1}^{4}(u_5 \cdot v_j)\left(\prod_i^4 u_i \cdot l\right)(d_j - d_0) \;\; - \frac{1}{2}\sum_{j=1}^{4}(u_5 \cdot v_j)\left(\prod_i^4 u_i \cdot l\right)(q_j^2 - m_j^2 + m_0^2)$$

$$\tilde{e}_{01234}(l) = \tilde{e}_{01234}^{(0)}$$

# Parameter counting

The numerators are simple polynomials of the loop momentum components of the corresponding trivial space.

For a given 5,-4-,3-,2-,1 denominator 1, 5, 10, 10, 5 parameters;

Pentagon (rank five): $\mathbf{D_P = 4\,, D_T = 0 + 1\,, l_T^2 = (ln_\epsilon)^2} = \text{constant terms} + \mathcal{O}(\mathbf{d_i})$

Box (rank four): $\mathbf{D_P = 3\,, D_T = 1 + 1\,, l_T^2 = (ln_4)^2 + (ln_\epsilon)^2} = \text{const} + \mathcal{O}(\mathbf{d_i})$

Triangle (rank three): $\mathbf{D_P = 2\,, D_T = 3 + 1\,, l_T^2 = (ln_3)^2 + (ln_4)^2 + (ln_\epsilon)^2} = \text{const} + \mathcal{O}(\mathbf{d_i})$

Bubble (rank two): $\mathbf{D_P = 1\,, D_T = 3+1\,, l_T^2 = (ln_2)^2 + (ln_3)^2 + (ln_4)^2 + (ln_\epsilon)^2} = \text{const} + \mathcal{O}(\mathbf{d_i})$

Tadpole (rank one): $\mathbf{D_P = 0\,, D_T = 4+1\,, l_T^2 = (ln_1)^2 + (ln_2)^2 + (ln_3)^2 + (ln_4)^2 + (ln_\epsilon)^2} = \text{const} + \mathcal{O}(\mathbf{d_i})$

# Parameter counting

The numerators are simple polynomials of the loop momentum components of the corresponding trivial space.

For a given 5,-4-,3-,2-,1 denominator 1, 5, 10, 10, 5 parameters; $l_T^2 = l^2 - l_P^2\,, \quad l_P^\mu = (lq_i)v_i^\mu$

Pentagon (rank five): $\mathbf{D_P = 4\,, D_T = 0+1\,, l_T^2 = (ln_\epsilon)^2} = \text{constant terms} + \mathcal{O}(\mathbf{d_i})$

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# Parameter counting

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$$\mathbf{\tilde{e}_{01234}(l) = \tilde{e}^{(0)}_{01234}}$$

# Parameter counting

$$\tilde{\mathbf{e}}_{\mathbf{01234}}(\mathbf{l}) = \tilde{\mathbf{e}}^{(\mathbf{0})}_{\mathbf{01234}}$$

$$\tilde{\mathbf{d}}_{\mathbf{0123}}(\mathbf{l}) = \tilde{\mathbf{d}}_{\mathbf{0}} + \tilde{\mathbf{d}}_{\mathbf{1}}(\mathbf{l}\cdot\mathbf{n}_{\mathbf{4}}) + \tilde{\mathbf{d}}_{\mathbf{2}}(\mathbf{l}\cdot\mathbf{n}_{\epsilon})^{\mathbf{2}} + \tilde{\mathbf{d}}_{\mathbf{3}}(\mathbf{l}\cdot\mathbf{n}_{\epsilon})^{\mathbf{2}}(\mathbf{l}\cdot\mathbf{n}_{\mathbf{4}}) + \tilde{\mathbf{d}}_{\mathbf{4}}(\mathbf{l}\cdot\mathbf{n}_{\epsilon})^{\mathbf{4}},$$

# Parameter counting

$$\tilde{\mathbf{e}}_{01234}(\mathbf{l}) = \tilde{\mathbf{e}}^{(0)}_{01234}$$

$$\tilde{\mathbf{d}}_{0123}(\mathbf{l}) = \tilde{\mathbf{d}}_0 + \tilde{\mathbf{d}}_1(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_2(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{d}}_3(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_4(\mathbf{l}\cdot\mathbf{n}_\epsilon)^4,$$

$$\begin{aligned}\tilde{\mathbf{c}}_{012}(\mathbf{l}) = \tilde{\mathbf{c}}_0 + \tilde{\mathbf{c}}_1(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{c}}_2(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{c}}_3((\mathbf{l}\cdot\mathbf{n}_3)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) + \tilde{\mathbf{c}}_4(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{c}}_5(\mathbf{l}\cdot\mathbf{n}_3)^3 \\ + \tilde{\mathbf{c}}_6(\mathbf{l}\cdot\mathbf{n}_4)^3 + \tilde{\mathbf{c}}_7(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{c}}_8(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{c}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4)\end{aligned}$$

# Parameter counting

$$\tilde{\mathbf{e}}_{01234}(\mathbf{l}) = \tilde{\mathbf{e}}^{(0)}_{01234}$$

$$\tilde{\mathbf{d}}_{0123}(\mathbf{l}) = \tilde{\mathbf{d}}_0 + \tilde{\mathbf{d}}_1(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_2(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{d}}_3(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_4(\mathbf{l}\cdot\mathbf{n}_\epsilon)^4,$$

$$\begin{aligned}\tilde{\mathbf{c}}_{012}(\mathbf{l}) = {} & \tilde{\mathbf{c}}_0+\tilde{\mathbf{c}}_1(\mathbf{l}\cdot\mathbf{n}_3)+\tilde{\mathbf{c}}_2(\mathbf{l}\cdot\mathbf{n}_4)+\tilde{\mathbf{c}}_3((\mathbf{l}\cdot\mathbf{n}_3)^2-(\mathbf{l}\cdot\mathbf{n}_4)^2)+\tilde{\mathbf{c}}_4(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{c}}_5(\mathbf{l}\cdot\mathbf{n}_3)^3 \\ & + \tilde{\mathbf{c}}_6(\mathbf{l}\cdot\mathbf{n}_4)^3 + \tilde{\mathbf{c}}_7(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{c}}_8(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{c}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4)\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{b}}_{01}(\mathbf{l}) = {} & \tilde{\mathbf{b}}_0+\tilde{\mathbf{b}}_1(\mathbf{l}\cdot\mathbf{n}_2)+\tilde{\mathbf{b}}_2(\mathbf{l}\cdot\mathbf{n}_3)+\tilde{\mathbf{b}}_3(\mathbf{l}\cdot\mathbf{n}_4)+\tilde{\mathbf{b}}_4((\mathbf{l}\cdot\mathbf{n}_2)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) + \tilde{\mathbf{b}}_5((\mathbf{l}\cdot\mathbf{n}_3)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) \\ & + \tilde{\mathbf{b}}_6(\mathbf{l}\cdot\mathbf{n}_2)(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{b}}_7(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{b}}_8(\mathbf{l}\cdot\mathbf{n}_2)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{b}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2,\end{aligned}$$

# Parameter counting

$$\tilde{\mathbf{e}}_{01234}(\mathbf{l}) = \tilde{\mathbf{e}}^{(0)}_{01234}$$

$$\tilde{\mathbf{d}}_{0123}(\mathbf{l}) = \tilde{\mathbf{d}}_0 + \tilde{\mathbf{d}}_1(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_2(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{d}}_3(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_4(\mathbf{l}\cdot\mathbf{n}_\epsilon)^4,$$

$$\begin{aligned}\tilde{\mathbf{c}}_{012}(\mathbf{l}) = {} & \tilde{\mathbf{c}}_0+\tilde{\mathbf{c}}_1(\mathbf{l}\cdot\mathbf{n}_3)+\tilde{\mathbf{c}}_2(\mathbf{l}\cdot\mathbf{n}_4)+\tilde{\mathbf{c}}_3((\mathbf{l}\cdot\mathbf{n}_3)^2-(\mathbf{l}\cdot\mathbf{n}_4)^2)+\tilde{\mathbf{c}}_4(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{c}}_5(\mathbf{l}\cdot\mathbf{n}_3)^3 \\ & + \tilde{\mathbf{c}}_6(\mathbf{l}\cdot\mathbf{n}_4)^3 + \tilde{\mathbf{c}}_7(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{c}}_8(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{c}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4)\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{b}}_{01}(\mathbf{l}) = {} & \tilde{\mathbf{b}}_0+\tilde{\mathbf{b}}_1(\mathbf{l}\cdot\mathbf{n}_2)+\tilde{\mathbf{b}}_2(\mathbf{l}\cdot\mathbf{n}_3)+\tilde{\mathbf{b}}_3(\mathbf{l}\cdot\mathbf{n}_4)+\tilde{\mathbf{b}}_4((\mathbf{l}\cdot\mathbf{n}_2)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) + \tilde{\mathbf{b}}_5((\mathbf{l}\cdot\mathbf{n}_3)^2 - (\mathbf{l}\cdot\mathbf{n}_4)^2) \\ & + \tilde{\mathbf{b}}_6(\mathbf{l}\cdot\mathbf{n}_2)(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{b}}_7(\mathbf{l}\cdot\mathbf{n}_3)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{b}}_8(\mathbf{l}\cdot\mathbf{n}_2)(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{b}}_9(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2,\end{aligned}$$

$$\tilde{\mathbf{a}}_\mathbf{i}(\mathbf{l}) = \tilde{\mathbf{a}}_0 + \tilde{\mathbf{a}}_1(\mathbf{l}\cdot\mathbf{n}_1) + \tilde{\mathbf{a}}_2(\mathbf{l}\cdot\mathbf{n}_2) + \tilde{\mathbf{a}}_3(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{a}}_4(\mathbf{l}\cdot\mathbf{n}_4)$$

# Parameter counting

$$\tilde{\mathbf{e}}_{01234}(\mathbf{l}) = \tilde{\mathbf{e}}_{01234}^{(0)}$$

$$\tilde{\mathbf{d}}_{0123}(\mathbf{l}) = \tilde{\mathbf{d}}_0 + \tilde{\mathbf{d}}_1(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_2(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{d}}_3(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n}_4) + \tilde{\mathbf{d}}_4(\mathbf{l}\cdot\mathbf{n}_\epsilon)^4,$$

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$$\tilde{\mathbf{a}}_\mathbf{i}(\mathbf{l}) = \tilde{\mathbf{a}}_0 + \tilde{\mathbf{a}}_1(\mathbf{l}\cdot\mathbf{n}_1) + \tilde{\mathbf{a}}_2(\mathbf{l}\cdot\mathbf{n}_2) + \tilde{\mathbf{a}}_3(\mathbf{l}\cdot\mathbf{n}_3) + \tilde{\mathbf{a}}_4(\mathbf{l}\cdot\mathbf{n}_4)$$

The coefficients $\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{e}}_{01234}$ are independent from the loop momenta and in all numerator functions we can replace $(\mathbf{l}\cdot\mathbf{n_i})$ $\mathtt{\mathbf{with}}$ $(\mathbf{l_T}\cdot\mathbf{n_i})$

Note that the integration over the transverse space is trivial

$$\int \mathrm{d}^{\mathbf{D_1}}\mathbf{l}_\perp \delta(\mathbf{l}_\perp^2 - \mu_\mathbf{0}^2)\,(\mathbf{l}_\perp^\mu, \mathbf{l}_\perp^\mu \mathbf{l}_\perp^\nu) = \int \mathrm{d}^{\mathbf{D_1}}\mathbf{l}_\perp \delta(\mathbf{l}_\perp^2 - \mu_\mathbf{0}^2)\left(\mathbf{0},\ \frac{\mathbf{g}_\perp^{\mu\nu}}{\mathbf{D_1}}\,\mathbf{l}_\perp^2\right), \qquad \mathbf{D_1} = \mathbf{D} - \mathbf{1}$$

Only the constant terms and some of the terms depending on $(\mathbf{l} \cdot \mathbf{n}_\epsilon)^{\mathbf{2}}$ give non-vanishing integrals

