0. MOTIVATION

Scattering amplitudes are the central observables in QFTs in particular for gauge theories. Computational formalism in terms of Feynman Amplitudes dates back to the 1970s. Outsiders impression: Fixed set of rules, purely algorithmical problem, by now practically all relevant QCD & SM amplitudes must be known at higher loop level.

NOT the case: Computational complexity grows dramatically with # legs and # loops!

Problems:

* Too many diagrams, many diagrams related by gauge invariance
* Individual diagrams are highly complicated
* Enormous number of terms dependent on all kinematical variables.

Still: Final expression when expressed in suitable formalism often rather simple!

In recent 10 years remarkable advances in understanding and ability to compute scattering amplitudes in gauge theories due to "ON-SHELL" methods:

* Recursion relations: Build higher point amplitudes from lower point ones. (tree-level)

* Generalized unitarity: Construct one-loop amplitudes from studies of analytical structure across cuts. Generalization of the optical theorem

* Symmetries: Obvious and hidden can strongly constrain the form of amplitudes at tree- and loop level.

Central message: Work with full amplitudes as building blocks which are gauge invariant and on-shell, rather than gauge variant and off-shell quantities...

Still: Knowledge of underlying Feynman diagrammatic structure is always vital!

I. INTRODUCTION AND RECAP

I.1 LORENTZ & POINCARÉ GROUP & THEIR REPRESENTATIONS

Fundamental symmetry of all relativistic QFTs: Lorentz invariance

Lorentz transformations:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \qquad \text{with} \qquad x'^2 = x^2 = x^{\mu} x^{\nu} \eta_{\mu\nu} \qquad (I.1)$$

Linear homogenous coord. transf which leaves interval invariant

$\eta_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$; $p_{\mu} p^{\mu} = p_0^2 - \vec{p}^{\,2}$.

(I.1) implies $$\boxed{\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\kappa} = \eta_{\rho\kappa}} \qquad (I.2)$$

Infinitesimally $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu} + \mathcal{O}(\omega^2)$

(I.2) $\Rightarrow$ $\omega_{\mu\nu} = -\omega_{\nu\mu}$ (antisymm.)

In Quantum Theory symmetries are represented by unitary operators: $\mathcal{U}(\Lambda)$. Should furnish a representation of the Lorentz group:

$$\mathcal{U}(\Lambda)\, \mathcal{U}(\Lambda') = \mathcal{U}(\Lambda\Lambda')$$

Infinitesimally: $\mathcal{U}(\mathbb{1}+\omega) = \mathbb{1} + \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}$

$M^{\mu\nu} = -M^{\nu\mu}$ is hermitian operators called "generators of L.G."

Lorentz-algebra:

$$\mathcal{U}(\Lambda)^{-1}\,\mathcal{U}(\Lambda')\,\mathcal{U}(\Lambda) = \mathcal{U}(\Lambda^{-1}\Lambda'\Lambda)$$

and let $\Lambda' = \mathbb{1} + \omega'$

$$\Rightarrow \quad \omega'_{\mu\nu}\,\mathcal{U}(\Lambda^{-1})\,M^{\mu\nu}\,\mathcal{U}(\Lambda) = (\underbrace{\Lambda^{-1}\omega'\Lambda}_{(\Lambda^{-1})_\mu{}^\rho\,\omega'_{\rho\kappa}\,\Lambda^\kappa{}_\nu})_{\mu\nu}\,M^{\mu\nu}$$

$$\Rightarrow \quad \omega'_{\mu\nu}\,\mathcal{U}(\Lambda^{-1})\,M^{\mu\nu}\,\mathcal{U}(\Lambda) = \omega'_{\rho\kappa}\,\Lambda^\rho{}_\mu\,\Lambda^\kappa{}_\nu\,M^{\mu\nu}$$

$$\forall\ \omega'_{\mu\nu} \Rightarrow \quad \boxed{\mathcal{U}(\Lambda^{-1})\,M^{\mu\nu}\,\mathcal{U}(\Lambda) = \Lambda^\mu{}_\rho\,\Lambda^\nu{}_\kappa\,M^{\rho\kappa}} \qquad (I.3)$$

$\Rightarrow$ Every component of $M^{\mu\nu}$ transforms with its own $\Lambda^\mu{}_\nu$ matrix. Hence we expect a vector P^μ to transform as

$$\boxed{\mathcal{U}(\Lambda^{-1})\,P^\mu\,\mathcal{U}(\Lambda) = \Lambda^\mu{}_\nu\,P^\nu}$$

Now choosing also $\Lambda = \mathbb{1} + \omega$ infinitesimal (I.3) implies

$$\frac{i}{2}\,\omega_{\rho\kappa}\,[M^{\mu\nu}, M^{\rho\kappa}] = \delta^\mu_\rho\,\omega^\nu{}_\kappa\,M^{\rho\kappa} + \delta^\nu_\kappa\,\omega^\mu{}_\rho\,M^{\rho\kappa}$$

LHS: $\omega_{\sigma\delta}\left(\delta^{\mu}_{\rho}\,\eta^{\nu\sigma}\,\delta^{\delta}_{\kappa}\,M^{\rho\kappa}+\delta^{\nu}_{\kappa}\,\eta^{\mu\sigma}\,\delta^{\delta}_{\rho}\,M^{\rho\kappa}\right)$

$$= \omega_{\sigma\delta}\left(M^{\mu\delta}\eta^{\nu\sigma}+M^{\delta\nu}\eta^{\mu\sigma}\right)$$

$$= \omega_{\rho\kappa}\left(M^{\mu\kappa}\eta^{\nu\rho}+M^{\kappa\nu}\eta^{\mu\rho}\right)$$

Strip off a/sym $\omega_{\rho\kappa}$ on LHS & RHS

$$\Rightarrow \boxed{[M^{\mu\nu},M^{\rho\kappa}]=i\left(\eta^{\nu\rho}M^{\mu\kappa}+\eta^{\mu\kappa}M^{\nu\rho}-\eta^{\nu\kappa}M^{\mu\rho}-\eta^{\mu\rho}M^{\nu\kappa}\right)} \qquad (I.4)$$

Lorentz-algebra

Similar argument yields

$$\boxed{[M^{\mu\nu},P^{\rho}]=-i\eta^{\mu\rho}P^{\nu}+i\eta^{\nu\rho}P^{\mu}} \qquad (I.5)$$

(I.4) + (I.5) "inhomogeneous Lorentz algebra" or

"Poincaré algebra".

(I.4) forms the SO(3,1) Lie algebra. Most general representation:

$$\boxed{M_{\mu\nu}=i(x_{\mu}\partial_{\nu}-x_{\nu}\partial_{\mu})+S_{\mu\nu}} \qquad (I.6)$$

where $(S_{\mu\nu})^i{}_j$ are matrices obeying (I.4) commutation relations and commute with $i(x_\mu\partial_\nu - x_\nu\partial_\mu)$.

SU(2) ⊗ SU(2) decomposition

Define $J_i := \frac{1}{2}\varepsilon_{ijk} M_{jk}$ (spatial or rotation components of $M_{\mu\nu}$) $i=1,2,3$

$K_i := M_{0i}$

Then one finds:

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \qquad (+)$$

$$[K_i, K_j] = -i\varepsilon_{ijk} J_k$$

$$[J_i, K_j] = i\varepsilon_{ijk} K_k$$

(+) obeys Lie algebra of SU(2) with known representation thy from QM.

Take the complex combination $N_i := \frac{1}{2}(J_i + iK_i)$

(N.B. N_i no longer hermitian!). $N_i^+ = \frac{1}{2}(J_i - iK_i)$

$$\Rightarrow \boxed{\begin{aligned} &[N_i, N_j] = i\varepsilon_{ijk} N_k \\ &[N_i^+, N_j^+] = i\varepsilon_{ijk} N_k^+ \end{aligned} \qquad [N_i, N_j^+] = 0} \qquad (I.7)$$

Two comuting copies of SU(2) algebras!

Representations labeled by (m,n) with $m,n = 0, 1/2, 1, 3/2, \ldots$

eigenvalues of $N_i N_i$ & $N_i^+ N_i^+$ respectively.

Since $J_3 = N_3 + N_3^+$ spin of rep. (m,n) is $\underline{m+n}$

$\underline{(0,0)}$ Spin 0 scalar field $\phi(x)$

$\underline{(\frac{1}{2},0)}$ Spin 1/2 left-handed Weyl spinor χ_α $\alpha = 1,2$

$\underline{(0,\frac{1}{2})}$ Spin 1/2 right-handed Weyl spinor $\bar{\xi}^{\dot{\alpha}}$

$\underline{(\frac{1}{2},\frac{1}{2})} = (\frac{1}{2},0) \otimes (0,\frac{1}{2})$ Spin 1 vector field $A_\mu(x)$

$\underline{(1,0)}$ Spin 1 self-dual rank 2 tensor

$$B_{\mu\nu} = -B_{\mu\nu} \qquad B_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu}{}^{\rho\kappa} B_{\rho\kappa}$$

$\underline{(0,1)}$ Spin 1 anti-self-dual rank 2 tensor

$$\tilde{B}_{\mu\nu} = -\tilde{B}_{\mu\nu} \qquad \hat{B}_{\mu\nu} = -\frac{1}{2} \varepsilon_{\mu\nu}{}^{\rho\kappa} \hat{B}_{\rho\kappa}$$

$(1,\frac{1}{2})$ Spin 3/2 gravitino

$(1,1)$ Spin 2 graviton

$$\Downarrow$$

$$B_{\mu\nu} \underset{\text{rep.}}{\hat{=}} (0,1) \oplus (1,0)$$

I.2 WEYL & DIRAC SPINORS, LAGRANGIANS

Build Lagrangian for $(\frac{1}{2},0)$ - field $\chi_\alpha(x)$

(Left-handed or Weyl spinor):

Hermitian conjugation: $(\chi_\alpha)^+ = \tilde{\chi}_{\dot{\alpha}}$ (Anti-commuting field!)

☐ Lagrangian invariant under Poincaré - transformations:

$$\boxed{\mathcal{L}_W = i\,\tilde{\chi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\,\partial_\mu \chi_\alpha - \frac{1}{2} m\,\chi^\alpha \chi_\alpha - \frac{1}{2} m^*\,\tilde{\chi}_{\dot{\alpha}}\tilde{\chi}^{\dot{\alpha}}} \qquad (I.8)$$

☐ Conventions:

$$\chi^\alpha = \varepsilon^{\alpha\beta}\chi_\beta \qquad \tilde{\chi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\tilde{\chi}_{\dot{\beta}}$$

$$\bar{\sigma}_\mu = (\mathbb{1}, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

☐ Short, spinor index free form

$$\mathcal{L} = i\,\chi^+ \bar{\sigma}^\mu \partial_\mu \chi - \frac{1}{2} m\,\chi\chi - \frac{1}{2} m^*\,\chi^+\chi^+$$

☐ Equations of motion:

$$0 = -\frac{\delta S}{\delta \chi^+} = -i\,\bar{\sigma}^\mu \partial_\mu \chi + m^*\,\chi^+$$

$$\Rightarrow \quad \boxed{0 = -i(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu \chi_\alpha + m^*\,\tilde{\chi}^{\dot{\alpha}}} \qquad (I.9)$$

Take hermitian conjugate of (I.9) (or E.O.M : $0 = -\frac{\delta S}{\delta \chi}$):

$$0 = i (\bar{\sigma}^\mu)^{\alpha\dot{\alpha}} \partial_\mu \bar{\chi}_{\dot{\alpha}} - m \chi^\alpha$$

$\Big\downarrow$ raise and lower index

$$\boxed{0 = -i (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu \bar{\chi}^{\dot{\alpha}} + m \chi_\alpha} \qquad (I.10)$$

$(\sigma^\mu)_{\alpha\dot{\alpha}} := \varepsilon_{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}^\mu)^{\beta\dot{\beta}}$

$\sigma^\mu = (\mathbb{1}, \vec{\sigma}) \quad \sigma_\mu = (\mathbb{1}, -\vec{\sigma})$

Then (I.9) & (I.10) may be combined into one 4-compound eq:

$$0 = \begin{pmatrix} m\, \delta_\alpha^\gamma & -i (\sigma^\mu)_{\alpha\dot{\gamma}} \partial_\mu \\ -i (\bar{\sigma}^\mu)^{\dot{\alpha}\gamma} \partial_\mu & m^* \delta^{\dot{\alpha}}_{\dot{\gamma}} \end{pmatrix} \begin{pmatrix} \chi_\gamma \\ \bar{\chi}^{\dot{\gamma}} \end{pmatrix}$$

in the chiral rep.

Or using 4x4 Dirac matrices $\vee$ $\gamma^\mu := \begin{pmatrix} 0 & \sigma^\mu{}_{\alpha\dot{\gamma}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\gamma} & 0 \end{pmatrix}$ $\vee$

and setting $m = m^*$

$$\boxed{(-i \gamma^\mu \partial_\mu + m) \Psi = 0} \qquad \text{with} \qquad \Psi = \begin{pmatrix} \chi_\gamma \\ \bar{\chi}^{\dot{\gamma}} \end{pmatrix} \qquad (I.10)$$

Ψ is <u>MAJORANA</u> field $\left(\bar{\chi}^{\dot{\gamma}} = \varepsilon^{\dot{\gamma}\dot{\delta}} (\chi_\delta)^\dagger \right)$.

Introduce two independent Weyl spinors in Ψ to obtain <u>DIRAC</u> field

$$\Psi = \begin{pmatrix} \chi_\gamma \\ \bar{\xi}^{\dot{\gamma}} \end{pmatrix}$$

with 4 complex d.o.f.

DIRAC eq. $\boxed{(-i\gamma^\mu\partial_\mu + m)\psi = 0}$ follows from action

$$\boxed{\mathcal{L}_D = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi} \qquad (I.11)$$

with: $\bar{\psi} := \psi^\dagger\gamma^0$.