Only the constant terms and some of the terms depending on $(\mathbf{l} \cdot \mathbf{n}_\epsilon)^2$ give non-vanishing integrals

$$\mathbf{I_N} = \int \frac{\mathrm{d}^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathrm{Num}(\mathbf{l})}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} = \sum_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \tilde{\mathbf{e}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \mathbf{I_{i_1 i_2 i_3 i_4 i_5}}$$
$$+ \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4}} \mathbf{I_{i_1 i_2 i_3 i_4}} + \sum_{\mathbf{i_1,i_2,i_3}} \tilde{\mathbf{c}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3}} \mathbf{I_{i_1 i_2 i_3}} + \sum_{\mathbf{i_1,i_2}} \tilde{\mathbf{b}}^{(\mathbf{0})}_{\mathbf{i_1,i_2}} \mathbf{I_{i_1 i_2}} + \sum_{\mathbf{i_1}} \tilde{\mathbf{a}}^{(\mathbf{0})}_{\mathbf{i_1}} \mathbf{I_{i_1}} + \mathcal{R}$$

Only the constant terms and some of the terms depending on $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2$ give non-vanishing integrals

$$\mathbf{I_N} = \int \frac{\mathrm{d}^{\mathbf{D}}\mathbf{l}}{(\mathbf{2}\pi)^{\mathbf{D}}} \frac{\mathrm{Num}(\mathbf{l})}{\prod_{\mathbf{i}} \mathbf{d_i}(\mathbf{l})} = \sum_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \tilde{\mathbf{e}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \mathbf{I_{i_1 i_2 i_3 i_4 i_5}}$$
$$+ \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4}} \mathbf{I_{i_1 i_2 i_3 i_4}} + \sum_{\mathbf{i_1,i_2,i_3}} \tilde{\mathbf{c}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3}} \mathbf{I_{i_1 i_2 i_3}} + \sum_{\mathbf{i_1,i_2}} \tilde{\mathbf{b}}^{(\mathbf{0})}_{\mathbf{i_1,i_2}} \mathbf{I_{i_1 i_2}} + \sum_{\mathbf{i_1}} \tilde{\mathbf{a}}^{(\mathbf{0})}_{\mathbf{i_1}} \mathbf{I_{i_1}} + \mathcal{R}$$

Non-vanishing master integrals with $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2$ factors in the numerator

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = -\frac{D-4}{2} I^{D+2}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^4}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = \frac{(D-2)(D-4)}{4} I^{D+4}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2 i_3},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2}.$$

$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3 i_4} = 0,$$
$$\lim_{D\to 4} \frac{(D-4)(D-2)}{4} I^{(D+4)}_{i_1 i_2 i_3 i_4} = -\frac{1}{3},$$
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Only the constant terms and some of the terms depending on $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2$ give non-vanishing integrals

$$\mathbf{I_N} = \int \frac{\mathrm{d}^\mathbf{D}\mathbf{l}}{(\mathbf{2}\pi)^\mathbf{D}} \frac{\mathrm{Num}(\mathbf{l})}{\prod_\mathbf{i} \mathbf{d_i}(\mathbf{l})} = \sum_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \tilde{\mathbf{e}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4,i_5}} \mathbf{I}_{\mathbf{i_1 i_2 i_3 i_4 i_5}} + \sum_{\mathbf{i_1,i_2,i_3,i_4}} \tilde{\mathbf{d}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3,i_4}} \mathbf{I}_{\mathbf{i_1 i_2 i_3 i_4}} + \sum_{\mathbf{i_1,i_2,i_3}} \tilde{\mathbf{c}}^{(\mathbf{0})}_{\mathbf{i_1,i_2,i_3}} \mathbf{I}_{\mathbf{i_1 i_2 i_3}} + \sum_{\mathbf{i_1,i_2}} \tilde{\mathbf{b}}^{(\mathbf{0})}_{\mathbf{i_1,i_2}} \mathbf{I}_{\mathbf{i_1 i_2}} + \sum_{\mathbf{i_1}} \tilde{\mathbf{a}}^{(\mathbf{0})}_{\mathbf{i_1}} \mathbf{I}_{\mathbf{i_1}} + \mathcal{R}$$

Non-vanishing master integrals with $(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2$ factors in the numerator

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = -\frac{D-4}{2} I^{D+2}_{i_1 i_2 i_3 i_4},$$

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they contribute to the rational terms

$$\mathcal{R} = -\sum_{\mathbf{i_1,i_2,i_3,i_4}} \frac{\tilde{\mathbf{d}}^{(4)}_{\mathbf{i_1i_2i_3i_4}}}{\mathbf{6}} + \sum_{\mathbf{i_1,i_2,i_3}} \frac{\tilde{\mathbf{c}}^{(7)}_{\mathbf{i_1i_2i_3}}}{\mathbf{2}} + \sum_{\mathbf{i_1,i_2}} \left[\frac{\mathbf{m^2_{i_1}}+\mathbf{m^2_{i_2}}}{\mathbf{2}} - \frac{(\mathbf{q_{i_1}}-\mathbf{q_{i_2}})^2}{\mathbf{6}}\right] \tilde{\mathbf{b}}^{(9)}_{\mathbf{i_1i_2}}.$$

# Example: Projecting out individual (quadrupole) coefficients in D-dimension

$$\mathcal{I}_N(p_1, p_2, \ldots, p_N, l) = \frac{\mathcal{N}(p_1, p_2, \ldots, p_N; l)}{d_1 d_2 \cdots d_N} =$$

$$= \sum_{i_1 \leq i_2 \leq i_3 \leq i_4 \leq i_5 \leq n} \frac{\bar{e}_{i_1 i_2 i_3 i_4 i_5}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} +$$

$$\sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq N} \frac{\bar{d}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{1 \leq i_1 < i_2 < i_3 \leq N} \frac{\bar{c}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{1 \leq i_1 < i_2 \leq N} \frac{\bar{b}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{1 \leq i_1 \leq N} \frac{\bar{a}_{i_1}(l)}{d_{i_1}}$$

**Denote:**

$$\mathbf{RR(l) = Residuum_{0123}(Integrand)}$$
$$\mathbf{- pentagon\ contributions\ = \tilde{d}_{0123}(l)}$$

## Projecting out individual quadrupole coefficients in D-dimension:

$$\tilde{\mathbf{d}}_{\mathbf{0123}}(\mathbf{l}) = \tilde{\mathbf{d}}_{\mathbf{0}} + \tilde{\mathbf{d}}_{\mathbf{1}}(\mathbf{l}\cdot\mathbf{n_4}) + \tilde{\mathbf{d}}_{\mathbf{2}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{2}} + \tilde{\mathbf{d}}_{\mathbf{3}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{2}}(\mathbf{l}\cdot\mathbf{n_4}) + \tilde{\mathbf{d}}_{\mathbf{4}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{4}},$$

$$\mathbf{l}^\mu = \mathbf{V}^\mu + \mathbf{l}_\perp(\cos\phi\,\mathbf{n}_\mathbf{4}^\mu + \sin\phi\,\mathbf{n}_\epsilon^\mu), \qquad \mathbf{V}^\mu = -\frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{i}}^{\mathbf{3}} \mathbf{v}_\mathbf{i}^\mu\left(\mathbf{q}_\mathbf{i}^\mathbf{2} - \mathbf{m}_\mathbf{i}^\mathbf{2} + \mathbf{m}_\mathbf{0}^\mathbf{2}\right),$$

- Choose $\sin\phi = \mathbf{0}, \cos\phi = \pm\mathbf{1}$ denote $\mathbf{l}_\pm^\mu = \mathbf{V}^\mu \pm \mathbf{l}_\perp\mathbf{n}_\mathbf{4}^\mu,$

## Projecting out individual quadrupole coefficients in D-dimension:

$$\mathbf{\tilde{d}_{0123}(l) = \tilde{d}_0 + \tilde{d}_1(l\cdot n_4) + \tilde{d}_2(l\cdot n_\epsilon)^2 + \tilde{d}_3(l\cdot n_\epsilon)^2(l\cdot n_4) + \tilde{d}_4(l\cdot n_\epsilon)^4},$$

$$\mathbf{l}^\mu = \mathbf{V}^\mu + \mathbf{l}_\perp(\cos\phi\, \mathbf{n}_4^\mu + \sin\phi\, \mathbf{n}_\epsilon^\mu), \qquad \mathbf{V}^\mu = -\frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{i}}^{\mathbf{3}} \mathbf{v}_{\mathbf{i}}^\mu \left(\mathbf{q}_{\mathbf{i}}^{\mathbf{2}} - \mathbf{m}_{\mathbf{i}}^{\mathbf{2}} + \mathbf{m}_{\mathbf{0}}^{\mathbf{2}}\right),$$

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calculate the residuum of the integrand for these values

$$\mathbf{\tilde{d}_0} = \frac{\mathrm{RR}(\mathbf{l}_+) + \mathrm{RR}(\mathbf{l}_-)}{\mathbf{2}}, \qquad \mathbf{\tilde{d}_1} = \frac{\mathrm{RR}(\mathbf{l}_+) - \mathrm{RR}(\mathbf{l}_-)}{\mathbf{2l_\perp}}$$

## Projecting out individual quadrupole coefficients in D-dimension:

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- Choose $\cos\phi = \sin\phi = \pm\mathbf{1}/\sqrt{\mathbf{2}},$ denote $\tilde{\mathbf{l}}_\pm = \mathbf{V} \pm \mathbf{l}_\perp(\mathbf{n_4} + \mathbf{n}_\epsilon)/\sqrt{\mathbf{2}}$

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$$\tilde{\mathbf{d}}_{\mathbf{2}} + \tilde{\mathbf{d}}_{\mathbf{4}}\frac{\mathbf{l}_\perp^{\mathbf{2}}}{\mathbf{2}} = \mathbf{l}_\perp^{-\mathbf{2}}\left(\mathrm{RR}(\tilde{\mathbf{l}}_+) + \mathrm{RR}(\tilde{\mathbf{l}}_-) - \mathbf{2}\,\tilde{\mathbf{d}}_{\mathbf{0}}\right), \tilde{\mathbf{d}}_{\mathbf{3}} = \sqrt{\mathbf{2}}\,\mathbf{l}_\perp^{-\mathbf{3}}\left(\mathrm{RR}(\tilde{\mathbf{l}}_+) - \mathrm{RR}(\tilde{\mathbf{l}}_-) - \sqrt{\mathbf{2}}\tilde{\mathbf{d}}_{\mathbf{1}}\mathbf{l}_\perp\right).$$

## Projecting out individual quadrupole coefficients in D-dimension:

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## Projecting out individual quadrupole coefficients in D-dimension:

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- Choose $\mathbf{l}_\epsilon^\mu = \mathbf{V}^\mu + \mathbf{l}_\perp$

$$\tilde{\mathbf{d}}_2 + \tilde{\mathbf{d}}_4\mathbf{l}_\perp^2 = \frac{\mathrm{Num}(\mathbf{l}_\epsilon) - \tilde{\mathbf{d}}_0}{\mathbf{l}_\perp^2}.$$

## Comments on the rational part:

The origin is UV divergent tensor integrals. Reduction requires regularization. After reduction: D-dimensional finite tensor integrals. In the limiting case D=4 they provide finite constants independent from the kinematics.

It may happen that the numerator has manifest dependence on (D-4) coming from polarization sum. These terms can only contribute if it appears in a term leading to UV divergent integrals. There are only few UV divergent one-loop Feynman diagrams even for large number of external particles (OPP).