V: γ^μ obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

I.3 NON-ABELIAN GAUGE THEORIES

$\mathcal{L}_W$ of (I.8) and $\mathcal{L}_D$ of (I.11) are invariant under *global* $U(1)$ transformations

$$\chi_\alpha \to e^{-i\alpha}\chi_\alpha \quad \text{resp.} \quad \psi \to e^{-i\alpha}\psi$$

These can be turned into *local gauge* transformations $\alpha \to \alpha(x)$ upon introducing a $U(1)$ gauge field $A_\mu(x)$

Replace $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu(x)$ $\qquad$ e: coupling const. (charge of electron)

In particular

$$\boxed{\mathcal{L}_{QED} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \qquad (I.12)$$

Field strength tensor $\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{i}{e}[D_\mu . D_\nu]}$

$\mathcal{L}_{QED}$ is invariant under <u>local</u> gauge transformation $\alpha(x)$:

$$\psi \to e^{-ie\alpha(x)}\psi \qquad A_\mu \to A_\mu + \partial_\mu \alpha$$

<u>Formalizing</u>: Is a $\mathcal{U}(1)$ transformation $\mathcal{U}(x) = e^{i\alpha\alpha(x)}$

$\mathcal{U}^+(x)\mathcal{U}(x) = 1$

$$\psi \to \mathcal{U}(x)\psi \qquad D_\mu \to \mathcal{U}(x) D_\mu \mathcal{U}^+(x)$$ then $D_\mu\psi \to \mathcal{U} D_\mu \psi$

which implies

$$\boxed{A_\mu(x) \to \mathcal{U}(x) A_\mu(x) \mathcal{U}^+(x) + \frac{i}{e}\mathcal{U}(x)\partial_\mu \mathcal{U}^+(x)} \qquad (\text{I}.13)$$

<u>Abelian gauge symmetry</u>: $i\bar\psi \not{D}\psi$ & $\bar\psi\psi$ are invariant

<u>GENERALIZATION</u>: NON-ABELIAN GAUGE SYMMETRY.

Consider N spinor fields:

$$\chi_{i,\alpha} \quad \text{or} \quad \psi_{i,A} \qquad \alpha = 1,2 ; \quad A = (\alpha, \dot\alpha)$$

Then

$$\mathcal{L}_N = \sum_{i=1}^{N} (i\bar\psi_i \not{\partial}\psi_i - m\bar\psi_i\psi_i)$$

is invariant under <u>global</u> SU(N) or SO(N) transformations: $$\boxed{\psi_i(x) \to U_{ij}\,\psi_j(x)} \qquad (I.14)$$

SU(N): special-unitary: $U^\dagger U = U U^\dagger = \underline{\mathbb{1}}$

$\det U = 1$

Comment: Can consider also other Lie groups: SO(N), Sp(2N), G_2, F_4, E_6, E_7 & E_8.

In these lectures will focus exclusively on $\boxed{SU(N)}$.

Wish to lift this global sym. to a local sym.!

$U_{ij} \to U_{ij}(x)$

Problem: $\mathcal{L}_N$ no longer gauge invariant, needs non-abelian generalisation of D_μ.

Infinitesimal g. transformation: $U_{ij}(x) = \delta_{ij} - i g\, \theta^a(x) (T^a)_{ij}$

g: coupling constant,

$$\boxed{\begin{array}{ll} (T^a)_{ij} & a = 1, \ldots, N^2-1 \\ & i,j = 1, \ldots, N \\ \multicolumn{2}{l}{\text{"Generators" of SU(N) group.}} \end{array}}$$

$\theta^a(x)$ local transf. parameters $\Leftrightarrow$ $\alpha(x)$ for N=1

$(T^a)_{ij}$: Hermitian & traceless matrix (consequence of

$U^\dagger U = \mathbb{1}$ & det $U = 1$). Obey commutation relations:

$$\boxed{[T^a, T^b] = i\sqrt{2}\, f^{abc}\, T^c} \quad (I.15) \quad (\sqrt{2} \text{ normal. conv.})$$

f^{abc} : structure constants ; $f^{abc} = -f^{bac}$

Normalization: $$\boxed{\mathrm{tr}(T^a T^b) = \delta^{ab}} \quad (I.16)$$

Examples:

- ☐ SU(2): $T^a = \frac{1}{\sqrt{2}} \sigma^a$ σ^a: Pauli matrices $a = 1,2,3$
- ☐ SU(3): $T^a = \frac{1}{\sqrt{2}} \lambda^a$ λ^a: Gell-Mann matrices $a = 1,2,\dots,8$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \\ 1 & & \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & \\ i & & \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

Obey $\mathrm{tr}(\sigma^a \sigma^b) = \mathrm{tr}(\lambda^a \lambda^b) = 2\,\delta^{ab}$.

$-\frac{1}{N}\delta_{i_1 j_1}\delta_{i_2 j_2}$ ✓

SU(N) identity: $(T^a)_{i_1 j_1} (T^a)_{i_2 j_2} = \delta_{i_1 j_2} \delta_{i_2 j_1}$

rep of $(T^a)_{ij}$

☐ Explicit via 't Hooft twist matrices for example → (Lüscher, Nicolai de Wit).

We have

$$\boxed{f^{abc} = -\frac{i}{\sqrt{2}} \operatorname{Tr}\left(T^a [T^b, T^c]\right)} \qquad (I.17)$$

In generalization of the $U(1)$ transformation law (I.13) we now define an $SU(N)$ gauge field $A_\mu(x)$, a traceless, hermitian $N\times N$ matrix, with transformation law:

$$\boxed{A_\mu(x) \to \mathcal{U}(x) A_\mu(x) \mathcal{U}^\dagger(x) + \frac{i}{g} \mathcal{U}(x) \partial_\mu \mathcal{U}^\dagger(x)} \qquad (I.18)$$

with $\mathcal{U}(x) = \exp\left[-ig\, \theta^a(x)\, T^a\right]$ ($\theta^a(x)$ no longer infinitesimal)

☐ Covariant derivative: $$\boxed{(D_\mu)_{ij} = \delta_{ij}\partial_\mu - ig(A_\mu)_{ij}(x)} \qquad (I.19)$$

Transforms as $D_\mu \to \mathcal{U} D_\mu \mathcal{U}^\dagger$.

Then
$$(D_\mu \psi)_i = (D_\mu)_{ij}\psi_j$$
$$= \partial_\mu \psi_i - ig(A_\mu)_{ij}(x)\psi_j$$

transforms homogeneously

$$(D_\mu \psi)_i \to \mathcal{U}_{ij} (D_\mu \psi)_j$$

and the gauged action

$$\mathcal{L}'_N = \bar{\Psi}_i \not{D}_{ij} \Psi_j - m \bar{\Psi}_i \Psi_i$$

is invariant under local gauge transformations.

☐ Kinetic term for gauge field $A_\mu(x)$?

Define non-abelian field strength tensor $F_{\mu\nu}(x)$:

$$F_{\mu\nu} := \frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]$$

Transforms covariantly under gauge transf

$$F_{\mu\nu} \to \mathcal{U} F_{\mu\nu} \mathcal{U}^+$$

Hence

$$\mathcal{L}_{KIN} = -\frac{1}{4} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$$

is gauge and Lorentz invariant.

☐ Expansion in terms of generators T^a:

$$A_\mu(x) = A^a_\mu(x) T^a \iff A^a_\mu = \mathrm{Tr}(T^a A_\mu(x))$$

as $A_\mu(x)$ is hermitian, traceless $N \times N$ matrix.

Similarly: $F_{\mu\nu}(x) = F^a_{\mu\nu}(x) T^a$

and

$$F_{\mu\nu}(x) = \partial_\mu A_\nu^c T^c - \partial_\nu A_\mu^c T^c - i g A_\mu^a A_\nu^b \underbrace{[T^a, T^b]}_{i\sqrt{2} f^{abc} T^c}$$

$$= \left(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c + \sqrt{2} g f^{abc} A_\mu^a A_\nu^b\right) T^c$$

$$\Rightarrow \quad F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + \sqrt{2} g f^{abc} A_\mu^a A_\nu^b$$

and $\mathcal{L}_{KIN} = -\frac{1}{4}\left(F_{\mu\nu}^c F^{c\mu\nu}\right)$.

Specific example: Quantum Chromodynamics (QCD)

Gauge group $SU(3)$: 8 gluons $A_\mu^a(x)$ $a=1,\dots,8$

6 flavors of quarks $\psi_{i,I}$

$i=1,2,3$; $I=1,\dots,6$

(up, down, strange, charm, bottom, top)

$$\mathcal{L}_{QCD} = i \bar{\psi}_{i,I} \not{D}_{ij} \psi_{j,I} - m_I \bar{\psi}_{i,I} \psi_{i,I} - \frac{1}{4} F_{\mu\nu}^c F^{c\mu\nu}$$

$$m_I \approx \begin{array}{l} \{2, \; 5, \; 101, \; 1270, \; 4190, \; 172\,000\} \text{ MeV} \\ \;\; u \;\;\; d \;\;\;\; s \;\;\;\;\; c \;\;\;\;\;\; b \;\;\;\;\;\;\; t \end{array}$$

REPRESENTATIONS

Given structure constants f^{abc} of a non-abelian gauge group, can search for a representation of the group via $d_R \times d_R$ Matrices $(T^a_R)_{IJ}$ obeying: (d_R is dim. of representation)

$$[T^a_R, T^b_R] = i\sqrt{2}\, f^{abc}\, T^c_R \qquad (I.20)$$

So far have discussed the fundamental or defining representation of SU(N) in form of N×N hermitian, traceless matrices.

As f^{abc} is real, if T^a_R is rep then $T^a_{\bar{R}} := -T^{a\,*}_R$ (complex conjugate) is also a representation.

Important further representation is the adjoint representation:

$$(T^a_A)^{bc} = -i\sqrt{2}\, f^{abc}$$

$(T^a_{\bar{A}})^{bc} = (T^a_A)^{bc}$: the adjoint rep. is real.

Gauge fields transform in the adjoint rep.

$$\delta A^a_\mu = \partial_\mu \theta^a + g\sqrt{2}\, f^{abc}\, \theta^b A^c_\mu$$

$$= \partial_\mu \theta^a - \sqrt{2}\, g\, \theta^b (T^b_A)^{ac} A^c_\mu$$

whereas quarks transform in the <u>fundamental</u> rep.

$$\delta \psi_i = \theta^a (T^a_N)_{ij} \, \psi_j .$$

I.4 FEYNMAN-RULES OF NON-ABELIAN GAUGE THEORY

Consider pure Yang-Mills theory

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = \frac{1}{4} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b_\mu A^c_\nu \right)^2$$

$$= -\frac{1}{2} \partial^\mu A^a_\nu \, \partial_\mu A^{a\nu} + \frac{1}{2} \partial^\mu A^{a\nu} \, \partial_\nu A^a_\mu$$
$$- g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^c_\nu - \frac{1}{4} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu$$

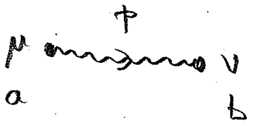
R_ξ - gauge fixing term

$$\mathcal{L}_{G.F.} = -\frac{1}{2} \xi^{-1} \left(\partial^\mu A^a_\mu \right)^2$$

$$\Rightarrow \mathcal{L}_{YM} + \mathcal{L}_{G.F} = \frac{1}{2} A^{a\mu} \left(\eta_{\mu\nu} \partial^2 - (1 - \xi^{-1}) \partial_\mu \partial_\nu \right) A^{a\nu}$$
$$- g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^c_\nu - \frac{1}{4} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu$$

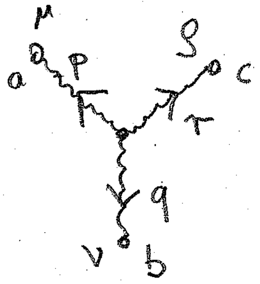
Plus ghost terms $\mathcal{L}_{GHOST}$ from the Fadeev-Popov gauge fixing procedure. These will not play a rôle for us.

Momentum space Feynman rules:

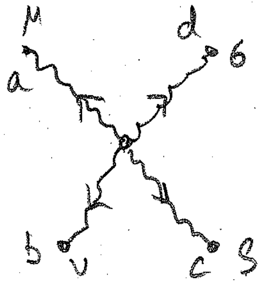


$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2+i\varepsilon}\left(\eta_{\mu\nu} + (\xi-1)\frac{k_\mu k_\nu}{k^2}\right)$$

will choose $\xi = 1$ (Feynman gauge) from now on.

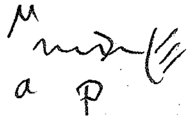


$$iV^{abc}_{\mu\nu\varrho}(p,q,r) = g f^{abc}\left[(q-r)_\mu \eta_{\nu\varrho} + (r-p)_\nu \eta_{\varrho\mu} + (p-q)_\varrho \eta_{\mu\nu}\right]$$

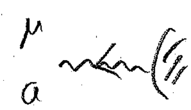


$$iV^{abcd}_{\mu\nu\varrho\sigma} = -i g^2\left[f^{abe}f^{cde}(\eta_{\mu\varrho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\varrho}) + f^{ace}f^{dbe}(\eta_{\mu\sigma}\eta_{\varrho\nu} - \eta_{\mu\nu}\eta_{\varrho\sigma}) + f^{ade}f^{bce}(\eta_{\mu\nu}\eta_{\sigma\varrho} - \eta_{\mu\varrho}\eta_{\sigma\nu})\right]$$

Polarization vectors: s: polarization

Ingoing: 

$\hat{=}\ \varepsilon^*_{s,\mu}(p)$ with $p\cdot\varepsilon^*_s(p)=0$, $p^2=0$

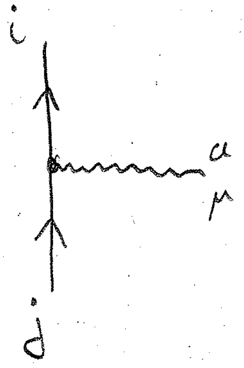
Outgoing: 

$\hat{=}\ \varepsilon_{s\mu}(p)$ with $p\cdot\varepsilon_s(p)=0$, $p^2=0$

Coupling to quarks:

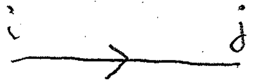
(Matter in fundamental representation)

$$\mathcal{L}_q = i \bar{\psi}_i \not{\partial}_{ij} \psi_j - m \bar{\psi}_i \psi_i + g A^a_\mu \bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j$$



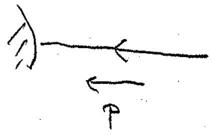
$$i V^{\mu a}_{ij} = i g \gamma^\mu T^a_{ij}$$

Quark propagator.