Sophisticated recursion relations (BDK) as an option in Black Hat

# Comment on N=4 sYM amplitudes

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- $\mathcal{N} = 4$ sYM scattering amplitudes are free from UV divergences.
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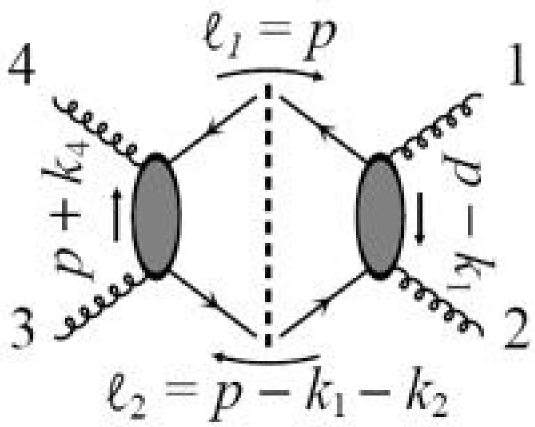
**Excercise (not easy):**

Find proper parametrization for the bubble numerator $\tilde{b}_{01}(l)$ in case of light-like inflow momentum.

## Lecture 3: Unitarity method and amplitudes

## Constraints from Unitarity: $M^\dagger - M = -iM^\dagger M$




$$-i\,\mathrm{Disc}\,A_4(1,2,3,4)\Big|_{s-\mathrm{cut}} = \int \frac{d^4p}{(2\pi)^4}\, 2\pi\delta^{(+)}(\ell_1^2 - m^2)\, 2\pi\delta^{(+)}(\ell_2^2 - m^2) \times A_4^{\mathrm{tree}}(-\ell_1, 1, 2, \ell_2)\, A_4^{\mathrm{tree}}(-\ell_2, 3, 4, \ell_1)\,,$$

Imaginary part of NLO an amplitude is calculated from tree amplitudes.

- Non-linear relation, iterative in the coupling.
- Iterative in amplitudes. Building blocks are amplitudes and not Feynman diagrams
- Manifestly gauge invariant.

# Unitarity and Cutkosky rule:

## Unitarity and Cutkosky rule:

Scattering amplitudes of scalar particles are functions of the external momenta $\mathbf{A_N(\{p_i\})}$

In case of 2 to 2 scattering with equal masses the scattering amplitude $\mathbf{A_4(s,t)}$ depends on Lorenz-invariant Mandelstam-variables s,t , $\mathbf{s = (p_1+p_2)^2, t = (p_1+p_3)^2}$,

$$\mathbf{s = 4E_{\mathrm{cm}}^2}, \qquad \mathbf{t = -\frac{(s-4m^2)}{2}\,(1-\cos\theta_{\mathrm{cms}})}$$

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The scattering amplitude as analytic function of s at fixed $\theta_{\rm cms}$ ?

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**Unitarity**:

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The scattering amplitude as analytic function of s at fixed $\theta_{\mathrm{cms}}$ ?

**Unitarity**: $\mathbf{S^\dagger S = 1, \; S = 1 + iT,}$

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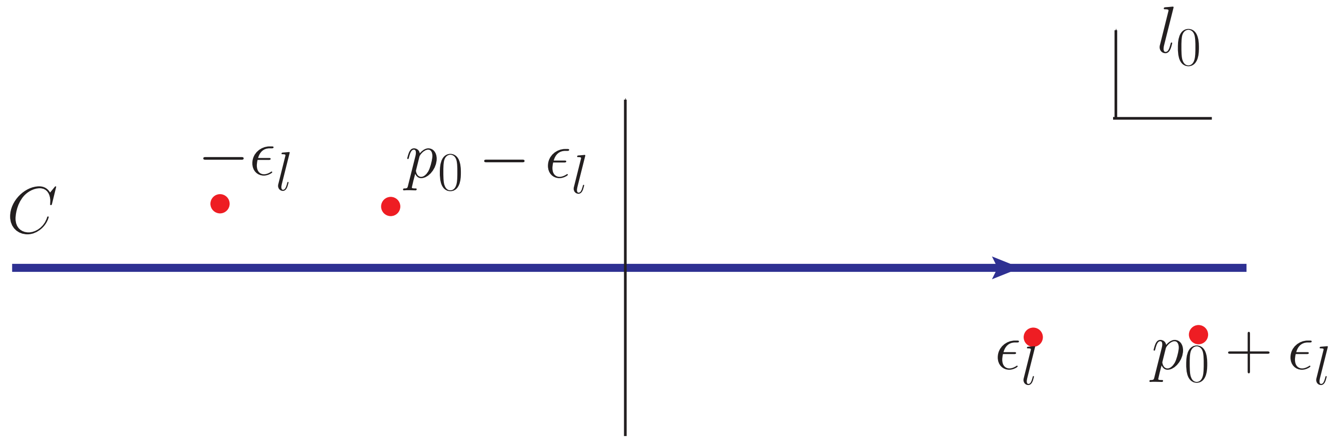


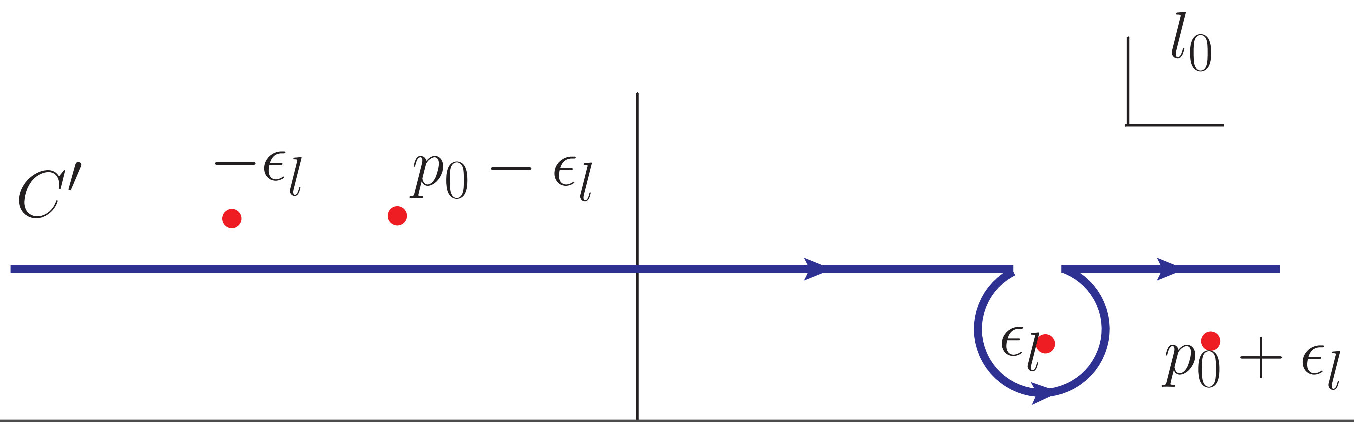

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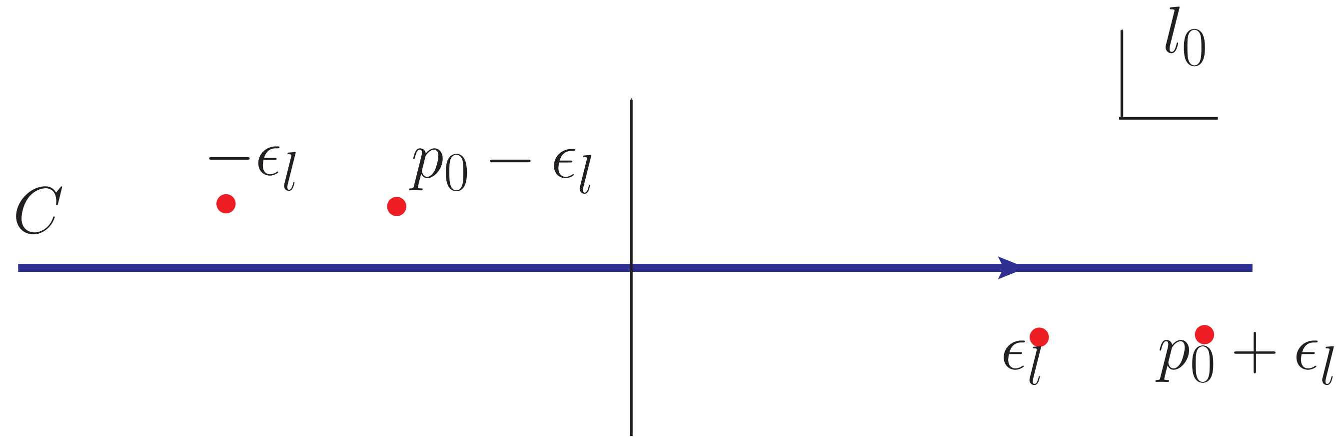


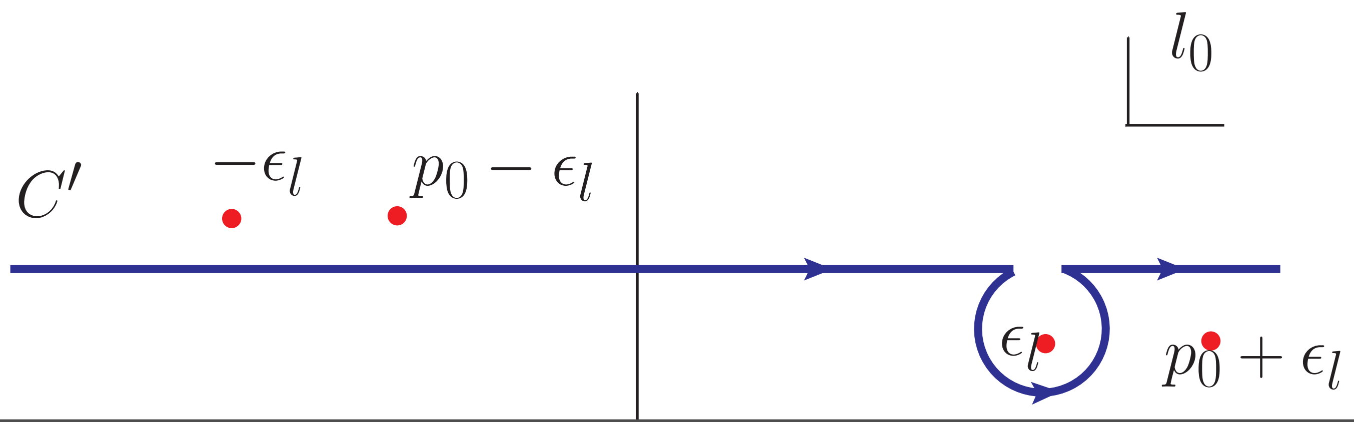

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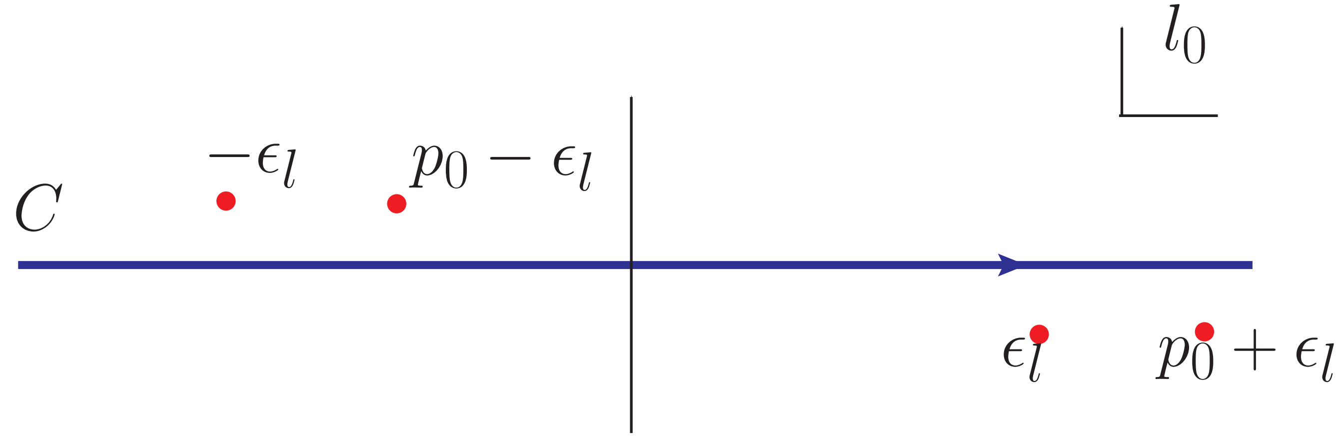
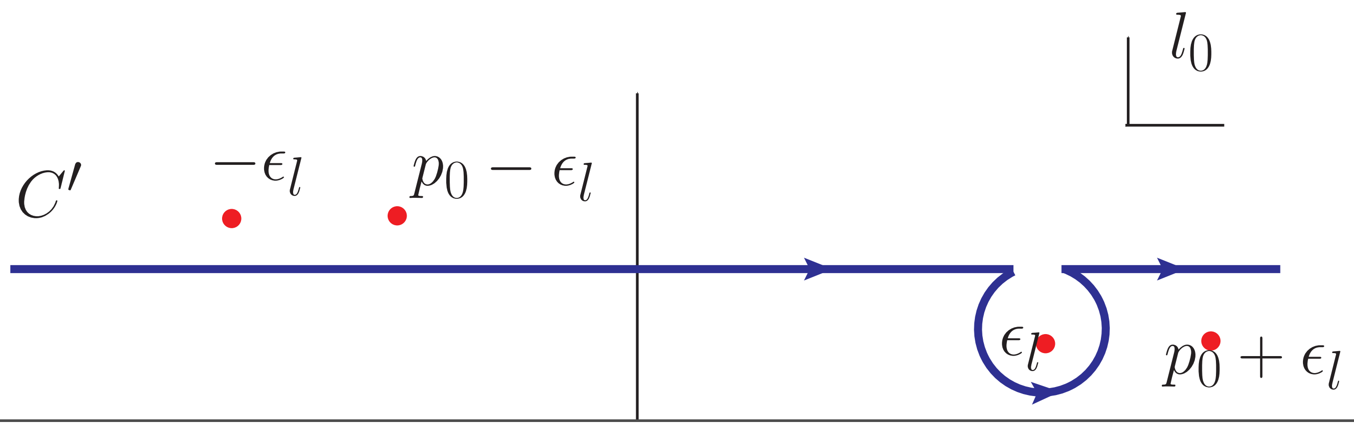
The distance of the poles $a_1, b_2$ can vanish if $p_0 > 2m$ when these poles can pinch the contour.




One can avoid pinching the contour by moving the first pole above the contour.
To compensate the difference for the integral we add an integral over a closed small circle around the first pole

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$$\mathcal{I}^{(2)} = \int \frac{\mathrm{d}^3\vec{l}}{(2\pi)^3}\left[\int_{C'} \frac{\mathrm{d}l_0}{(2\pi)}\cdots + \oint_{\gamma_{a_1}} \frac{\mathrm{d}l_0}{(2\pi)}\cdots\right]$$

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Necessary condition for singularity: $D_1(l^*) = 0\,,\ D_2(l^*) = 0$ and so $D(l^*) = 0$

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Excercise: Work out the Landau equations and solve them for the self energy and two different internal mass and find the location of the branch cut.

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Unitarity: Non-linear relations between scattering amplitudes.
It can be used to compute the discontinuities of scattering amplitudes at a given order in PT in terms of amplitudes at lower order.

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$$\mathrm{Disc}_{l\to l_{n_1 n_2 n_3}}\left[A_n^{1-\mathrm{loop}}\right] = \sum_{\mathrm{states}} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \prod_{i=1}^{3}(-2\pi i)\delta_+(l_i^2 - m_i^2)$$

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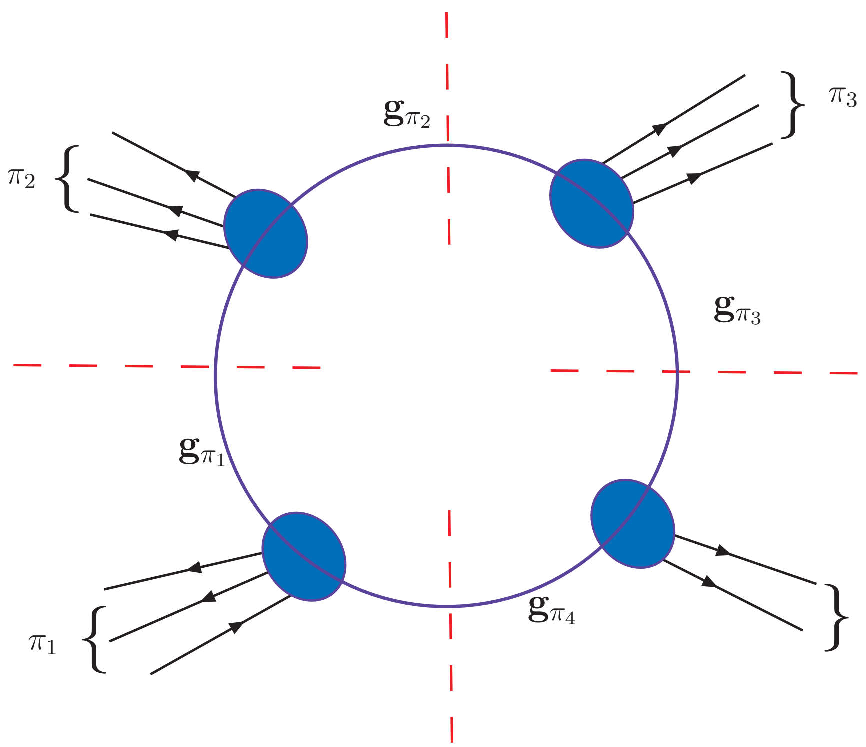

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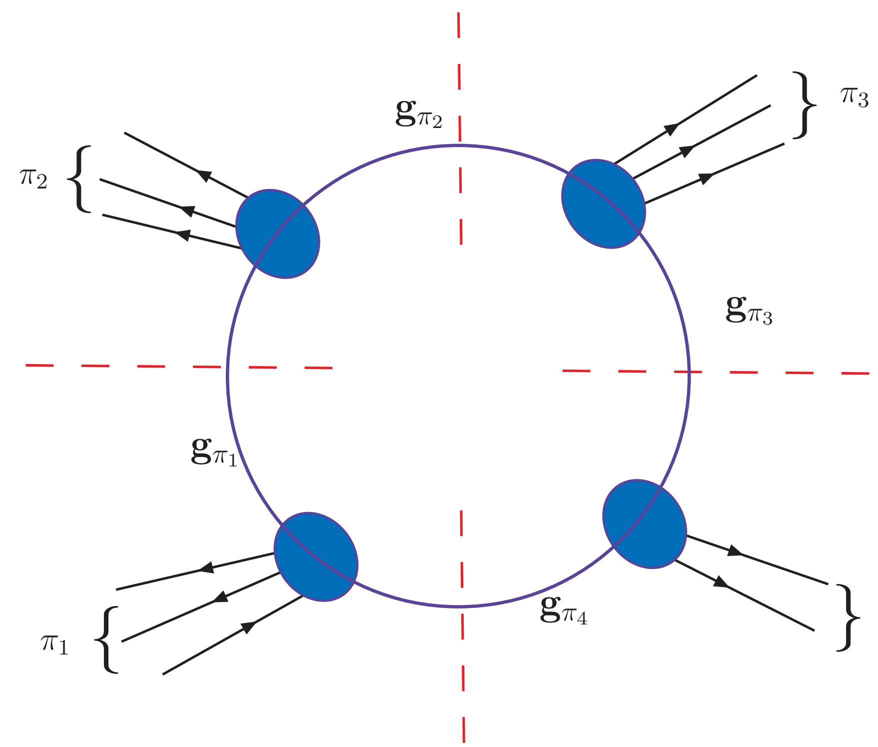

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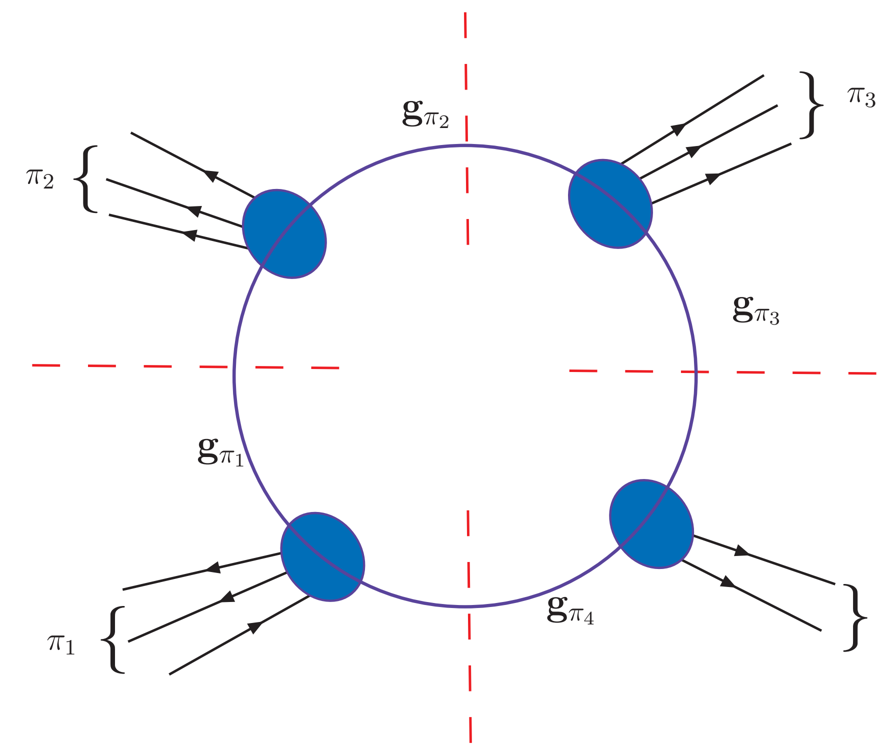

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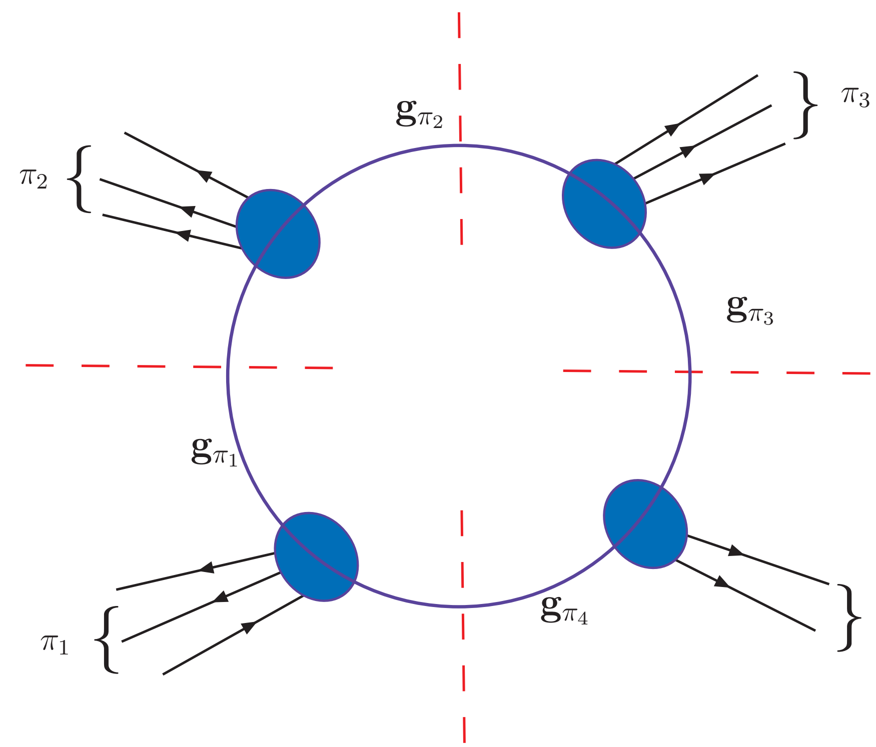