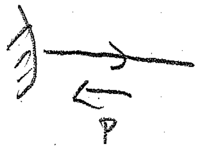


$$S_{ij}(p) = \frac{\not{p} - m}{p^2 - m^2 + i\varepsilon}$$

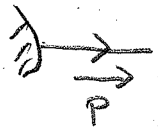
External lines:



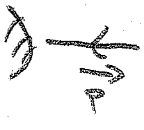
$u_s(p)$ incoming fermion with spin s



$v_s(p)$ incoming anti-fermion " " "



$\bar{u}_s(p)$ outgoing fermion with spin s



$\bar{v}_s(p)$ " anti-fermion with spin s

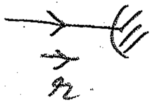
Massless particles : Helicity & Polarization

Spin projection on 3-momentum axis of massless particles is Lorentz invariant quantity : Helicity

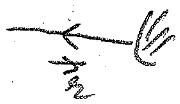
☒ Spin ½ fermions : (massless quarks) : helicity ±½

For $m=0$: $u_\pm(k) = v_\mp(k)$ polarizations

→ External fermion leg is labeled by momentum k with $k^2=0$ and helicity ±½



or



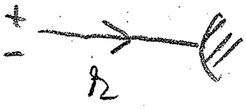
☒ Spin 1 gauge field (gluon): helicity : ±1

Polarization vector $\varepsilon_{\pm\mu}(k)$ obey

$$k^\mu \varepsilon_{\pm\mu}(k) = 0 \qquad \varepsilon_+\cdot\varepsilon_- = 0 \qquad \varepsilon_+\cdot\varepsilon_+ = 1 = \varepsilon_-\cdot\varepsilon_-$$

$$(\varepsilon_{\pm\mu})^* = \varepsilon_{\mp\mu}$$

→ External gluon leg carries momentum k with $k^2=0$ and helicity ±1



I.5 SPINOR HELICITY

Goal: Provide a uniform formulation of the kinematic data (momentum, polarization) of external scattering states (gluons, fermions, scalars).

☐ Write 4-momentum p^μ as bispinor ($\alpha=1,2$; $\dot\alpha=1,2$)

$$p^\mu \to p^{\alpha\dot\alpha} = \bar\sigma_\mu^{\alpha\dot\alpha}\, p^\mu \qquad \bar\sigma_\mu^{\alpha\dot\alpha} = (\mathbb{1}, \vec\sigma)$$

Then

$$p_i^{\alpha\dot\alpha}\, p_j^{\beta\dot\beta}\, \varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta} = \underbrace{\bar\sigma_\mu^{\alpha\dot\alpha}\,\bar\sigma_\nu^{\beta\dot\beta}\,\varepsilon_{\dot\alpha\dot\beta}\,\varepsilon_{\alpha\beta}}_{2\eta_{\mu\nu}}\, p_i^\mu\, p_j^\nu = 2\, p_i\cdot p_j$$

☐ Mass-shell condition:

$$p_i^2 = 0 \quad \Leftrightarrow \quad p_i\cdot p_i = \frac{1}{2}\, p_i^{\alpha\dot\alpha}\, p_i^{\beta\dot\beta}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta} = \det(p_i^{\alpha\dot\alpha}) = 0$$

$\Rightarrow$ 2x2 matrix $p_i^{\alpha\dot\alpha}$ should have rank 1: $\boxed{p_i^{\alpha\dot\alpha} = \lambda_i^\alpha\,\tilde\lambda_i^{\dot\alpha}}$ (I.21)

☒ Helicity spinors λ_i^α and $\tilde\lambda_i^{\dot\alpha}$ are commuting Weyl spinors in the $(\frac{1}{2},0)$ resp. $(0,\frac{1}{2})$ representation.

Reality of momentum translates into the condition $(\lambda_i^\alpha)^* = \pm\,\tilde\lambda_i^{\dot\alpha}$.

In fact sign is determined by sign of energy component p^0.

☒ From $\{\lambda_i, \hat{\lambda}_i\}$ we can build the Lorentz invariant quantities $i = 1, \ldots, n$:

$$\langle \lambda_i, \lambda_j \rangle := \lambda_i^{\alpha} \lambda_{j\alpha} = \varepsilon_{\alpha\beta} \lambda_i^{\alpha} \lambda_j^{\beta} = -\langle \lambda_j \lambda_i \rangle = \langle ij \rangle$$

for short

$$[\lambda_i, \lambda_j] := \tilde{\lambda}_{i\dot\alpha} \hat{\lambda}_j^{\dot\alpha} = -\varepsilon_{\dot\alpha\dot\beta} \lambda_i^{\dot\alpha} \hat{\lambda}_j^{\dot\beta} = -[\lambda_i, \lambda_j] = -[ij]$$

☒ Then the Mandelstam invariants may be written as

$$s_{ij} = (p_i + p_j)^2 = 2 p_i \cdot p_j = p_i^{\alpha\dot\alpha} p_{j\alpha\dot\alpha} = \langle ij \rangle [ji]$$

For positive energy states $p_i^0 > 0$, $p_j^0 > 0$ one shows

$$\langle ij \rangle = \sqrt{|s_{ij}|}\, e^{i\phi_{ij}} \qquad \cos\phi_{ij} = \frac{\lambda_i^1 \lambda_j^+ - \lambda_j^1 \lambda_i^+}{\sqrt{|s_{ij}|}\, \lambda_i^+ \lambda_j^+}$$

$$[ij] = -\sqrt{|s_{ij}|}\, e^{-i\phi_{ij}}$$

$$\sin\phi_{ij} = \frac{\lambda_i^2 \lambda_j^+ - \lambda_j^2 \lambda_i^+}{\sqrt{|s_{ij}|}\, \lambda_i^+ \lambda_j^+}$$

$$\lambda^{\pm} = \lambda^0 \pm \lambda^3 \quad \to \underline{Ex}$$

☒ __Note__: λ_i^{α} & $\hat{\lambda}_i^{\dot\alpha}$ are not completely fixed by choice of momentum $p_i^{\alpha\dot\alpha}$ in (I.21) as the rescalings

$$\lambda_i^{\alpha} \to e^{-i\varphi_i} \lambda_i^{\alpha} \quad ; \quad \hat{\lambda}_i^{\dot\alpha} \to e^{+i\varphi_i} \hat{\lambda}_i^{\dot\alpha}$$

leave p_i invariant.

In fact these rescalings are generated by an $\mathcal{U}(1)$ generator

$$h = \frac{1}{2}\sum_{i=1}^{n}\left[-\lambda_i^\alpha \frac{\partial}{\partial\lambda_i^\alpha} + \tilde\lambda_i^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}\right] \qquad (I.22)$$

Convention: λ helicity $-1/2$; $\tilde\lambda$ helicity $1/2$.

Thus external helicity spinor states carry information on both: momentum & helicity of external leg!

Helicity spinors solve the massless Dirac equation:

(I.9): $(\bar\sigma^\mu)^{\dot\alpha\alpha} p_\mu \lambda_\alpha = \lambda^\alpha\tilde\lambda^{\dot\alpha}\cdot\lambda_\alpha = \tilde\lambda^{\dot\alpha}\langle\lambda\lambda\rangle = 0$

(I.10) $(\sigma^\mu)_{\alpha\dot\alpha} p_\mu \tilde\lambda^{\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}\tilde\lambda^{\dot\alpha} = \lambda_\alpha[\lambda\lambda] = 0$

Hence f. external fermion states (all momenta outgoing)

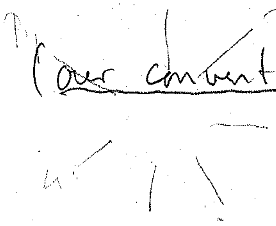
Identities

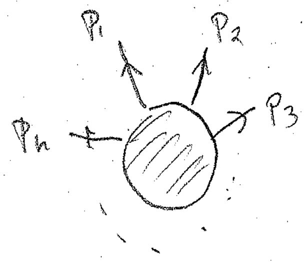
i) Schouten: $\langle\lambda_1\lambda_2\rangle\lambda_3^\alpha + \langle\lambda_2\lambda_3\rangle\lambda_1^\alpha + \langle\lambda_3\lambda_1\rangle\lambda_2^\alpha = 0$

or $\langle 12\rangle\langle 3n\rangle + \langle 23\rangle\langle 1n\rangle + \langle 31\rangle\langle 2n\rangle = 0$

ii) Total momentum conservation in scattering amplitudes

(our convention: All momenta are outgoing)





external legs of any massless particle (fermion, gluon, scalar):

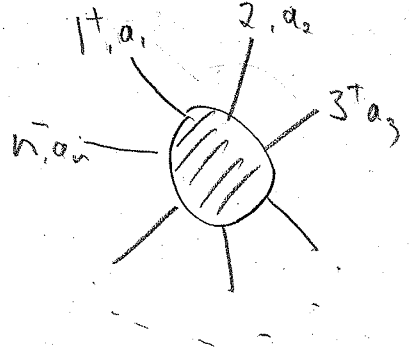
$$\sum_{i=1}^{n} p_i^\mu = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{n} \lambda_i^\alpha \tilde{\lambda}_i^{\dot\alpha} = 0 \quad \Leftrightarrow \quad \sum_i \langle a | i \rangle [i | b] = 0$$

for any λ_a & $\tilde{\lambda}_b$.

GLUON POLARIZATIONS

Next step is to introduce a bispinor representation for the polarization vector of a massless gauge boson of definite helicity ±1:

Gluon amplitudes depend on momenta, helicities and color of external states: e.g



$$A_n = \varepsilon_+^{\mu_1} \varepsilon_-^{\mu_2} \varepsilon_+^{\mu_3} \cdots \varepsilon_-^{\mu_n} \, A^{a_1 \dots a_n}_{\mu_1 \dots \mu_n}(p_1 \dots p_n)$$

Polarization vectors $\varepsilon_\pm^\mu$ in terms of helicity spinors:

$$\varepsilon_{+,i}^{\alpha\dot\alpha} = -\sqrt{2}\, \frac{\tilde{\lambda}_i^{\dot\alpha} \mu_i^\alpha}{\langle \lambda_i \mu_i \rangle}$$

$$\varepsilon_{-,i}^{\alpha\dot\alpha} = \sqrt{2}\, \frac{\lambda_i^\alpha \tilde{\mu}_i^{\dot\alpha}}{[\tilde{\lambda}_i \tilde{\mu}_i]}$$

with arbitrary reference spinors μ_i^α & $\hat{\mu}_i^{\dot\alpha}$.

Easily shown, that this choice obeys the desired properties:

1) $$p\cdot\varepsilon_\pm = \frac{1}{2}\lambda_\alpha\tilde\lambda_{\dot\alpha}\,\varepsilon_\pm^{\alpha\dot\alpha} = +\frac{1}{\sqrt{2}}\begin{cases}[\tilde\lambda\tilde\lambda]\\ \langle\lambda\lambda\rangle\end{cases} = 0$$

2) $$(\varepsilon_+)^* = \varepsilon_-$$

3) $$\varepsilon_+\cdot(\varepsilon_+)^* = \varepsilon_+\cdot\varepsilon_- = -\frac{2}{2}\,\frac{\tilde\lambda^{\dot\alpha}\mu^\alpha\lambda_\alpha\hat\mu_{\dot\alpha}}{\langle\lambda\mu\rangle[\tilde\lambda\hat\mu]} = -\frac{\langle\mu\lambda\rangle[\hat\mu\tilde\lambda]}{\langle\lambda\mu\rangle[\tilde\lambda\hat\mu]}$$

$$= \underline{\underline{-1}}$$

4) $$\varepsilon_+\cdot(\varepsilon_-)^* = \varepsilon_+\cdot\varepsilon_+ = \frac{2}{2}\,\frac{\tilde\lambda^{\dot\alpha}\mu^\alpha\tilde\lambda_{\dot\alpha}\mu_\alpha}{\langle\lambda\mu\rangle^2} = 0.$$

- Freedom of choosing reference momenta $\xi_i^{\alpha\dot\alpha} = \mu_i^\alpha\hat\mu_i^{\dot\alpha}$ corresponds to a gauge transformation:

Shift $\mu \to \mu + \delta\mu$

$$\delta\varepsilon_+^{\alpha\dot\alpha} = -\sqrt{2}\left(\frac{\tilde\lambda^{\dot\alpha}\delta\mu^\alpha}{\langle\lambda\mu\rangle} - \tilde\lambda^{\dot\alpha}\mu^{\alpha}\frac{\langle\lambda\,\delta\mu\rangle}{\langle\lambda\mu\rangle^2}\right)$$

$$= -\sqrt{2}\,\frac{1}{\langle\lambda\mu\rangle^2}\,\tilde\lambda^{\dot\alpha}\left(\delta\mu^\alpha\langle\lambda\mu\rangle - \mu^\alpha\langle\lambda\,\delta\mu\rangle\right)$$

$$\underset{\text{Schouten id.}}{=} -\sqrt{2}\,\lambda^\alpha\tilde\lambda^{\dot\alpha}\,\frac{\langle\mu\,\delta\mu\rangle}{\langle\lambda\mu\rangle^2} \sim p^{\alpha\dot\alpha}$$

Hence $\delta\varepsilon_+^\mu(p)\,A^{a a_1\dots a_{n-1}}_{\mu\nu_1\dots\nu_{n-1}}(p, q_1, \dots q_{n-1}) \sim \langle p^\mu A_\mu(p)\dots\rangle = 0$

→ Amplitudes depend on $\{\lambda_i, \tilde{\lambda}_i, \pm_i\}$ only!