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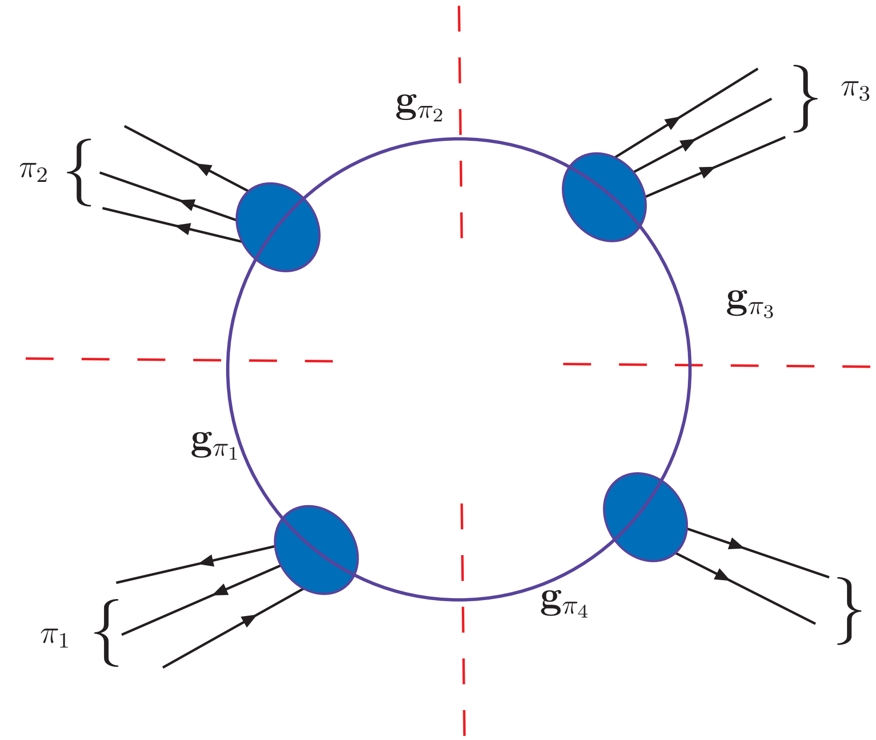


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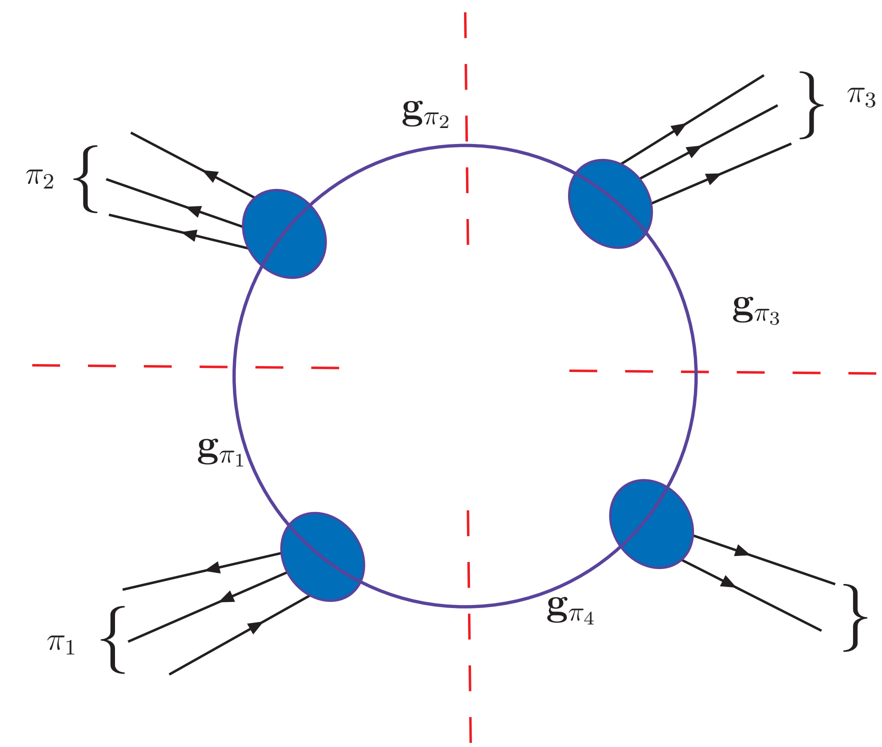


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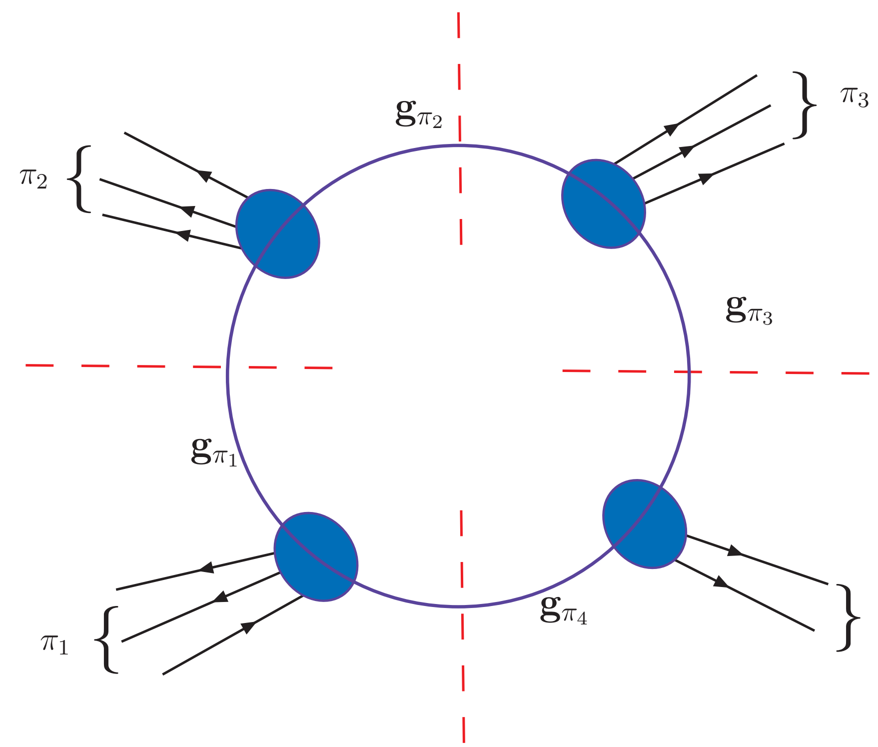


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$$\tilde{\mathbf{d}}_{\mathbf{0123}}(\mathbf{l}) = \tilde{\mathbf{d}}_{\mathbf{0}} + \tilde{\mathbf{d}}_{\mathbf{1}}(\mathbf{l}\cdot\mathbf{n_4}) + \tilde{\mathbf{d}}_{\mathbf{2}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{2}} + \tilde{\mathbf{d}}_{\mathbf{3}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{2}}(\mathbf{l}\cdot\mathbf{n_4}) + \tilde{\mathbf{d}}_{\mathbf{4}}(\mathbf{l}\cdot\mathbf{n}_\epsilon)^{\mathbf{4}},$$

$$\mathbf{l}^\mu = \mathbf{V}^\mu + \mathbf{l}_\perp(\cos\phi\,\mathbf{n}_{\mathbf{4}}^\mu + \sin\phi\,\mathbf{n}_\epsilon^\mu), \qquad \mathbf{V}^\mu = -\frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{i}}^{\mathbf{3}} \mathbf{v}_{\mathbf{i}}^\mu\left(\mathbf{q}_{\mathbf{i}}^{\mathbf{2}} - \mathbf{m}_{\mathbf{i}}^{\mathbf{2}} + \mathbf{m}_{\mathbf{0}}^{\mathbf{2}}\right),$$

- Choose $\sin\phi = \mathbf{0}, \cos\phi = \pm\mathbf{1}$ denote $\mathbf{l}_\pm^\mu = \mathbf{V}^\mu \pm \mathbf{l}_\perp \mathbf{n}_{\mathbf{4}}^\mu,$

calculate the numerator for these values

$$\tilde{\mathbf{d}}_{\mathbf{0}} = \frac{\mathrm{RR}(\mathbf{l}_+) + \mathrm{RR}(\mathbf{l}_-)}{\mathbf{2}}, \qquad \tilde{\mathbf{d}}_{\mathbf{1}} = \frac{\mathrm{RR}(\mathbf{l}_+) - \mathrm{RR}(\mathbf{l}_-)}{\mathbf{2}\mathbf{l}_\perp}$$

- Choose $\cos\phi = \sin\phi = \pm\mathbf{1}/\sqrt{\mathbf{2}},$ denote $\tilde{\mathbf{l}}_\pm = \mathbf{V} \pm \mathbf{l}_\perp(\mathbf{n_4} + \mathbf{n}_\epsilon)/\sqrt{\mathbf{2}}$

$$\tilde{\mathbf{d}}_{\mathbf{2}} + \tilde{\mathbf{d}}_{\mathbf{4}}\frac{\mathbf{l}_\perp^{\mathbf{2}}}{\mathbf{2}} = \mathbf{l}_\perp^{-\mathbf{2}}\left(\mathrm{RR}(\tilde{\mathbf{l}}_+) + \mathrm{RR}(\tilde{\mathbf{l}}_-) - \mathbf{2}\,\tilde{\mathbf{d}}_{\mathbf{0}}\right), \tilde{\mathbf{d}}_{\mathbf{3}} = \sqrt{\mathbf{2}}\,\mathbf{l}_\perp^{-\mathbf{3}}\left(\mathrm{RR}(\tilde{\mathbf{l}}_+) - \mathrm{RR}(\tilde{\mathbf{l}}_-) - \sqrt{\mathbf{2}}\tilde{\mathbf{d}}_{\mathbf{1}}\mathbf{l}_\perp\right).$$

## Projecting out individual quadrupole coefficients in D-dimension:

$$\tilde{\mathbf{d}}_{0123}(\mathbf{l}) = \tilde{\mathbf{d}}_0 + \tilde{\mathbf{d}}_1(\mathbf{l}\cdot\mathbf{n_4}) + \tilde{\mathbf{d}}_2(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2 + \tilde{\mathbf{d}}_3(\mathbf{l}\cdot\mathbf{n}_\epsilon)^2(\mathbf{l}\cdot\mathbf{n_4}) + \tilde{\mathbf{d}}_4(\mathbf{l}\cdot\mathbf{n}_\epsilon)^4,$$

$$\mathbf{l}^\mu = \mathbf{V}^\mu + \mathbf{l}_\perp(\cos\phi\,\mathbf{n}_4^\mu + \sin\phi\,\mathbf{n}_\epsilon^\mu), \qquad \mathbf{V}^\mu = -\frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{i}}^{\mathbf{3}} \mathbf{v}_\mathbf{i}^\mu\left(\mathbf{q}_\mathbf{i}^2 - \mathbf{m}_\mathbf{i}^2 + \mathbf{m}_\mathbf{0}^2\right),$$

- Choose $\sin\phi = \mathbf{0}, \cos\phi = \pm\mathbf{1}$ denote $\mathbf{l}_\pm^\mu = \mathbf{V}^\mu \pm \mathbf{l}_\perp\mathbf{n}_4^\mu,$

  calculate the numerator for these values

$$\tilde{\mathbf{d}}_0 = \frac{\mathrm{RR}(\mathbf{l}_+) + \mathrm{RR}(\mathbf{l}_-)}{\mathbf{2}}, \qquad \tilde{\mathbf{d}}_1 = \frac{\mathrm{RR}(\mathbf{l}_+) - \mathrm{RR}(\mathbf{l}_-)}{\mathbf{2l}_\perp}$$