FERMION POLARIZATIONS

We saw in exercise 1.2 that in the chiral representation of the Dirac matrices (multiplied by -1):

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ +\sigma^i & 0 \end{pmatrix}$$

the polarization of massless fermions read

$$u_+(k) = v_-(k) = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix} = |k\rangle_\alpha \qquad u_-(k) = v_+(k) = \begin{pmatrix} 0 \\ \tilde{\lambda}^{\dot\alpha} \end{pmatrix} = {}^{\dot\alpha}|k]$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu{}_{\alpha\dot\gamma} \\ \bar{\sigma}^\mu{}_{\dot\gamma\alpha} & 0 \end{pmatrix} \qquad \begin{aligned} (\sigma^\mu)_{\alpha\dot\gamma} &= (\mathbb{1}, -\vec{\sigma}) \\ (\bar{\sigma}^\mu)_{\dot\gamma\dot\alpha} &= (\mathbb{1}, \vec{\sigma}) \end{aligned}$$

$$\bar{u}_+(k) = \bar{v}_-(k) = (\lambda^*_\alpha \;\; 0) \begin{pmatrix} 0 & +\mathbb{1} \\ +\mathbb{1} & 0 \end{pmatrix} = (0 \;\; \tilde{\lambda}_{\dot\alpha}) = [k|_{\dot\alpha}$$

$$\bar{u}_-(k) = \bar{v}_+(k) = (0 \;\; \tilde{\lambda}^{\dot\alpha *}) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (\lambda^\alpha \;\; 0) = \langle k|^\alpha$$

One immediately sees that

$$\langle k|\gamma^\mu|p\rangle = 0 \quad \text{and} \quad [k|\gamma^\mu|p] = 0 \qquad (\text{I.24})$$

as well as

$$\langle p|\gamma^\mu|p] = \lambda^\alpha \sigma^\mu_{\alpha\dot\gamma} \tilde{\lambda}^{\dot\gamma} = 2p^\mu$$

$$[p|\gamma^\mu|p\rangle = 2p^\mu .$$

Fierz rearrangement: (check)

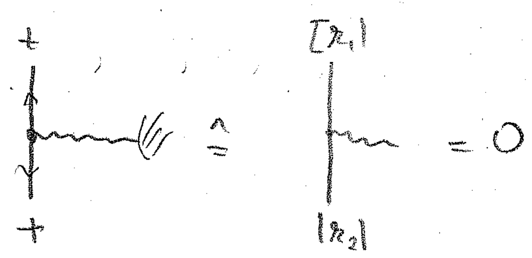
$$[i|\gamma^\mu|j\rangle\,[k|\gamma_\mu|l\rangle = 2\,[ik]\,\langle lj\rangle .$$

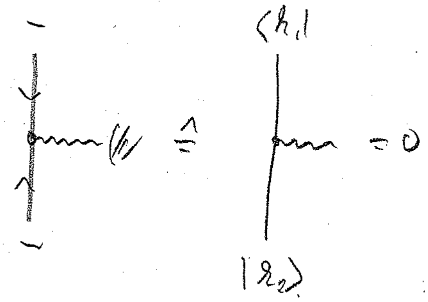
Our convention: All external 4-momenta are outgoing:

In this convention $|r\rangle$ & $\langle r|$ represent $-\frac{1}{2}$ helicity states

$|p]$ & $[p|$ " $+\frac{1}{2}$ helicity states.

Hence: External fermions meeting at a vertex need to have opposite helicity.





due to (I.24).

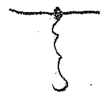
Exercise: Compute $e^+e^- \to \gamma\gamma$ & $e^+e^- \to e^+e^-$ in QED using the spinor helicity formalism

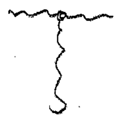
I.6 COLOR DECOMPOSITION

Goal: Disentangle the color degrees of freedom from the kinematical ones.

$SU(N_c)$ gauge theory: Color dependence of Feynman graphs via

$$(T^a)_i{}^j \quad \text{and} \quad f^{abc} \quad , \quad \delta^{ab} \quad \& \quad f^{abe} f^{cde}$$





Using

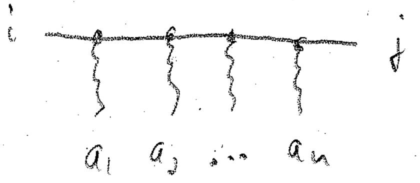
$$\boxed{f^{abc} = -\frac{i}{\sqrt{2}}\, \mathrm{Tr}\left(T^a \left[T^b, T^c\right]\right)} \qquad (I.15)$$

a generic graph depends on a large number of traces many sharing generators T^a with contracted indices such as

$$\mathrm{Tr}(\dots T^a \dots)\ \mathrm{Tr}(\dots T^a \dots)$$

External quark lines lead to strings of T^a's with open fundamental indices like $(T^{a_1} T^{a_2} \dots T^{a_n})_i{}^j$

$$\Downarrow$$



SU(N_c) identity:

$$\boxed{(T^a)_{i_1}^{\ \ j_1}(T^a)_{i_2}^{\ \ j_2} = \delta_{i_1}^{\ \ j_2}\delta_{i_2}^{\ \ j_1} - \frac{1}{N_c}\delta_{i_1}^{\ \ j_1}\delta_{i_2}^{\ \ j_2}} \qquad (I.25)$$

Proof: Consider $U(N_c) = SU(N_c)\times U(1)$ by augmenting the N_c^2-1 $(T^a)_i^{\ j}$ with $(T^\emptyset)_i^{\ j} = \frac{1}{\sqrt{N_c}}\delta_i^{\ j}$ proportional to the unit matrix.

Obviously $f^{\emptyset ab} = f^{a\emptyset b} = f^{ab\emptyset} = 0$

(Photon decoupling).

The $\{T^\emptyset, T^a\}$ form a complete set of hermitian $N\times N$ matrices and

$$\sum_{A=\{\emptyset,a\}}(T^A)_{i_1}^{\ \ j_1}(T^A)_{i_2}^{\ \ j_2} = \delta_{i_1}^{\ \ j_2}\delta_{i_2}^{\ \ j_1}$$

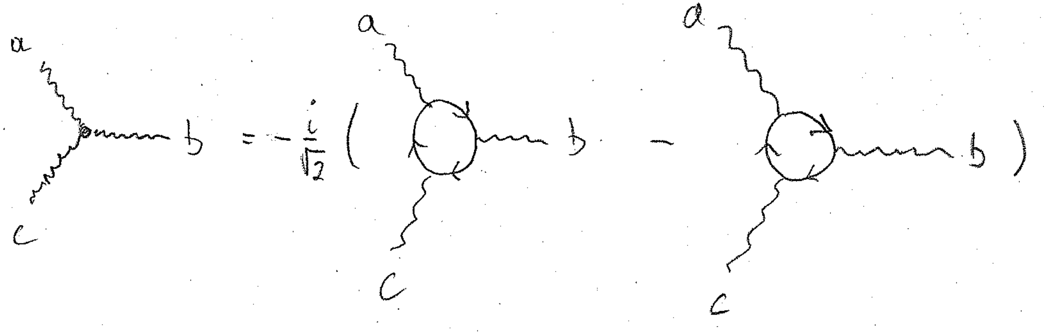
is the completeness relation taking into account the normalization condition

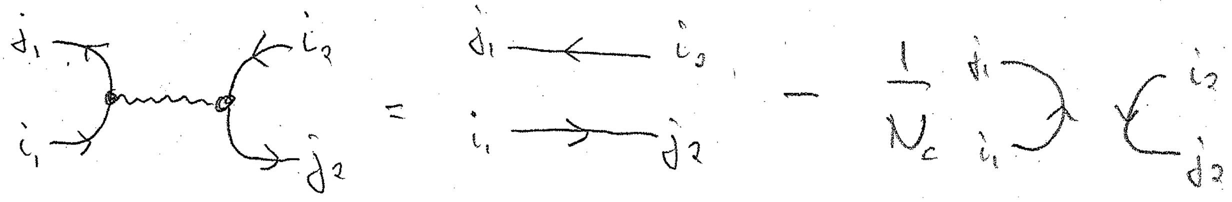
$$\mathrm{Tr}(T^A T^B) = \delta^{AB}$$

Comment: For pure gluon amplitudes the $SU(N_c)$ and $U(N_c)$ theories are identical, as the $U(1)$ photon part decouples. Not true for external quark amplitudes or gluon loop amplitudes when quarks are present in the loop.

Graphical representations of (I.15) and (I.25):

f^{abc} :





☒ Then any pure gluon tree diagram reduces to a single trace structure and takes the <u>color decomposed</u> form.

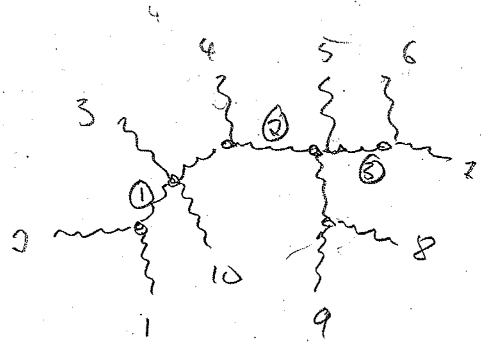
$$\boxed{\mathcal{A}^{\text{tree}}_{n\text{-gluon}}\left(\{a_i, h_i, p_i\}\right) = g^{n-2} \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}\left(T^{a_{\sigma_1}} T^{a_{\sigma_2}} \dots T^{a_{\sigma_n}}\right) \quad A^{\text{tree}}_n\left(p_{\sigma_1}, h_{\sigma_1}; p_{\sigma_2}, h_{\sigma_2}; \dots ; p_{\sigma_n}, h_{\sigma_n}\right)} \qquad (\text{I.26})$$

S_n/Z_n : Set of all <u>non cyclic</u> permutations of $(1, 2, \dots, n)$.

Is equivalent to S_{n-1}.

$A_n^{\text{tree}}(\{p_i, h_i\})$: partial or color-ordered amplitudes

Partial amplitudes are simpler than full amplitudes $\mathcal{A}_n^{\text{tree}}$ as they only receive contributions from a fixed cyclic ordering of gluons. In particular the poles of A_n^{tree} can only arise in channels of cyclically adjacent momenta $(p_i + p_{i+1} + \dots + p_{i+s})^2$:



Possible poles e.g. :

① : $(p_1+p_2)^2$

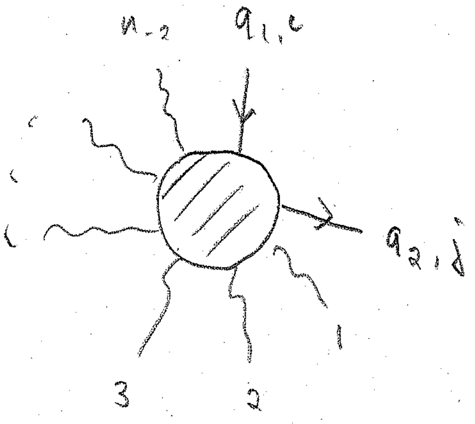
② : $(p_{10}+p_1+\dots+p_4)^2$

③ $(p_6+p_7)^2$

▨ For $(g^n q\bar{q})$-amplitudes at tree-level one has

$$\mathcal{A}^{\text{tree}}_{g^n q\bar{q}} = g^{n-2} \sum_{\sigma \in S_{n-2}} \left(T^{a_{\sigma_1}} T^{a_{\sigma_2}} \dots T^{a_{\sigma_{n-2}}}\right)_i^{\ j} \tag{I.27}$$

$$A_n^{\text{tree}}(p_{\sigma_1}, h_{\sigma_1}; \dots; p_{\sigma_{n-2}}, h_{\sigma_{n-2}}; q_1, h_{q_1}; q_2, h_{q_2})$$



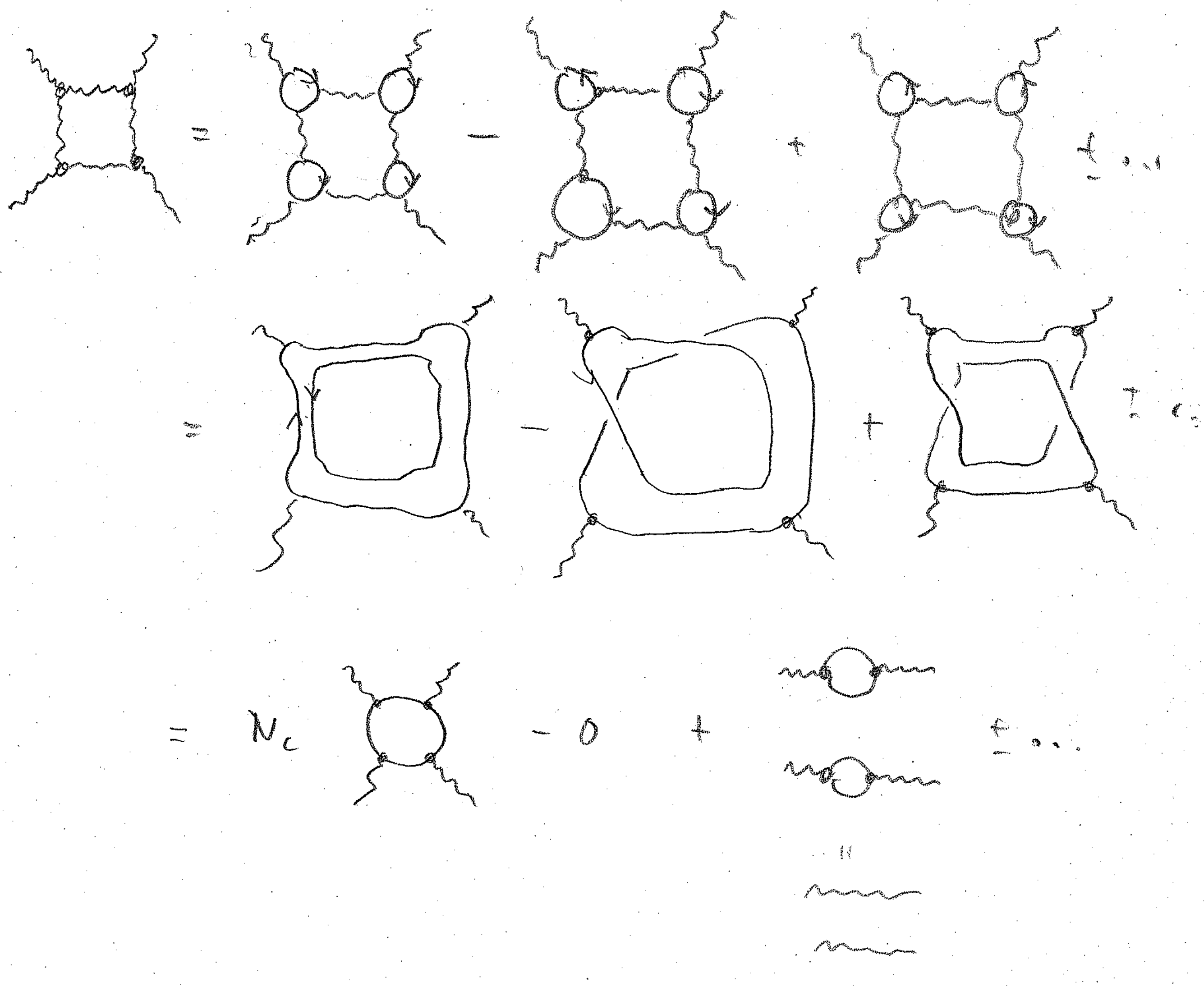
At the loop level higher powers of traces emerge. E.g. one loop gluon amplitude:

$$\mathcal{A}^{1\text{-loop}}_{g^n} = g^n \Big\{ N_c \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma_1}}\dots T^{a_{\sigma_n}})\, A_n^{1\text{-loop}}(\sigma_1^{h_{\sigma_1}},\dots,\sigma_n^{h_{\sigma_n}})$$

$$+ \sum_{c=2}^{[n/2]+1} \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma_1}}\dots T^{a_{\sigma_{c-1}}})\, \mathrm{Tr}(T^{a_{\sigma_c}}\dots T^{a_{\sigma_n}})$$

$$A_{n;c}^{1\text{-loop}}(\sigma_1^{h_{\sigma_1}}\dots\sigma_n^{h_{\sigma_n}}) \qquad (\mathrm{I}.28)$$

as can be seen from the 4-gluon graph:



GENERAL PROPERTIES OF COLOR ORDERED AMPLITUDES

- CYCLICITY: $A(1,2,\dots,n) = A(2,3,\dots,n,1)$

- PARITY: Reverse all helicities in A:

$$A(1,2,\dots,n) = A(\bar{1},\bar{2},\dots,\bar{n})$$

- CHARGE CONJUGATION: Flip helicity on a quark line:

$$A(1_q,2_{\bar{q}},3,\dots,n) = A(1_{\bar{q}},2_q,3,\dots,n)$$

$$= A(\bar{1}_q,\bar{2}_{\bar{q}},3,\dots,n)$$

- REFLECTION:

$$\overset{\text{tree}}{A}(1,2,\dots,n) = (-)^n \overset{\text{tree}}{A}(n,n-1,\dots,1)$$

Follows from a/sym of color ordered Feynman rules (→ to be discussed). Also holds in the presence of quark lines

- PHOTON DECOUPLING:

For pure gluon trees:

$$A(1,2,\dots,n) + A(2,1,3,\dots,n) + A(2,3,1,4,\dots,n)$$

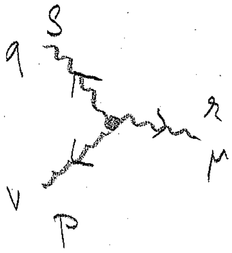
$$+ \dots A(2,3,\dots,n-1,1,n) = 0$$

Follows from (I.26): Pure glue amplitude containing a U(1) photon must vanish. Take (I.26) with one U(1) piece and collect all color trace terms of identical structure.

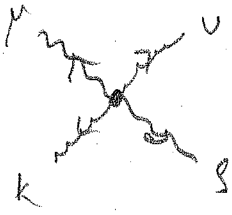
Ex: Use these rules to determine the independent set of partial amplitudes for 4 & 5 gluon scattering.

COLOR ORDERED FEYNMAN RULES

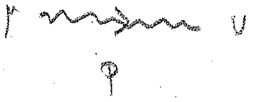
May be established and used to evaluate partial amplitudes:



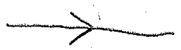
$$\hat{=} -\frac{i}{\sqrt{2}}\left[\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-r)_\nu + \eta_{\mu\nu}(r-p)_\rho\right]$$



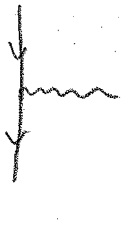
$$\hat{=} \frac{i}{\sqrt{2}}\left[2\eta_{\mu\rho}\eta_{\nu\kappa} - \eta_{\mu\nu}\eta_{\rho\kappa} - \eta_{\mu\kappa}\eta_{\nu\rho}\right]$$



$$\hat{=} -\frac{i}{p^2}\eta_{\mu\nu}$$

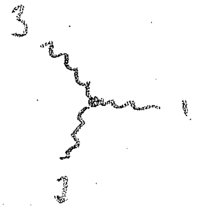


$$\hat{=} \frac{i}{\not p}$$



$$\hat{=} -\frac{i}{\sqrt{2}}\gamma^\mu$$

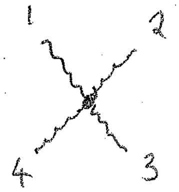
Compact notation of vertices with polarizations attached:



$$\hat{=}\ -\frac{i}{\sqrt{2}}\Big[(\varepsilon_1\cdot\varepsilon_2)(p_{12}\cdot\varepsilon_3)+(\varepsilon_2\cdot\varepsilon_3)(p_{23}\cdot\varepsilon_1)+(\varepsilon_3\cdot\varepsilon_1)(p_{31}\cdot\varepsilon_2)\Big]$$

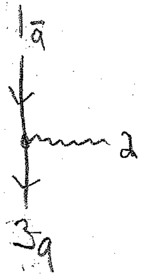
$$p_{ij}=p_i-p_j$$

Obeys $A(1,2,3) = -A(3,2,1)$ and cyclicity.

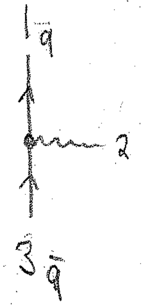


$$\hat{=}\ \frac{i}{\sqrt{2}}\Big[2(\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4)-(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_4)-(\varepsilon_1\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3)\Big]$$

Obeys $A(1,2,34) = +A(4,3,2,1)$ and cyclicity.

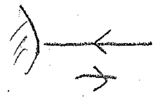


$$\hat{=}\ -\frac{i}{\sqrt{2}}[3|\not{\varepsilon}_2|1\rangle$$

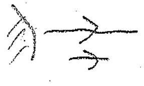


$$\hat{=}\ -\frac{i}{\sqrt{2}}[1|\not{\varepsilon}_2|3\rangle$$

Read fermion arrows backwards. Remember all momenta are outgoing:



$\hat{=}\ \lambda$



$\hat{=}\ \tilde{\lambda}$

I.7 VANISHING TREE-AMPLITUDES

Our freedom to choose arbitrary reference momentum $q = \mu\bar{\mu}$ the polarizations $\varepsilon_{\pm,i}$ may be used to show the vanishing of large classes of tree amplitudes.

From (I.23) we have the products:

$$\varepsilon_{+,i} \cdot \varepsilon_{+,j} = \frac{\langle \mu_i \mu_j \rangle [\lambda_j \lambda_i]}{\langle \lambda_i \mu_i \rangle \langle \lambda_j \mu_j \rangle}$$

N.B: Only restrictions are

$\mu_i \neq \lambda_i$

$\bar{\mu}_i \neq \tilde{\lambda}_i$

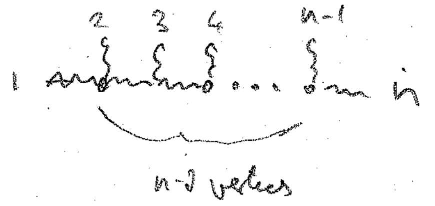
$$\varepsilon_{+,i} \cdot \varepsilon_{-,j} = - \frac{\langle \mu_i \lambda_j \rangle [\mu_j \lambda_i]}{\langle \lambda_i \mu_i \rangle [\lambda_j \mu_j]}$$

$$\varepsilon_{-,i} \cdot \varepsilon_{-,j} = \frac{\langle \lambda_i \lambda_j \rangle [\mu_j \mu_i]}{[\lambda_i \mu_i] [\lambda_j \mu_j]}$$

Uniform choice $\underline{q = q_1 = \ldots = q_n}$ with $q^2 = 0$

yields $\boxed{\varepsilon_{+i} \cdot \varepsilon_{+j} = 0 = \varepsilon_{-i} \cdot \varepsilon_{-j}}$ $\forall i, j$.

n-gluon tree amplitude consists of maximally $(n-2)$ vertices:



$\Rightarrow$ Any graph has at least <u>one</u> $\varepsilon_i \cdot \varepsilon_j$ contraction.

Thm: (p_i: external momenta; q_i: reference momenta)

i) $\boxed{A_n^{\text{tree}}(1^+, 2^+, \ldots, n^+) = 0}$ with choice (I.29)

$q_1 = q_2 = \ldots = q_n = q$ arbitrary

with $q \neq p_i \ \forall i$.

as all $\varepsilon_{+i} \cdot \varepsilon_{+j} = 0$

ii) $\boxed{A_n^{\text{tre}}(1^-, 2^+, \ldots, n^+) = 0}$ with choice (I.30)

$q_1 = q \neq p_1$

$q_2 = q_3 = \ldots = q_n = p_1$

as all $\varepsilon_{+i} \cdot \varepsilon_{+j} = 0$ $(i,j \in \{2, \ldots, n\})$

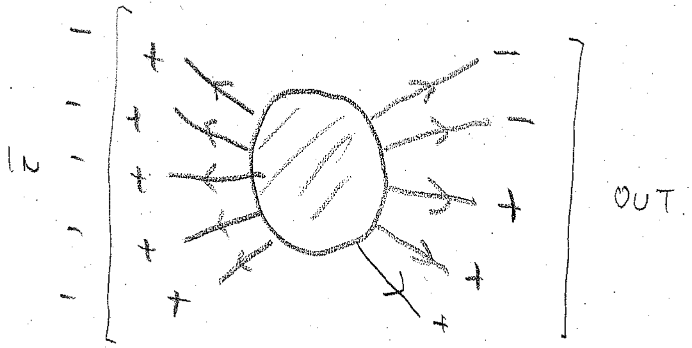
and $\varepsilon_{+i} \cdot \varepsilon_{-1} = 0$

By parity this implies $\boxed{A_n^{\text{tree}}(1^\pm, 2^-, \ldots, n^-) = 0}$ (I.31)

Hence the first non-trivial pure gluon tree amplitudes are the

$$A_n^{\text{tree}}(1^-, 2^-, 3^+, \ldots, n^+)$$

"maximally helicity violating" amplitudes



iii) Similarly the $\bar{q}qg^{n-2}$-amplitude

$$\boxed{A_n^{\text{tree}}(1^-_{\bar{q}}, 2^+_q, 3^+, 4^+, \ldots, n^+) = 0} \qquad (\text{I}.32)$$

vanishes.

Now there ist at least one contraction

$$[2|\not{\varepsilon}_{+i}|1\rangle = \tilde{\lambda}_{2\dot{\alpha}}\, \varepsilon_{+i}^{\dot{\alpha}\alpha}\, \lambda_{1\alpha} \qquad i \in \{3,\ldots,n\}$$

in the amplitude. Choosing the reference momenta of the gluons $q_i = \mu_i \tilde{\mu}_i = \lambda_1 \tilde{\lambda}_1 \ \forall i \in \{3,\ldots,n)$ sets $[2|\not{\varepsilon}_{+i}|1\rangle = 0$

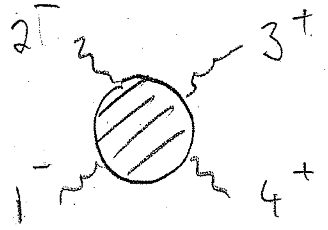
Similarly $$\boxed{A_n^{\text{tree}}(1^-_{\bar{q}}, 2^+_q, 3^-, \ldots, n^-) = 0} \qquad (\text{I}.33)$$

by choosing $q_i = \mu_i \tilde{\mu}_i = \lambda_2 \tilde{\lambda}_2 \ \forall i \in \{3,\ldots,n)$.

The vanishing of the amplitudes (I.29-33) actually derives from a *hidden* *supersymmetry* in gluon-quark tree amplitudes, which will be discussed later on.

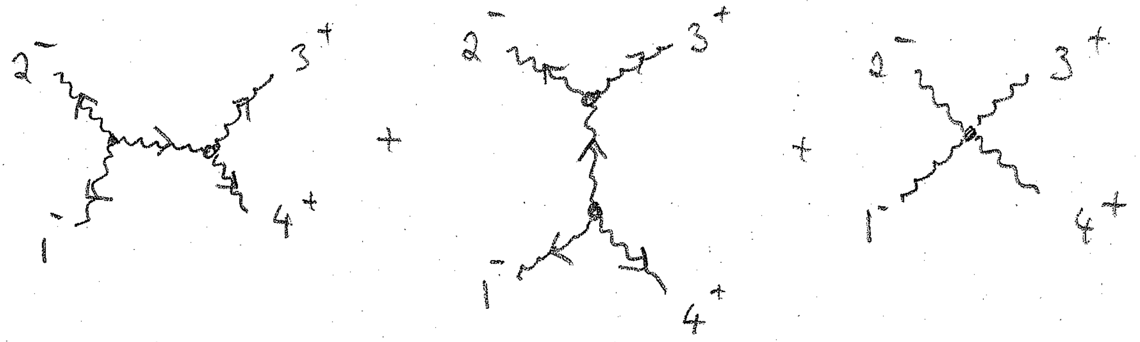
I.8 4-GLUON TREE-AMPLITUDE

Simplest non-trivial gluon amplitude:



$$= A(1^-, 2^-, 3^+, 4^+)$$

Diagrams from color-ordered Feynman rules:



Choose reference momenta:

$q_1 = q_2 = p_4$ & $q_3 = q_4 = p_1$

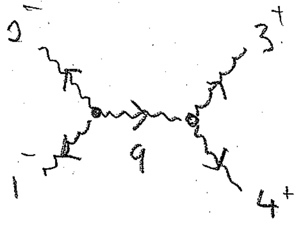
Then $\varepsilon_1^- \cdot \varepsilon_2^- = 0 = \varepsilon_3^+ \cdot \varepsilon_4^+$ & $\varepsilon_1^- \cdot \varepsilon_4^+ = 0 = \varepsilon_2^- \cdot \varepsilon_4^+$

$$\varepsilon_1^- \cdot \varepsilon_3^+ = \varepsilon_1^- \cdot \varepsilon_4^+ = 0$$

only $\varepsilon_2^- \cdot \varepsilon_3^+ = - \dfrac{\langle \mu_3 \lambda_2 \rangle [\mu_2 \lambda_3]}{\langle \lambda_3 \mu_3 \rangle [\lambda_2 \mu_2]}$

$$= - \frac{\langle 12 \rangle [34]}{\langle 13 \rangle [24]}$$

is nonvanishing.