- Choose $\cos\phi = \sin\phi = \pm\mathbf{1}/\sqrt{\mathbf{2}},$ denote $\tilde{\mathbf{l}}_\pm = \mathbf{V} \pm \mathbf{l}_\perp(\mathbf{n_4} + \mathbf{n}_\epsilon)/\sqrt{\mathbf{2}}$

$$\tilde{\mathbf{d}}_2 + \tilde{\mathbf{d}}_4\frac{\mathbf{l}_\perp^2}{\mathbf{2}} = \mathbf{l}_\perp^{-2}\left(\mathrm{RR}(\tilde{\mathbf{l}}_+) + \mathrm{RR}(\tilde{\mathbf{l}}_-) - \mathbf{2}\,\tilde{\mathbf{d}}_0\right), \tilde{\mathbf{d}}_3 = \sqrt{\mathbf{2}}\,\mathbf{l}_\perp^{-3}\left(\mathrm{RR}(\tilde{\mathbf{l}}_+) - \mathrm{RR}(\tilde{\mathbf{l}}_-) - \sqrt{\mathbf{2}}\tilde{\mathbf{d}}_1\mathbf{l}_\perp\right).$$

- Choose $\mathbf{l}_\epsilon^\mu = \mathbf{V}^\mu + \mathbf{l}_\perp$

## Projecting out individual quadrupole coefficients in D-dimension:

$$\mathbf{\tilde{d}_{0123}(l) = \tilde{d}_0 + \tilde{d}_1 (l \cdot n_4) + \tilde{d}_2 (l \cdot n_\epsilon)^2 + \tilde{d}_3 (l \cdot n_\epsilon)^2 (l \cdot n_4) + \tilde{d}_4 (l \cdot n_\epsilon)^4},$$

$$\mathbf{l}^\mu = \mathbf{V}^\mu + \mathbf{l}_\perp (\cos\phi\, \mathbf{n}_4^\mu + \sin\phi\, \mathbf{n}_\epsilon^\mu), \qquad \mathbf{V}^\mu = -\frac{\mathbf{1}}{\mathbf{2}} \sum_{\mathbf{i}}^{\mathbf{3}} \mathbf{v}_\mathbf{i}^\mu \left(\mathbf{q}_\mathbf{i}^2 - \mathbf{m}_\mathbf{i}^2 + \mathbf{m}_\mathbf{0}^2\right),$$

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calculate the numerator for these values

$$\tilde{\mathbf{d}}_\mathbf{0} = \frac{\mathrm{RR}(\mathbf{l}_+) + \mathrm{RR}(\mathbf{l}_-)}{\mathbf{2}}, \qquad \tilde{\mathbf{d}}_\mathbf{1} = \frac{\mathrm{RR}(\mathbf{l}_+) - \mathrm{RR}(\mathbf{l}_-)}{\mathbf{2l}_\perp}$$

- Choose $\cos\phi = \sin\phi = \pm\mathbf{1}/\sqrt{\mathbf{2}},$ denote $\tilde{\mathbf{l}}_\pm = \mathbf{V} \pm \mathbf{l}_\perp (\mathbf{n}_4 + \mathbf{n}_\epsilon)/\sqrt{\mathbf{2}}$

$$\tilde{\mathbf{d}}_\mathbf{2} + \tilde{\mathbf{d}}_\mathbf{4} \frac{\mathbf{l}_\perp^2}{\mathbf{2}} = \mathbf{l}_\perp^{-2} \left(\mathrm{RR}(\tilde{\mathbf{l}}_+) + \mathrm{RR}(\tilde{\mathbf{l}}_-) - \mathbf{2}\,\tilde{\mathbf{d}}_\mathbf{0}\right), \tilde{\mathbf{d}}_\mathbf{3} = \sqrt{\mathbf{2}}\,\mathbf{l}_\perp^{-3} \left(\mathrm{RR}(\tilde{\mathbf{l}}_+) - \mathrm{RR}(\tilde{\mathbf{l}}_-) - \sqrt{\mathbf{2}}\tilde{\mathbf{d}}_\mathbf{1} \mathbf{l}_\perp\right).$$

- Choose $\mathbf{l}_\epsilon^\mu = \mathbf{V}^\mu + \mathbf{l}_\perp$

$$\tilde{\mathbf{d}}_\mathbf{2} + \tilde{\mathbf{d}}_\mathbf{4} \mathbf{l}_\perp^2 = \frac{\mathrm{Num}(\mathbf{l}_\epsilon) - \tilde{\mathbf{d}}_\mathbf{0}}{\mathbf{l}_\perp^2}.$$

# The parameters are fixed by linear algebraic equations

$$\text{Res}_{ij\cdots k}\left[F(l)\right] \equiv \left[d_i(l)d_j(l)\cdots d_k(l)F(l)\right]\Big|_{l=l_{ij\cdots k}} \;.$$

$$\bar{d}_{ijkl}(l) = \text{Res}_{ijkl}\big(\mathcal{A}_N(l)\big)$$ $d_i=d_j=d_k=d_l=0$ two solutions

$$\bar{c}_{ijk}(l) = \text{Res}_{ijk}\left(\mathcal{A}_N(l) - \sum_{l\neq i,j,k}\frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l}\right)$$ $d_i=d_j=d_k=0$ infinite # of solutions

$$\bar{b}_{ij}(l) = \text{Res}_{ij}\left(\mathcal{A}_N(l) - \sum_{k\neq i,j}\frac{\bar{c}_{ijk}(l)}{d_i d_j d_k} - \frac{1}{2!}\sum_{k,l\neq i,j}\frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l}\right)$$ $d_i=d_j=0$ infinite # of solutions

**unitarity: the residues factorize into the products of tree amplitudes**

**we fully reconstruct the integrand in terms of product of tree amplitudes**

**no Feynman diagrams**

## Unitarity method: one-loop amplitudes from tree amplitudes + scalar integral functions

- Decompose the amplitude in terms of basic set of scalar integral functions and read out the coefficients using unitarity cuts ('98) (BDK)

- Consider the integrand, the amplitude is parametric integral over the loop momentum OPP('06) ( EGK ('07))

- Use generalized cuts, read out the coefficients in terms of tree amplitudes at cut-momenta (complex) BCF/BDK('05)

- Rational part is obtained by carrying out the algorithm in two different integer D>4 dimensions GKM (08) (see also Badger,BDK, OPP)

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Bern, Dixon, Kosower (BDK) ; Ellis, Giele, ZK, Melnikov (EGKM); Britto, Cachazo, Feng (BCF)

# Managing color of amplitudes

Unitarity: instead of Feynman diagrams amplitudes
Tree and one loop Feynman diagrams: color and space time part

$$\mathcal{A}_{\mathbf{n}}^{\mathrm{tree}}(\{\mathbf{k}\},\{\mathbf{h}\},\{\mathbf{a}\}),\ \mathcal{A}_{\mathbf{n}}^{1-\mathrm{loop}}(\{\mathbf{k}\},\{\mathbf{h}\},\{\mathbf{a}\})$$

➡ Color decomposition of amplitudes with the help of a basis color space (T-based, F-based, color flow based}

$$[T^a,T^b] = -F^a_{bc}T^c\,, \qquad \mathrm{Tr}(T^aT^b) = \delta_{ab}.$$

$$[F^a,F^b] = -F^a_{bc}F^c\,, \quad F^a_{bc} = -i\sqrt{2}f^{abc}\,, \quad \mathrm{Tr}(F^aF^b) = 2N_c\delta_{ab}.$$
$$(F^{a_1})_{a_2a_3} = -\frac{1}{2N_c}\mathrm{Tr}\left([F^{a_1},F^{a_2}]F^{a_3}\right) = -\mathrm{Tr}\left([T^{a_1},T^{a_2}]T^{a_3}\right).$$

$$(T^a)_{i_1}^{\bar{j}_1}(T^a)_{i_2}^{\bar{j}_2} = (\delta)_{i_1}^{\bar{j}_2}(\delta)_{i_2}^{\bar{j}_1} - \frac{1}{N_c}(\delta)_{i_1}^{\bar{j}_1}(\delta)_{i_2}^{\bar{j}_2}$$

$$\begin{aligned}(F^{a_2}F^{a_3}\ldots F^{a_{(n-2)}}F^{a_{(n-1)}})_{a_1a_n} &= \frac{1}{2N_c}\mathrm{Tr}\left([[\ldots[[F^{a_1},F^{a_2}],F^{a_3}],\ldots,F^{a_{n-2}}]\,[F^{a_{n-1}},F^{a_n}]\right)\\ &= \mathrm{Tr}\left([[\ldots[[T^{a_1},T^{a_2}],T^{a_3}],\ldots,T^{a_{n-2}}]\,[T^{a_{n-1}},T^{a_n}]\right).\end{aligned}$$

# Color ordered n-gluon tree sub-amplitudes

$$\mathcal{A}_{\mathbf{n}}^{\rm tree} = \frac{\mathbf{g_s^{n-2}}}{\mathbf{2N_c}} \sum_{\sigma\in \mathbf{S_n/Z_n}} {\rm Tr}\left(\mathbf{F}^{\mathbf{a}_{\sigma(1)}}\mathbf{F}^{\mathbf{a}_{\sigma(2)}}\mathbf{F}^{\mathbf{a}_{\sigma(3)}}\dots\mathbf{F}^{\mathbf{a}_{\sigma(n)}}\right)\mathbf{A}_{\mathbf{n},\sigma}^{\rm tree},$$

(n-1)! color ordered sub-amplitudes

$$\mathcal{A}_{\mathbf{n}}^{\rm tree} = \mathbf{g_s^{n-2}} \sum_{\sigma\in \mathbf{S_n/Z_n}} {\rm Tr}\left(\mathbf{T}^{\mathbf{a}_{\sigma(1)}}\mathbf{T}^{\mathbf{a}_{\sigma(2)}}\mathbf{T}^{\mathbf{a}_{\sigma(3)}}\dots\mathbf{T}^{\mathbf{a}_{\sigma(n)}}\right)\mathbf{A}_{\mathbf{n},\sigma}^{\rm tree}$$

$$\mathbf{A}_{\mathbf{n},\sigma}^{\rm tree} = \mathbf{m_n}(\mathbf{g}_{\sigma(\mathbf{1})},\mathbf{g}_{\sigma(\mathbf{2})},\mathbf{g}_{\sigma(\mathbf{3})},\dots,\mathbf{g}_{\sigma(\mathbf{n})})$$

Some properties of sub-amplitudes:

$$\mathbf{m_n}(\mathbf{g_1},\mathbf{g_2},\mathbf{g_3},\dots,\mathbf{g_n}) = \mathbf{m_n}(\mathbf{g_2},\mathbf{g_3},\dots,\mathbf{g_n},\mathbf{g_1})$$ (cyclic identity)

$$\mathbf{m_n}(\mathbf{g_1},\mathbf{g_2},\mathbf{g_3},\dots,\mathbf{g_{n-1}},\mathbf{g_n}) = (-\mathbf{1})^{\mathbf{n}}\mathbf{m_n}(\mathbf{g_n},\mathbf{g_{n-1}},\dots,\mathbf{g_2},\mathbf{g_1}),$$ (reflection identity)

$$\mathbf{m_n}(\mathbf{1},\underline{\mathbf{2},\dots\mathbf{n_1}},\overline{\mathbf{n_1}+\mathbf{1},\dots,\mathbf{n}}) \equiv \sum_{\sigma(\mathbf{n})}\mathbf{m_n}(\mathbf{g_1},\mathbf{g}_{\sigma(\mathbf{2})},\mathbf{g}_{\sigma(\mathbf{3})},\dots,\mathbf{g}_{\sigma(\mathbf{n})}) = \mathbf{0}$$ (Abelian identity)