$$= \left(-\frac{i}{\sqrt{2}}\right)^2 \frac{-i}{s_{12}} \left[\varepsilon_2^\mu (p_{2q}\cdot\varepsilon_1) + \varepsilon_1^\mu (p_{q1}\cdot\varepsilon_2) \right]$$

$$q = -p_1 - p_2$$
$$= p_3 + p_4$$

$$\left[\varepsilon_4^\mu (p_{4-q}\cdot\varepsilon_3) + \varepsilon_3^\mu (p_{-q3}\cdot\varepsilon_4) \right]$$

$$= \frac{i}{2s_{12}} (\varepsilon_2\cdot\varepsilon_3)\ (p_2 + p_1 + p_2)\cdot\varepsilon_1\ (-p_3 - p_4 - p_3)\cdot\varepsilon_4$$

$$= -\frac{2i}{s_{12}} (\varepsilon_2^-\cdot\varepsilon_3^+)(p_2\cdot\varepsilon_1^-)(p_3\cdot\varepsilon_4^+)$$

$$= -\frac{2i}{s_{12}} \left(-\frac{\langle 12\rangle[34]}{\langle 13\rangle[24]}\right)\left(\frac{1}{\sqrt{2}}\frac{\langle 12\rangle[24]}{[14]}\right)\left(\frac{1}{\sqrt{2}}\frac{\langle 13\rangle[34]}{\langle 41\rangle}\right)$$

$$\underline{= -i\ \frac{\langle 12\rangle[34]^2}{[12]\langle 14\rangle[41]}}$$

$$p_2\cdot\varepsilon_1^- = \frac{1}{2}\lambda_{2\alpha}\tilde{\lambda}_{2\dot{\alpha}}\,\varepsilon_1^{-\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}}\frac{\langle 12\rangle[2\mu_1]}{[1\mu_1]} \underset{\mu_1=4}{=} \frac{1}{\sqrt{2}} = \frac{\langle 12\rangle[24]}{[14]}$$

$$p_3\cdot\varepsilon_4^+ = \frac{1}{2}\lambda_{3\alpha}\hat{\lambda}_{3\dot{\alpha}}\,\varepsilon_4^{+\alpha\dot{\alpha}} = -\frac{1}{\sqrt{2}}\frac{\langle 3\mu_4\rangle[34]}{\langle 4\mu_4\rangle} \underset{\mu_4=1}{=} -\frac{1}{\sqrt{2}}\cdot\frac{\langle 31\rangle[34]}{\langle 41\rangle}$$

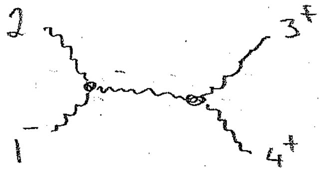
$$s_{12} = \langle 12\rangle[21]$$

Simplify: $s_{12} = \langle 12\rangle[21] = s_{34} = \langle 34\rangle[43]$

$$-i\,\frac{\langle 12\rangle[34]^2}{[12]\,\langle 14\rangle[41]}\cdot\frac{\langle 43\rangle}{\langle 43\rangle} = i\,\frac{\langle 12\rangle\,[34]\,\overbrace{[34]\langle 43\rangle}^{\langle 12\rangle[21]}}{[12]\,\langle 14\rangle[41]\,\langle 43\rangle}$$

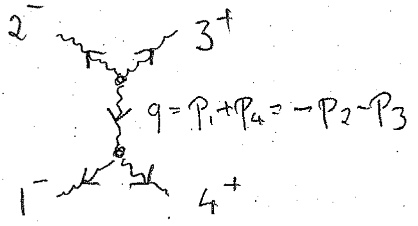
$$= -i\,\frac{\langle 12\rangle^2\;[34]}{\underbrace{\langle 14\rangle\langle 43\rangle[41]}_{-\langle 34\rangle[41] = \langle 32\rangle[21]}} = i\,\frac{\langle 12\rangle^2}{\langle 23\rangle\,\langle 41\rangle}\;\underbrace{\frac{[34]}{[21]}}_{\frac{\langle 12\rangle}{\langle 43\rangle}} = -i\,\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

Hence:



$$= -i\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

Graph II:



$$= \left(-\frac{i}{\sqrt{2}}\right)^2\frac{i}{s_{23}}\Big((\varepsilon_2\cdot\varepsilon_3)(p_{23}\cdot\varepsilon_q) + (\varepsilon_3\cdot\varepsilon_q)(p_{3q}\cdot\varepsilon_2) + (\varepsilon_q\cdot\varepsilon_2)(p_{q2}\cdot\varepsilon_3)\Big)$$

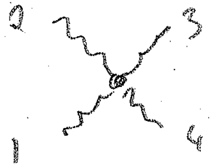
$$\Big(\underbrace{(\varepsilon_4\cdot\varepsilon_1)}_{=0}(p_{41}\cdot\varepsilon_q) + (\varepsilon_1\cdot\varepsilon_q)\,(p_{1-q}\cdot\varepsilon_4) + (\varepsilon_q\cdot\varepsilon_4)(p_{-q,4}\cdot\varepsilon_1)\Big)$$

$$= \left(\frac{+i}{2\,s_{23}}\right)\Big[(\varepsilon_2\cdot\varepsilon_3)\Big[(p_{23}\cdot\varepsilon_1)\underbrace{(p_{1-q}\cdot\varepsilon_4)}_{(p_1+p_1+p_4)\cdot\varepsilon_4 = 2p_1\cdot\varepsilon_4} + (p_{23}\cdot\varepsilon_4)\underbrace{(p_{-q,4}\cdot\varepsilon_1)}_{(-p_1-p_4-p_4)\cdot\varepsilon_1 = -2p_4\cdot\varepsilon_1}\Big]$$

$$= \frac{-i}{S_{23}} \, \varepsilon_2\cdot\varepsilon_3 \Big[(p_2\cdot\varepsilon_1)(p_1\cdot\varepsilon_4) - (p_3\cdot\varepsilon_1)(p_1\cdot\varepsilon_4) - (p_2\cdot\varepsilon_4)(p_4\cdot\varepsilon_1) + (p_3\cdot\varepsilon_4)(p_4\cdot\varepsilon_1) \Big] = 0$$

But $p_1\cdot\varepsilon_4 = 0$ and $p_4\cdot\varepsilon_1 = 0$ by choice of reference momenta $q_1 = q_2 = p_4$ and $q_3 = q_4 = p_1$.

Graph III



$$\sim \; 2\underset{0}{\underset{\shortparallel}{(\varepsilon_2\cdot\varepsilon_4)}}\underset{0}{\underset{\shortparallel}{(\varepsilon_1\cdot\varepsilon_3)}} - (\varepsilon_2\cdot\varepsilon_3)\underset{0}{\underset{\shortparallel}{(\varepsilon_1\cdot\varepsilon_4)}} - \underset{0}{\underset{\shortparallel}{(\varepsilon_2\cdot\varepsilon_1)}}\underset{0}{\underset{\shortparallel}{(\varepsilon_3\cdot\varepsilon_4)}}$$

$$= 0$$

Hence:

$$\boxed{A(1^-, 2^-, 3^+, 4^+) = -i \, \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}} \qquad (I.34)$$

Check helicities:

$$h_1 \circ A = \tfrac{1}{2}\left(-\lambda_1 \frac{\partial}{\partial\lambda_1} + \tilde{\lambda}_1 \frac{\partial}{\partial\tilde{\lambda}_1}\right) A = (-4+2)\,\tfrac{1}{2} A = -A$$

$$h_2 \circ A = -A$$

$$h_3 \circ A = +A$$

$$h_4 \circ A = +A$$

Using the U(1) coupling identity we have:

$$A(1,2,3,4) = -A(1,2,4,3) - A(1,4,2,3)$$

Putting helicities

$$A(1^-,2^+,3^-,4^+) = -A(1^-,2^+,4^+,3^-) - A(1^-,4^+,2^+,3^-)$$

$$= i\left(\frac{\langle 31\rangle^4}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle} + \frac{\langle 31\rangle^4}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle}\right)$$

$$= -i\,\frac{\langle 31\rangle^3}{\langle 24\rangle}\left(\frac{1}{\langle 12\rangle\langle 34\rangle} + \frac{1}{\langle 14\rangle\langle 23\rangle}\right)$$

$$= +i\,\frac{\langle 31\rangle^3}{\langle 24\rangle}\;\frac{\overbrace{\langle 14\rangle\langle 23\rangle + \langle 12\rangle\langle 34\rangle}^{-\langle 13\rangle\langle 42\rangle}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

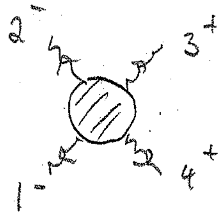
$$= -i\,\frac{\langle 31\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

Thus uniform structure:

$$\boxed{A_4(\ldots,i^-,\ldots,j^-,\ldots) = -i\,\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}} \qquad (\mathrm{I}.35)$$

Ex.

Reconstructing the full amplitude:



$$\hat{=}\ A_4^{\text{tree}}(1^-, a_1;\ 2^-, a_2;\ 3^+, a_3;\ 4^+, a_4)$$

$$\begin{aligned}
&= \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\ \underline{A(1,2,3,4)} \\
&+ \mathrm{Tr}(a_1\, a_3\, a_2\, a_4)\ \underline{A(1,3,2,4)} \\
&+ \mathrm{Tr}(a_1\, a_3\, a_4\, a_2)\ \underline{A(1,3,4,2)} \\
&+ \mathrm{Tr}(a_1\, a_2\, a_4\, a_3)\ \underline{A(1\,2\,4\,3)} \\
&+ \mathrm{Tr}(a_1\, a_4\, a_2\, a_3)\ \underline{A(1,4,2,3)} \\
&+ \mathrm{Tr}(a_1\, a_4\, a_3\, a_2)\ \underline{A(1,4,3,2)}
\end{aligned}$$

$$\begin{aligned}
&= \left[\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \mathrm{Tr}(T^{a_4}T^{a_3}T^{a_2}T^{a_1})\right] \times A(1,2,3\,4) \\
&+ \left[\mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4}) + \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3})\right] \times A(1,3,2\,4) \\
&+ \left[\mathrm{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) + \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})\right] \times A(1,3,4,2)
\end{aligned}$$

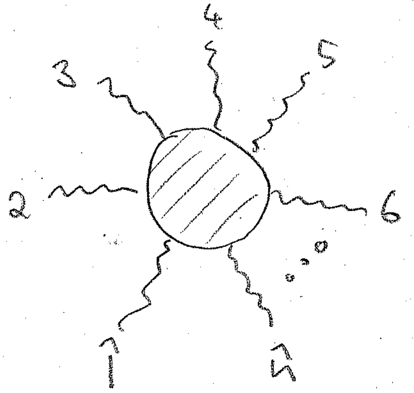
$$\begin{aligned}
\doteq -i\langle 12\rangle^4 \Big\{ &\left[\mathrm{Tr}(a_1a_2a_3a_4) + \mathrm{Tr}(a_1a_2a_3a_4)^*\right] \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \\
+ &\left[\mathrm{Tr}(a_1a_3a_2a_4) + \mathrm{Tr}(a_1a_3a_2a_4)^*\right] \frac{1}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle} \\
+ &\left[\mathrm{Tr}(a_1a_3a_4a_2) + \mathrm{Tr}(a_1a_3a_4a_2)^*\right] \frac{1}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 21\rangle} \Big\}
\end{aligned}$$

II. TREE LEVEL TECHNIQUES

II.1 BRITTO-CACHAZO-FENG-WITTEN (BCFW) ONSHELL RECURSION

Very efficient recursive relation to generate higher-point partial amplitudes from lower point ones: Knowledge of 4-gluon amplitudes allows construction of all n-gluon amplitudes.

Consider an n-gluon tree amplitude



Perform the complex shift for two neighboring legs:

$$\lambda_1 \to \hat{\lambda}_1(z) = \lambda_1 - z\,\lambda_n$$
$$\tilde{\lambda}_n \to \hat{\tilde{\lambda}}_n(z) = \tilde{\lambda}_n + z\,\tilde{\lambda}_1 \qquad z \in \mathbb{C}$$

but leave $\tilde{\lambda}_1$ and λ_n inert. Is a complexification of momenta

$$\boxed{\begin{aligned} p_1^{\alpha\dot{\alpha}} &\to \hat{p}_1^{\alpha\dot{\alpha}}(z) = (\lambda_1 - z\,\lambda_n)^{\alpha}\,\tilde{\lambda}_1^{\dot{\alpha}} \\ p_n^{\alpha\dot{\alpha}} &\to \hat{p}_n^{\alpha\dot{\alpha}}(z) = \lambda_n^{\alpha}\,(\tilde{\lambda}_n + z\,\tilde{\lambda}_1)^{\dot{\alpha}} \end{aligned}}$$

which preserves on-shellness and momentum conservation.

$$\hat{p}_1(z)^2 = 0 \qquad \hat{p}_n(z)^2 = 0 \qquad \hat{p}_1(z) + \hat{p}_n(z) = p_1 + p_2$$

Question: What are the analytical properties in z of the deformed amplitude $A_n \to A_n(z)$?

$$\mathcal{A}_n(z) = \delta^{(4)}\Big(\sum_{i=1}^{n} p_i\Big)\, A_n(z)$$

$A_n(z)$ is rational function of $\{\lambda_i, \tilde{\lambda}_i\}$ and z. As $A_n(z=0)$ only has poles in region momenta

$$A_n(z=0) \underset{\text{poles}}{\sim} \frac{1}{(p_i + p_{i+1} + \dots + p_j)^2}$$

the deformed amplitude $A_n(z)$ will have only simple poles in z of the form:

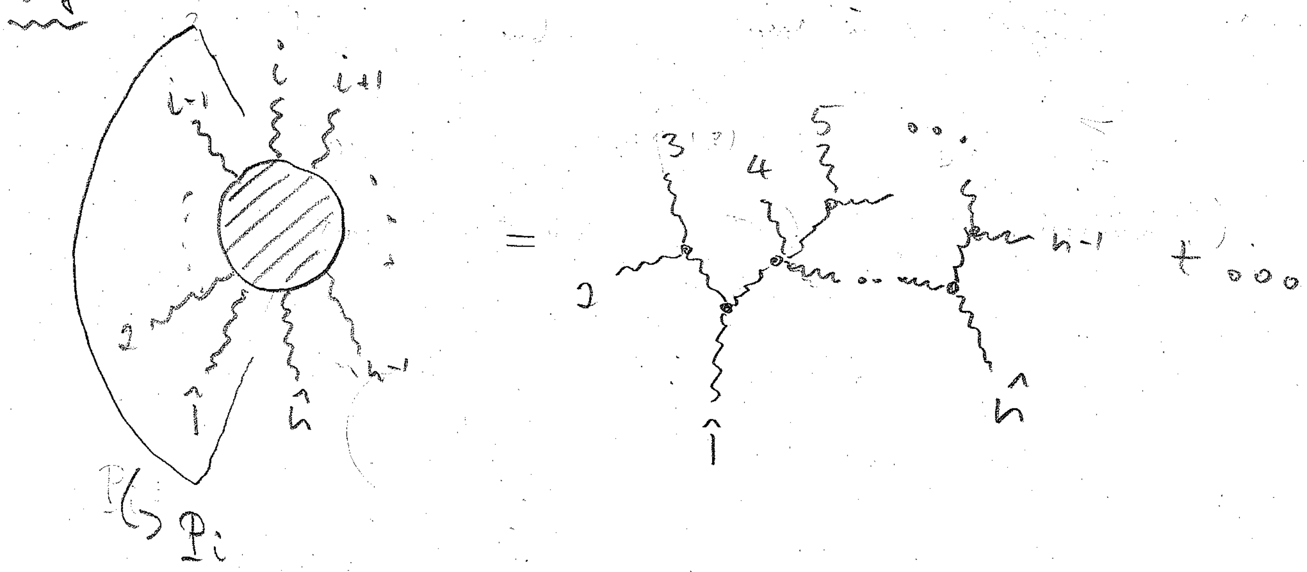
$$\frac{1}{\hat{P}_i(z)} := \frac{1}{(\hat{p}_1(z) + p_2 + \dots + p_{i-1})^2} = \frac{1}{(p_i + p_{i+1} + \dots + \hat{p}_n(z))^2}$$

$$= \frac{1}{P_i^2 - z\,\langle n|P_i|1]}$$

with $P_i := p_1 + p_2 + \dots + p_{i-1}$ and

$$\langle n|P_i|1] = \lambda_{n\alpha} P_i^{\alpha\dot{\alpha}} \tilde{\lambda}_{1\dot{\alpha}} .$$

Why?



Any region momentum containing $\hat 1$ and $\hat n$ is z-independent!

⇒ $A_n(z)$ has only simple poles at $\underline{z_{P_i} = \frac{P_i^2}{\langle n|P_i|1]}}$

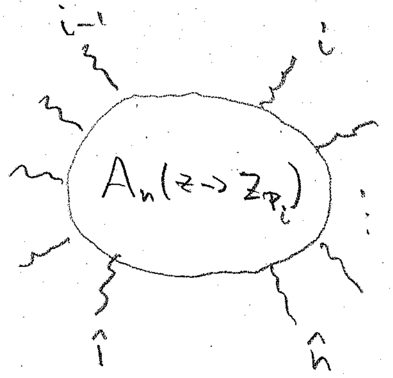
∀ $i \in [3, n-1]$

☒ Near the pole z_{P_i} the amplitude $A_n(z)$ factorises

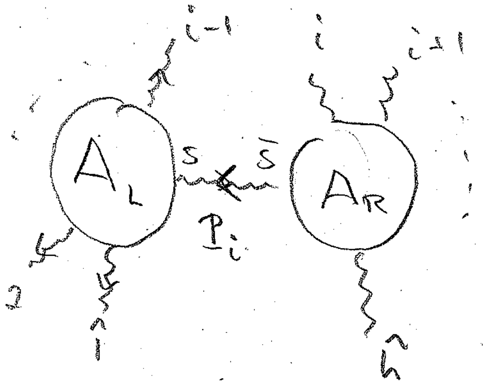
$$\lim_{z \to z_{P_i}} A_n(z) = \frac{1}{z - z_{P_i}} \frac{-1}{\langle n|P_i|1]} \sum_{\textcircled{s}} A_L(\hat 1(z_{P_i}), 2, \dots, i-1, -\hat{P}^{s}(z_{P_i})) \, A_R(\hat{P}^{\bar s}(z_{P_i}), i, \dots, n-1, \hat n(z_{P_i}))$$

☒ Internal propagator goes on-shell ⇒ Factorization of A_n into A_L and A_R on-shell subamplitudes

☒ Sum over S: Sum of d.o.f. For gluons $s = \{+, -\}$



$$\sim \frac{1}{z-z_p}$$



Now consider the function $\frac{A_n(z)}{z}$. This behaves as

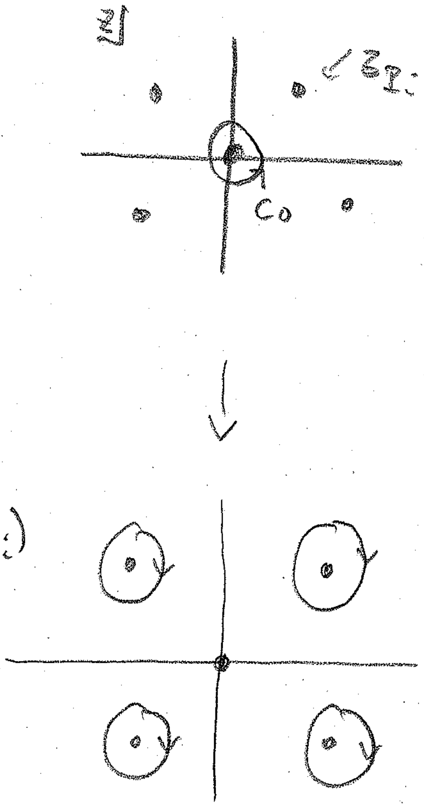
$$\lim_{z\to z_{P_i}} \frac{A(z)}{z} = -\frac{1}{z-z_{P_i}} \sum_s A_L^s(z_{P_i}) \frac{1}{P_i^2} A_R^{\bar{s}}(z_{P_i})$$

with $A_L^s(z_{P_i}) = A_L(\hat{1}(z_{P_i}), 2, \dots, i-1, -\hat{P}^s(z_{P_i}))$

$A_R^{\bar{s}}(z_{P_i}) = A_R(\hat{P}_i, i, \dots, \hat{n})$

Of course we are interested in the original Amplitude: $A_n = A_n(z=0)$ which may be written as

$$A_n = A_n(z=0) = \oint_{C_0} \frac{dz}{2\pi i} \frac{A(z)}{z}$$



$$= \sum_{i=2}^{n-1} \sum_s A_L^s(z_{P_i}) \frac{1}{P_i^2} A_R^{\bar{s}}(z_{P_i})$$

$+ \mathrm{Res}(z=\infty)$

We will show that $\mathrm{Res}(z=\infty)$ vanishes for gauge theories. Assuming this we have derived the BCFW recursion relation (2005): (PRL 94, 181602 (2005))

$$\boxed{\begin{aligned} A_n(1,2,\ldots,n) = \sum_{i=3}^{n-1} \sum_{s=\{+,-\}} & A_i\left(\hat{1}(z_{\underline{P}_i}), 2, \ldots, i-1, -\hat{\underline{P}}_i(z_{\underline{P}_i})\right) \\ & \frac{1}{\underline{P}_i^2} A_{n+2-i}\left(\hat{\underline{P}}(z_{\underline{P}_i}), i, \ldots, n-1, \hat{n}(z_{\underline{P}_i})\right) \end{aligned}}$$

Is constructive: Can build higher point amplitudes from lower point ones.

Comments:

☐ We chose neighboring legs $\hat{1}$ & $\hat{n}$ for complex shifts. Can generalize to non-neighboring ones

☐ Also shifts of more than 2 legs have been considered in the literature.

Open issue: Large z behaviour of $A_n(z)$

For $\oint_\infty \frac{dz}{2\pi i} \frac{A_n(z)}{z} = 0$ we need a falloff $\lim_{z\to\infty} A_n(z) \sim \frac{1}{z}$

as $\oint_\infty \frac{dz}{2\pi i} \frac{A_n(z)}{z} \underset{\substack{z=1/\omega \\ dz=-\frac{d\omega}{\omega^2}}}{=} \oint_0 \frac{d\omega}{2\pi i} \frac{A_n(\omega)}{\omega} \to 0$ iff $\lim_{\omega\to 0} A_n(\omega) \sim \omega$

Falloff depends on polarizations of 1 and n:

E.g. 4-gluon MHV amplitude:

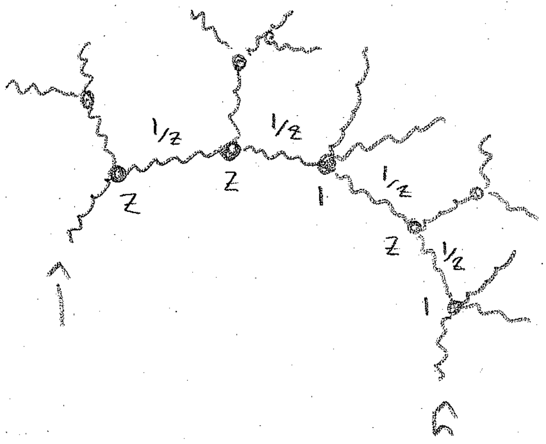
$$A_4(\hat{1}^-, 2^+, 3^+, \hat{4}^-) = \frac{\langle \hat{1}\hat{4}\rangle^4}{\langle \hat{1}2\rangle\langle 23\rangle\langle 3\hat{4}\rangle\langle \hat{4}\hat{1}\rangle} = \frac{\langle 14\rangle^4}{\langle \hat{1}2\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \sim \frac{1}{z}$$

$$\langle \hat{4}\hat{1}\rangle \underset{|\hat{4}\rangle = |4\rangle}{=} \langle 4\hat{1}\rangle = \langle 41\rangle + z\langle 44\rangle = \langle 41\rangle$$

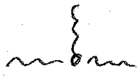
$$A_4(\hat{1}^-, 2^-, 3^+, \hat{4}^+) = \frac{\langle \hat{1}2\rangle^4}{\langle \hat{1}2\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \sim z^3$$

Detailed analysis:

☒ Consider a generic graph contributing to the tree-level n-gluon amplitude ($\hat{1}$ & $\hat{n}$ are neighbors)

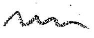


z-dependance occurs only along path $\hat{1}$ to $\hat{n}$:



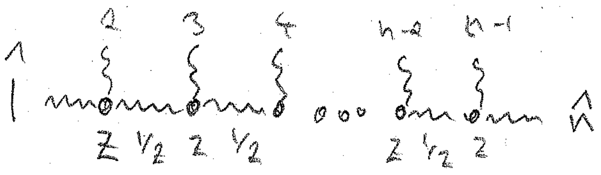
$\sim z$ (at most)!

~ 1



$\sim \frac{1}{z}$

☒ Worst case scenario: Line $\hat{1} \to \hat{n}$ contains only 3-valent vertices:



$\sim z$

▣ But z-dependence also arises from polarization vectors at legs 1 & n:

$$\varepsilon_1^{+\dot\alpha\alpha} = -\sqrt{2}\,\frac{\tilde\lambda_1^{\dot\alpha}\mu_1^{\alpha}}{\langle\hat\lambda_1(z)\,\mu_1\rangle} \sim \frac{1}{z} \qquad \varepsilon_1^{-\dot\alpha\alpha} = \sqrt{2}\,\frac{\hat\lambda_1^{\alpha}(z)\,\tilde\mu_1^{\dot\alpha}}{[\hat{\tilde\lambda}_1\,\tilde\mu_1]} \sim z$$

$$\varepsilon_n^{+\dot\alpha\alpha} = -\sqrt{2}\,\frac{\hat{\tilde\lambda}_n^{\dot\alpha}(z)\,\mu_n^{\alpha}}{\langle\hat\lambda_n\,\mu_n\rangle} \sim z \qquad \varepsilon_n^{-\dot\alpha\alpha} = \sqrt{2}\,\frac{\lambda_n^{\alpha}\,\tilde\mu_n^{\dot\alpha}}{[\hat{\tilde\lambda}_n(z)\,\tilde\mu_n]} \sim \frac{1}{z}$$

Summary: Individual graphs scale at worst as

$$\boxed{A(\hat 1^+\hat n^-) \sim \frac{1}{z}} \quad \checkmark \qquad \underline{A(\hat 1^+\hat n^+) \sim z}$$

$$\underline{A(\hat 1^-\hat n^-) \sim z} \qquad A(\hat 1^-\hat n^+) \sim z^3$$

It is always possible to find a $\{\hat 1^+, \hat n^-\}$ pair by cyclicity and one has a valid on-shell recursion relation.

Can show: Also shifts $A(\hat 1^+, \hat n^+)$ and $A(\hat 1^-, \hat n^-)$ lead to overall $1/z$ scaling once sum over all Feynman graphs is performed.

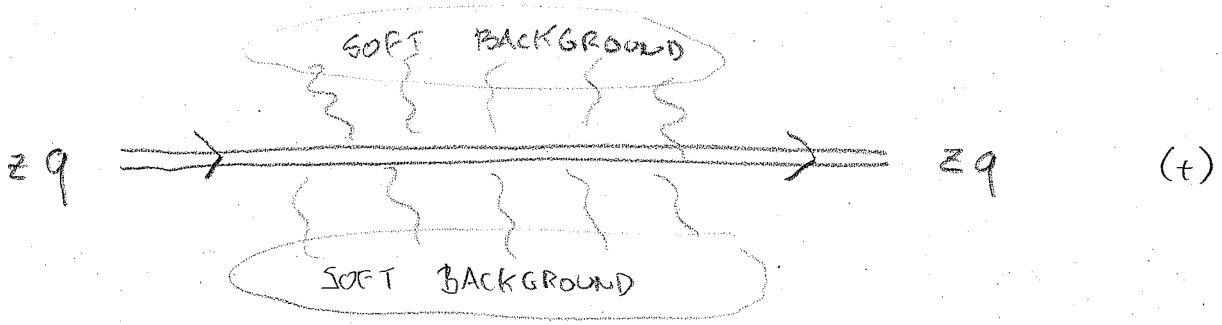
Only: $A(\hat 1^-, \hat n^+)$ shift gives non-vanishing Res_∞

$\Rightarrow$ No BCFW recursion for these shifts!