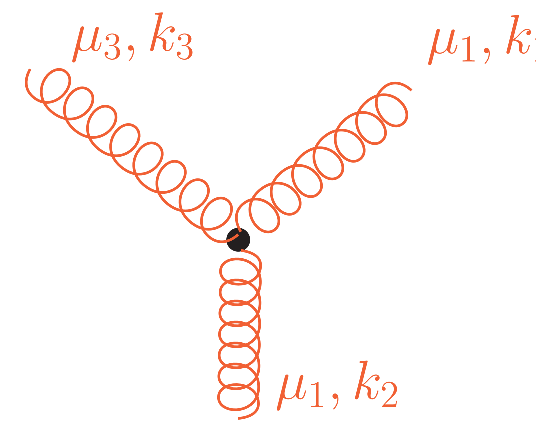
~ Kleiss-Kuijf relations

$$\mathcal{A}_{\mathbf{n}}^{\rm tree}(\mathbf{1},\mathbf{2},\mathbf{3},\dots,\mathbf{n}) = \mathbf{g_s^{n-2}}\sum_{\sigma=\mathcal{P}(\mathbf{2},\mathbf{3},\dots,\mathbf{n-1})}\left(\mathbf{F}^{\mathbf{a}_{\sigma(2)}}\dots\mathbf{F}^{\mathbf{a}_{\sigma(n-1)}}\right)_{\mathbf{a_1a_n}}\mathbf{m_n}(\mathbf{g_1},\mathbf{g}_{\sigma(\mathbf{2})},\mathbf{g}_{\sigma(\mathbf{3})},\dots,\mathbf{g}_{\sigma(\mathbf{n-1})},\mathbf{g_n}).$$

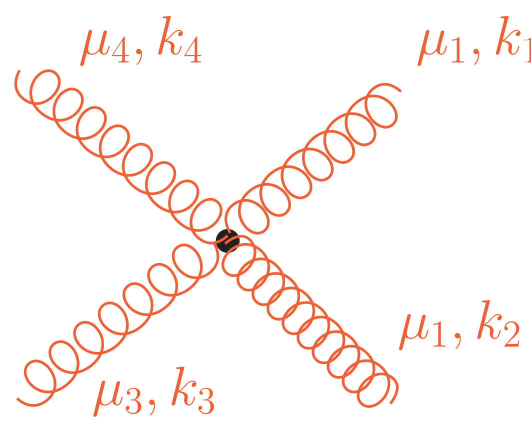
(n-2)! color ordered sub-amplitudes (see also BCJ relations)

Unitary color basis: on each pole of the tree amplitude the color factor of a given colorless amplitude also factorizes three gluon vertex

$$i\frac{g}{\sqrt{2}}\left((k_1-k_2)_{\mu_3}g_{\mu_1\mu_2}+(k_2-k_3)_{\mu_1}g_{\mu_2\mu_3}+(k_3-k_1)_{\mu_2}g_{\mu_3\mu_1}\right.$$

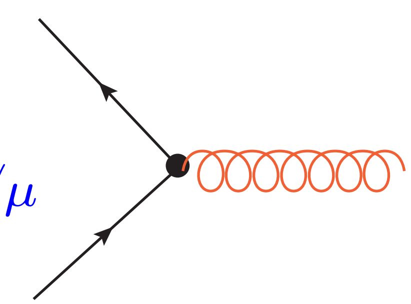
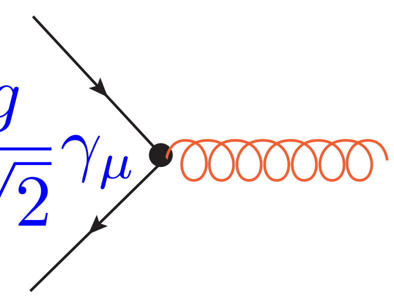



four gluon vertex




$$i\frac{g^2}{2}\left(2g_{\mu_1\mu_3}g_{\mu_2\mu_4}-g_{\mu_1\mu_4}g_{\mu_2\mu_3}-g_{\mu_1\mu_2}g_{\mu_3\mu_4}\right)$$

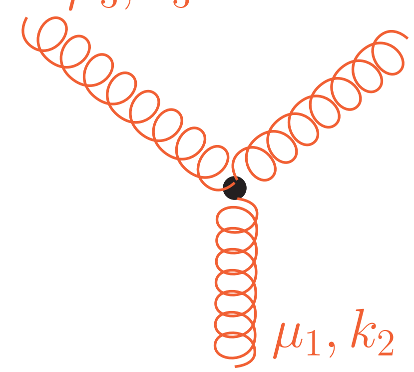
gluon-quark-antiquark vertex

$$i\frac{g}{\sqrt{2}}\gamma_\mu$$  - $$i\frac{g}{\sqrt{2}}\gamma_\mu$$ 

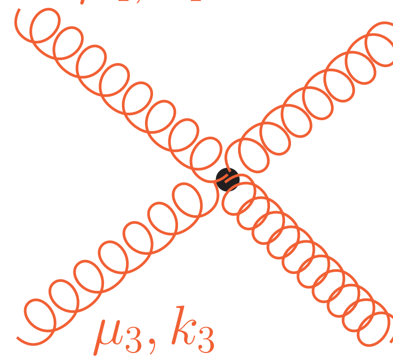
sign depends on orientation

# Color stripped Feynman rules for color ordered sub-amplitudes

three gluon vertex




$$i\frac{g}{\sqrt{2}}\left((k_1-k_2)_{\mu_3}g_{\mu_1\mu_2}+(k_2-k_3)_{\mu_1}g_{\mu_2\mu_3}+(k_3-k_1)_{\mu_2}g_{\mu_3\mu_1}\right.$$

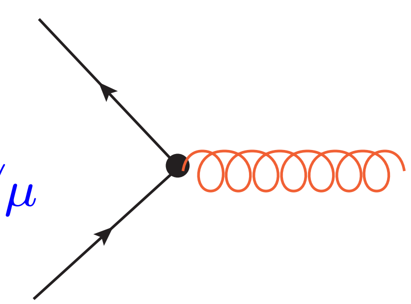



four gluon vertex

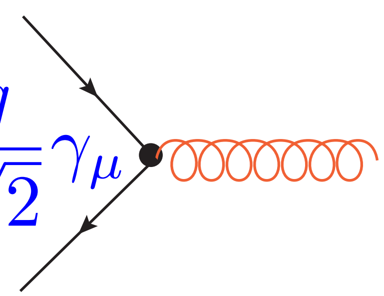
$$i\frac{g^2}{2}\left(2g_{\mu_1\mu_3}g_{\mu_2\mu_4}-g_{\mu_1\mu_4}g_{\mu_2\mu_3}-g_{\mu_1\mu_2}g_{\mu_3\mu_4}\right)$$

gluon-quark-antiquark vertex

$$i\frac{g}{\sqrt{2}}\gamma_\mu$$



$$-\ i\frac{g}{\sqrt{2}}\gamma_\mu$$



sign depends on orientation

# Color ordered n-gluon one-loop sub-amplitudes

$$\mathcal{A}_{\mathbf{n}}^{1-\mathrm{loop}} = \mathbf{g_s^n c_\Gamma} \sum_{\sigma \in \mathbf{S_{n-1}}} \mathbf{A_n^{(1)}}(\mathbf{g_1}, \mathbf{g}_{\sigma(\mathbf{2})}, \ldots, \mathbf{g}_{\sigma(\mathbf{n})}),$$

$$\mathbf{c_\Gamma} = \frac{\mathbf{1}}{(4\pi)^{\mathbf{2}-\epsilon}} \frac{\mathbf{\Gamma}(\mathbf{1}+\epsilon)\mathbf{\Gamma^2}(\mathbf{1}-\epsilon)}{\mathbf{\Gamma}(\mathbf{1}-\mathbf{2}\epsilon)}$$

Consider a double cut between external line k and k+1 and n and 1

$$\begin{aligned}
\mathrm{Im}_{(\mathbf{k},\mathbf{n})}\left[\mathbf{A_n^{(1)}}(\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_n})\right] &= \{\mathbf{a_1 a_2} \cdots \mathbf{a_k}\}_{\mathbf{vu}}\{\mathbf{a_{k+1}} \cdots \mathbf{a_n}\}_{\mathbf{uv}} \\
&\quad \times \mathbf{m_{k+2}}(\mathbf{g_v}, \mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_k}, \mathbf{g_u})\mathbf{m_{n-k+2}}(\mathbf{g_u}, \mathbf{g_{k+1}}, \ldots, \mathbf{g_n}, \mathbf{g_v}) \\
&= \{\mathbf{a_1 a_2} \cdots \mathbf{a_n}\}\, \mathrm{Im}_{(\mathbf{k},\mathbf{n})}\left[\mathbf{m_n^{(1)}}(\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_n})\right]
\end{aligned}$$

$$\mathcal{A}_{\mathbf{n}}^{1-\mathrm{loop}} = \mathbf{g_s^n c_\Gamma} \sum_{\mathcal{P}(\mathbf{2},\cdots,\mathbf{n})/\mathcal{R}} \mathrm{Tr}(\mathbf{F^{a_1}}, \ldots, \mathbf{F^{a_n}})\mathbf{m_n^{(1)}}(\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_n})$$

one loop color order sub-amplitude

A reflection transformation is factored out. The cyclic property and reflection symmetry remain valid. The number of independent one-loop amplitudes is (n-1)!/2.

# Color ordered n-gluon one-loop sub-amplitudes for large $\mathbf{N_c}$

Decomposition in T-basis:

$$\mathcal{A}_{\mathbf{n}}^{1-\mathrm{loop}} = \mathbf{g_s^n N_c c_\Gamma} \sum_{\mathcal{P}(\mathbf{2},\cdots,\mathbf{n})/\mathcal{R}} \mathrm{Tr}(\mathbf{T^{a_1}}, \ldots, \mathbf{T^{a_n}}) \mathbf{m_{1,n}^{(1)}} (\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_n})$$

$$+\mathbf{g_s^n c_\Gamma} \sum_{\mathbf{r=2}}^{[\mathbf{n}/\mathbf{2}]+\mathbf{1}} \left( \sum_{\mathcal{P}(\mathbf{2},\cdots,\mathbf{n})/\mathcal{Z}_{\mathbf{r-1}} \times \mathcal{Z}_{\mathbf{n-r+1}}} \mathrm{Tr}\left(\mathbf{T^{a_1}}, \ldots, \mathbf{T^{a_{r-1}}}\right) \mathrm{Tr}\left(\mathbf{T^{a_r}}, \ldots, \mathbf{T^{a_n}}\right) \mathbf{m_{2,n}^{(1)}} (\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_n}) \right)$$

- ➡ The single-trace color structures have an explicit factor of Nc out.
- ➡ They dominate in the large Nc limit.
- ➡ The planar L-loop color decomposition formula remains the same .
- ➡ The decomposition remains the same also for the N=4sYM theory.
- ➡ T-based color decomposition is preferred.

Two particle unitarity gives color decomposition of a quark-loop to n-gluon amplitudes

$$\mathcal{A}_{\mathbf{n;n_f}}^{1-\mathrm{loop}} = \mathbf{g_s^n c_\Gamma n_f} \sum_{\sigma \in \mathbf{S_{n-1}}} \mathrm{Tr}(\mathbf{T^{a_1}} \mathbf{T^{a_{\sigma(2)}}} \ldots \mathbf{T^{a_{\sigma n}}}) \, \mathbf{m_{n;n_f}^{(1)}} \left(\mathbf{g_1}, \mathbf{g}_{\sigma(\mathbf{2})}, \ldots, \mathbf{g}_{\sigma(\mathbf{n})}\right) ,$$

## $\bar{q}q + (n-2)g$ amplitudes and fully ordered primitive amplitudes

Color factors of Feynman diagrams $\left(T^{b_1} \dots T^{b_k} \dots\right)_{j\bar{\imath}} \times \left(F^{a_1} \dots F^{a_r}\right)_{b_1 a_{r+1}} \dots \left(F^{a_p} \dots F^{a_t - 1}\right)_{b_k a_t} \dots$

$$\mathcal{A}_n^{\rm tree}(\bar{q}_1, q_2, g_3, \dots, g_n) = g_s^{n-2} \sum_{\sigma \in S_{n-2}} \left(T^{a_{\sigma(3)}} T^{a_{\sigma(4)}} \dots T^{a_{\sigma(n)}}\right)_{i_2 \bar{\imath}_1} \; m_n(\bar{q}_1, q_2, g_{\sigma(3)}, \dots, g_{\sigma(n)}).$$

(n-2)! colorless color ordered tree sub-amplitudes

the quark labels do not participate in the permutation sum

Decomposition in mixed basis such that the quark is also in the permutation sum

$$\mathcal{A}_n^{\rm tree}(\bar{q}_1, q_2, g_3, \dots, g_n) = g_s^{n-2} (-1)^n \sum_{k=3}^{n} \sum_{\mathcal{P}(4,\dots,n)} \left(T^y \, T^{a_{k+1}} .. T^{a_n}\right)_{i_2 \bar{\imath}_1} {\rm Tr}\left(F^{a_4} \dots F^{a_k}\right)_{a_3 y}$$
$$\times \tilde{m}_n(\bar{q}_1, g_n, \dots, g_{k+1}, q_2, g_k, \dots, g_3)$$

Color ordered tree “left primitive amplitudes”

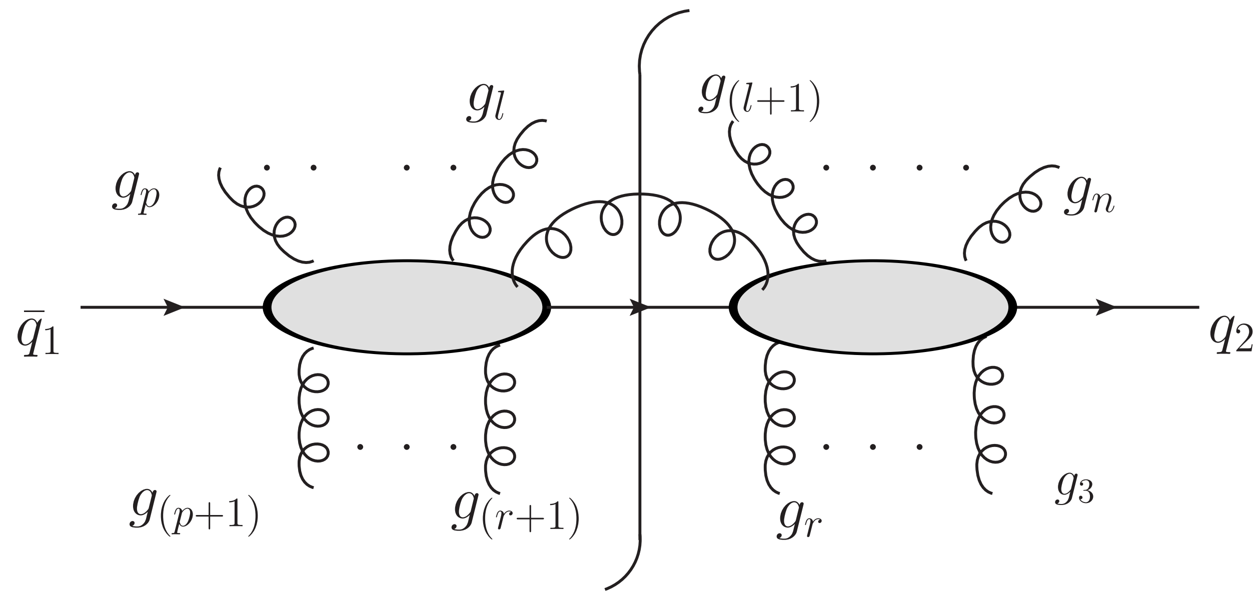
$$\tilde{m}_n(\bar{q}_1, g_n, .., g_{(k+1)}, q_2, g_k, .., g_3) = (-1)^k m_n(\bar{q}_1, q_2, \underline{k, \dots, 3}, \overline{(k+1), .., n})$$

**Excercise**: derive this relation using commutator identities

When anti-quark is in the permutation sum: “right primitive amplitude”

This mixed basis is “unitary”

## $\bar{\mathbf{q}}\mathbf{q} + (\mathbf{n}-\mathbf{2})\mathbf{g}$ **colorless one-loop fully ordered left primitive amplitude**




$$\mathcal{A}_{\mathbf{n}}^{1-\text{loop}}(\bar{\mathbf{q}}_{\mathbf{1}}, \mathbf{q}_{\mathbf{2}}, \mathbf{g}_{\mathbf{3}}, .., \mathbf{g}_{\mathbf{n}}) = \mathbf{g}_{\mathbf{s}}^{\mathbf{n}} \mathbf{c}_{\boldsymbol{\Gamma}} \sum_{\mathbf{p}=\mathbf{2}}^{\mathbf{n}} \sum_{\sigma \in \mathbf{S}_{\mathbf{n}-\mathbf{2}}} (\mathbf{T}^{\mathbf{x}_2} \mathbf{T}^{\mathbf{a}_{\sigma_3}} .. \mathbf{T}^{\mathbf{a}_{\sigma_\mathbf{p}}} \mathbf{T}^{\mathbf{x}_1})_{\mathbf{i}_2 \bar{\imath}_1} (\mathbf{F}^{\mathbf{a}_{\sigma_{\mathbf{p}+1}}} .. \mathbf{F}^{\mathbf{a}_{\sigma_\mathbf{n}}})_{\mathbf{x}_1 \mathbf{x}_2}$$
$$\times (-\mathbf{1})^{\mathbf{n}} \tilde{\mathbf{m}}_{\mathbf{n}}^{(1)}(\bar{\mathbf{q}}_{\mathbf{1}}, \mathbf{g}_{\sigma(\mathbf{p})}, .., \mathbf{g}_{\sigma(\mathbf{3})}, \mathbf{q}_{\mathbf{2}}, \mathbf{g}_{\sigma(\mathbf{n})}, .., \mathbf{g}_{\sigma(\mathbf{p}+\mathbf{1})})$$

Color ordered amplitudes for n-gluons and primitive amplitudes
for $\bar{\mathbf{q}}\mathbf{q} + (\mathbf{n}-\mathbf{2})\mathbf{g}$ can be calculated using colorless Feynman rules
Berends-Giele recursion relations

Comment: Leading color is good approximations for gluons only

# Amplitudes with multiple quarks

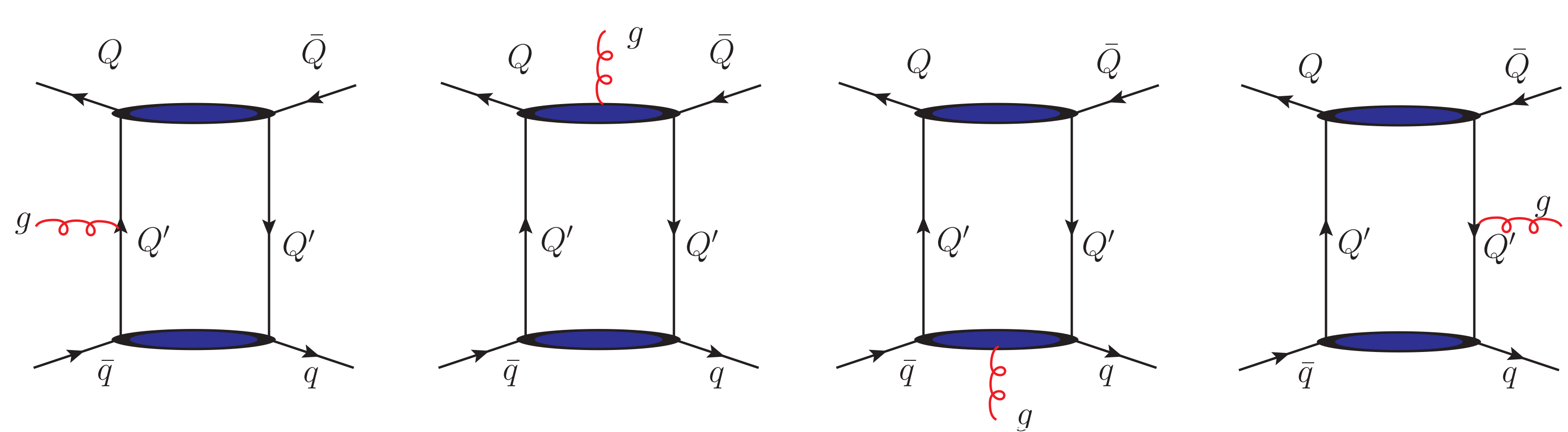
Colorless ordered amplitudes, primitive amplitudes, parent diagrams

$$\mathcal{B}^{\mathrm{tree}}(\bar{q}_1, q_2, \bar{Q}_3, Q_4, g_5) = g_s^3 \Big[ (T^{a_5})_{i_4\bar{\imath}_1}\delta_{i_2\bar{\imath}_3} B_{5;1}^{\mathrm{tree}} + \frac{1}{N_c}(T^{a_5})_{i_2\bar{\imath}_1}\delta_{i_4\bar{\imath}_3} B_{5;2}^{\mathrm{tree}} + (T^{a_5})_{i_2\bar{\imath}_3}\delta_{i_4\bar{\imath}_1} B_{5;3}^{\mathrm{tree}} + \frac{1}{N_c}(T^{a_5})_{i_4\bar{\imath}_3}\delta_{i_2\bar{\imath}_1} B_{5;4}^{\mathrm{tree}} \Big],$$

$$\mathcal{B}^{1-\mathrm{loop}}(\bar{q}_1, q_2, \bar{Q}_3, Q_4, g_5) = g_s^5 \Big[ N_c (T^{a_5})_{i_4\bar{\imath}_1}\delta_{i_2\bar{\imath}_3} B_{5;1} + (T^{a_5})_{i_2\bar{\imath}_1}\delta_{i_4\bar{\imath}_3} B_{5;2} + N_c (T^{a_5})_{i_2\bar{\imath}_3}\delta_{i_4\bar{\imath}_1} B_{5;3} + (T^{a_5})_{i_4\bar{\imath}_3}\delta_{i_2\bar{\imath}_1} B_{5;4} \Big]$$

$$B_{5;i} = B_{5;i}^{[1]} + \frac{n_f}{N_c} B_{5;i}^{[1/2]}, \qquad i = 1, 2, 3, 4,$$

$$B_{5;1}^{[1/2]} = -A_L^{[1/2]}(\bar{q}_1, g_5, Q_4, \bar{Q}_3, q_2), \quad B_{5;2}^{[1/2]} = -A_L^{[1/2]}(\bar{q}_1, Q_4, \bar{Q}_3, q_2, g_5),$$
$$B_{5;3}^{[1/2]} = -A_L^{[1/2]}(\bar{q}_1, Q_4, \bar{Q}_3, g_5, q_2), \quad B_{5;4}^{[1/2]} = -A_L^{[1/2]}(\bar{q}_1, Q_4, g_5, \bar{Q}_3, q_2).$$



## Amplitudes with multiple quarks

Colorless ordered amplitudes, primitive amplitudes, parent diagrams

More and more complicated.....

**End of color management**

# Singular behavior of one loop primitive amplitudes

**n-gluon:**

$$\mathbf{m_n^{(1)}(g_1, g_2, ...g_n) = -\sum_{i=1}^{n} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{11}{3n} + L_{i,i+1}\right)\right] m_n^{(0)}(g_1, g_2, ...g_n),}$$

$$\mathbf{L_{kn}} = \ln\left(\mu^{\mathbf{2}}/(-\mathbf{s_{kn}} - \mathbf{i0})\right)$$

$\mathbf{\bar{q}q + (n-2)g}$ :

$$\mathbf{\tilde{m}_n^{(1)}(\bar{q}_n, g_{k+1}, ..., g_{n-1}, q_2, g_3, ...g_k) =}$$

$$\mathbf{\tilde{m}_n(\bar{q}_n, g_{k+1}, ..., g_{n-1}, q_2, g_3, ...g_k)\left[-\frac{k}{\epsilon^2} - \frac{1}{\epsilon}\left(\frac{3}{2} + \sum_{i=1}^{k-1} L_{i\,i+1} + L_{kn}\right)\right]}$$

Color is eliminated. Important for testing the calculations.
Primitive tree amplitude is calculated with Berends-Giele recursion relations based on color stripped Feynman rules or BCFW recursion relations