THE $\frac{1}{z}$ - FALLOFF OF THE (+,+) & (−,−) SHIFTS

The limit $z \to \infty$ corresponds physically to the situation of a "hard" particle with very large (complex) momenta scattering off a background of $(n-2)$ "soft" particles:

$$p_1^{\alpha\dot\alpha}(z) \to -z\,\lambda_n^{\alpha}\,\tilde\lambda_1^{\dot\alpha} = -z\,q^{\alpha\dot\alpha}$$
$$p_n^{\alpha\dot\alpha}(z) \to +z\,\lambda_n^{\alpha}\,\tilde\lambda_1^{\dot\alpha} = +z\,q^{\alpha\dot\alpha}$$



Describe the soft particles as a background field B_μ and the hard particle as a fluctuation:

$$A_\mu = \underset{\substack{\uparrow \\ \text{soft}}}{B_\mu} + \underset{\substack{\uparrow \\ \text{hard}}}{a_\mu} \qquad \text{with} \qquad \begin{matrix} \partial_\mu a_\nu \sim z \\ \partial_\mu B_\nu \sim 1 \end{matrix}$$

The above scattering scenario then follows from the analysis of the two-point function for a_μ in the background B_μ ⇒ Need only quadratic piece of the Lagrangian

$$\left[\mathcal{L}^{\text{QUAD}} = -\frac{1}{4}\,\mathrm{Tr}\left(D_\mu a_\nu\, D^\mu a^\nu\right) + \frac{i}{2}\,g\,\mathrm{Tr}\left(G^{\mu\nu}\left[a_\mu, a_\nu\right]\right)\right] \qquad (\text{II}.3)$$

where we have added a gauge fixing term $(D_\mu a^\mu)^2$ and

$$D_\mu := \partial_\mu - i g B_\mu \qquad \text{(background field cov. derivative).}$$

First term in (II.3) responsible for leading z-behaviour, this term also has a higher symmetry, which is broken by subleading $Tr(G^{\mu\nu}[a_\mu, a_\nu])$ term: SO(1,3) rotation symmetry acting <u>only</u> on the fluctuating field a_μ:

$$\delta a_\mu = \omega_\mu{}^\nu a_\nu \qquad \delta B_{\mu\nu} = 0$$

To make this symmetry manifest introduce extra latin indices for fluctuating field: $a_\mu \to a_a$

$$\mathcal{L}^{QUADR} = -\frac{1}{4} Tr(D_\mu a_a D^\mu a_b)\eta^{ab} + \frac{i}{2} g\, Tr(G^{ab}[a_a, a_b])$$

<u>Then</u>: Leading contribution at large z from first term which exhibits global SO(1,3) symmetry. Sub-leading terms break this:

$$\boxed{A^{ab} := \lim_{z\to\infty} \langle a^a(-zq)\, a^b(zq)\rangle = z\cdot\eta^{ab}\cdot c + G^{ab}\cdot d + \frac{1}{z} B^{ab} + \dots}$$

with η^{ab} sym.; G^{ab}: a/sym; B^{ab} generic

In bispinor notation this may be written as

$$\boxed{\begin{aligned} A^{\alpha\dot\alpha\beta\dot\beta} &= \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\, z\cdot c + (\varepsilon^{\alpha\beta}\bar{S}^{\dot\alpha\dot\beta} + \varepsilon^{\dot\alpha\dot\beta} S^{\alpha\beta}) \\ &\quad + \frac{1}{z} B^{\alpha\dot\alpha\beta\dot\beta} + \ldots \end{aligned}} \qquad (\text{II}.4)$$

with $S^{\alpha\beta} = S^{\beta\alpha}$ & $\bar{S}^{\dot\alpha\dot\beta} = -S^{\dot\beta\dot\alpha}$ symmetries)

Why? ① $\eta^{\mu\nu}\,\sigma_\mu^{\alpha\dot\alpha}\sigma_\nu^{\beta\dot\beta} = 2\,\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}$

② $G^{\mu\nu} = g^{\mu\nu} + \bar{g}^{\mu\nu}$

with $g^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} g_{\mu\nu}$ (self-dual) (1,0) rep.

$\bar{g}^{\mu\nu} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{g}_{\mu\nu}$ (anti self-dual) (0,1) rep.

Now contracting the 2-point function $A^{\alpha\dot\alpha\beta\dot\beta} = \langle A^{\alpha\dot\alpha}(p_1)\, A^{\beta\dot\beta}(p_n)\rangle$ of (II.4) with the (+,+) polarization vectors

$$\varepsilon_{+,1}^{\alpha\dot\alpha}(z) = -\sqrt{2}\,\frac{\hat{\lambda}_1^{\dot\alpha}\mu_1^{\alpha}}{\langle\hat{\lambda}_1(z)\mu_1\rangle} \qquad \text{where} \qquad \hat{\lambda}_1(z) = \lambda_1 - z\lambda_n$$

$$\hat{\tilde\lambda}_n(z) = \tilde\lambda_n + z\tilde\lambda_1$$

$$\varepsilon_{+,n}^{\beta\dot\beta}(z) = -\sqrt{2}\,\frac{\hat{\tilde\lambda}_n^{\dot\beta}(z)\,\mu_n^{\beta}}{\langle\lambda_n\mu_n\rangle}$$

we see that for choice $\mu_1 = \mu_n = \mu$ the leading term in (II.4) does not contribute. The subleading term yields

$$(\varepsilon_{+1} A\, \varepsilon_{+n}) = 2\,\frac{[\lambda_1\hat{\lambda}_n]\; S^{\alpha\beta}\mu_\alpha\mu_\beta}{\langle\hat{\lambda}_1(z)\mu\rangle\langle\lambda_n\mu\rangle} \underset{[1\hat{n}]=[1n]}{=} \frac{[1n]\,\langle\mu|S|\mu\rangle}{\langle\hat{1}\mu\rangle\langle n\mu\rangle} \sim \frac{1}{z}$$

and by parity also $\langle \varepsilon_{-1} A\, \varepsilon_{-n} \rangle \sim \frac{1}{z}$. Note that the $1/z$ term in (II.4) will also only contribute at order ($1/z$) once contracted with the polarizations ε_{+n} & ε_{+1}.

In summary we find the scaling with z of the shifted legs:

$$\boxed{A(++) \sim \frac{1}{z} \quad , \quad A(--) \sim \frac{1}{z} \quad , \quad A(+-) \sim \frac{1}{z}} \qquad \text{(II.5)}$$

but $\quad A(-+) \sim z^3$

Thus shifts of $(++)$, $(--)$ & $(+-)$ lead to the BCFW recursion (II.2)

II.2 THE GLUON 3-POINT AMPLITUDE

For real momenta there exists <u>no</u> 3-pt amplitudes of massless particles:

$$p_1^\mu + p_2^\mu + p_3^\mu = 0 \quad \Rightarrow \quad p_1 \cdot p_2 = p_2 \cdot p_3 = p_3 \cdot p_1 = 0$$

All Mandelstam invariants vanish and there are no other Lorentz-scalars an amplitude could depend on.

$$p_i \cdot p_j = 0 \quad \Leftrightarrow \quad \langle ij \rangle [ji] = 0$$

Different situation for <u>complex momenta</u> $p_i \in \mathbb{C}$:

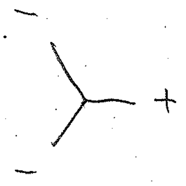
Then there is no relation between λ_i and $\tilde{\lambda}_i$ and

$$p_i \cdot p_j = 0 \quad \text{for either} \quad \langle ij\rangle = 0 \quad \forall\, i,j = 1,2,3$$
$$\text{or} \quad [ij] = 0$$

Hence either $\lambda_1^\alpha \propto \lambda_2^\alpha \propto \lambda_3^\alpha$ (colinear spinors) or $\tilde{\lambda}_1^{\dot\alpha} \propto \tilde{\lambda}_2^{\dot\alpha} \propto \tilde{\lambda}_3^{\dot\alpha}$

2 choices:

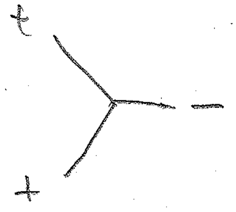
$$A_3^{MHV}(i^-,j^-) = i\,\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$



and $[12] = [23] = [31] = 0$

Or

$$A_3^{\overline{MHV}}(i^+,j^+) = i\,\frac{[ij]^4}{[12][23][31]} \qquad (II.6)$$



and $\langle 12\rangle = \langle 23\rangle = \langle 31\rangle$

Form follows from color ordered Feynman rules. Alternatively: This is the only form compatible with the helicity assignments.

Say w.l.o.g. $\langle ij\rangle = \langle 12\rangle$: $\quad A_3^{MHV} = A_3^{MHV}(\langle 12\rangle, \langle 23\rangle, \langle 31\rangle)$

$$= c \cdot \langle 12\rangle^\alpha \langle 23\rangle^\beta \langle 31\rangle^\gamma$$

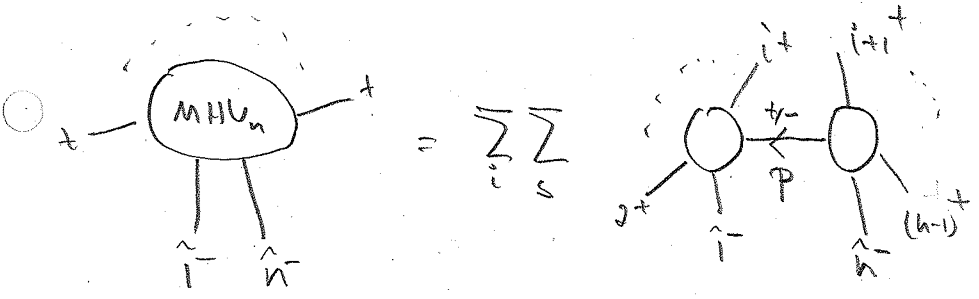
$h_{1,2}A_3^{MHV} = -A_3^{MHV}$ $\qquad h_3 A_3^{MHV} = A_3^{MHV}$

$$\Rightarrow -\tfrac{1}{2}(\alpha+\gamma) = -1 \qquad -\tfrac{1}{2}(\alpha+\beta) = -1 \qquad -\tfrac{1}{2}(\beta+\gamma) = 1$$

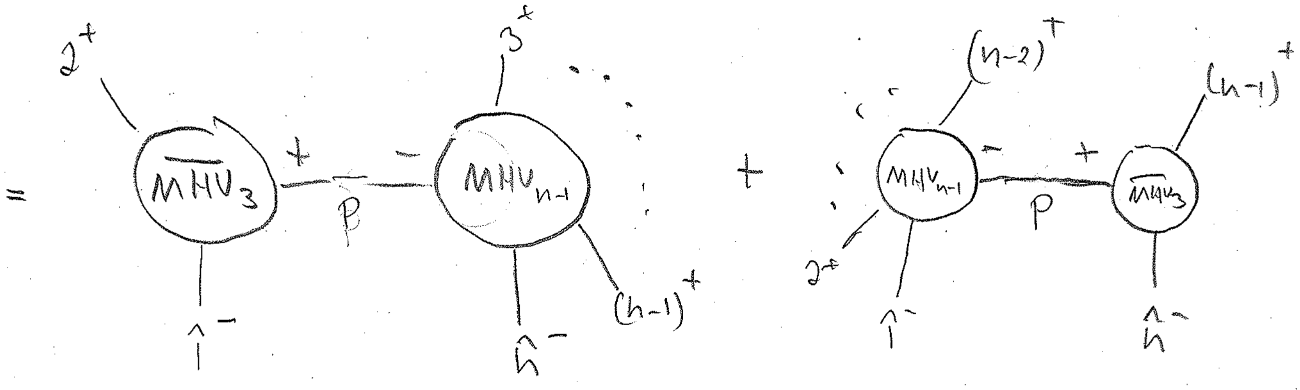
$$\Rightarrow \underline{\alpha = 3 \qquad \beta = -1 \qquad \gamma = -1}$$

- Remarkable result: Via BCFW recursion we can produce **all** gluon n-point tree-amplitudes from the 3-point amplitude. Their structure follows **solely** from kinematical considerations (helicity assignments & momentum conservation). The explicit form of the 4-pt vertex in YM theory is not needed!

- Apply BCFW recursion for MHV-amplitudes



Only two terms can contribute!



Recall shifts: $\lambda_1 \to \lambda_1 - z\lambda_n$

$\hat{\lambda}_n \to \hat{\lambda}_n + z\hat{\lambda}_1$

Subtle point: $\overline{MHV}_3$ amplitude at $\mathcal{A}_R$ not allowed.

$\overline{MHV}_3$ prescription implies $\langle \hat{n}, n-1\rangle = \langle n, n-1\rangle = 0$

$\Rightarrow p_n \cdot p_{n-1} = 0$ i.e. legs n and $(n-1)$ are colinear.

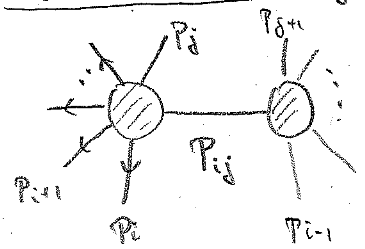
This is not generally true and implies a non-existing constraint on n-particle kinematics!

$\Rightarrow$ N.B. not a problem for $\overline{MHV}_3$ at $\mathcal{A}_L$ as then $\langle \hat{1} 2\rangle \neq \langle 12\rangle$ and $\langle \hat{1} 2\rangle = 0$ is OK.

II.3 FACTORIZATION PROPERTIES

Analytic structure of tree-level partial amplitudes: Poles

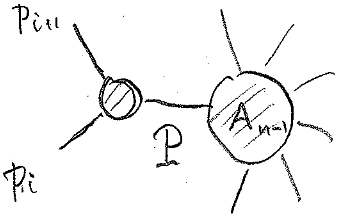
i) Region momenta go on-shell: $P_{i,j} = p_i + p_{i+1} + \dots + p_j$



$$A_n^{tree}(1,\dots,n) \underset{P_{ij}^2\to 0}{\sim} \sum_{\lambda} A_L(i,\dots,j,P^{\lambda}) \frac{1}{P_{ij}^2} A_R(P^{-\lambda},j+1,\dots,i-1)$$

"multiparticle pole"

⇒ Two-particle or collinear singularity: $p_i \sim p_{i+1}$



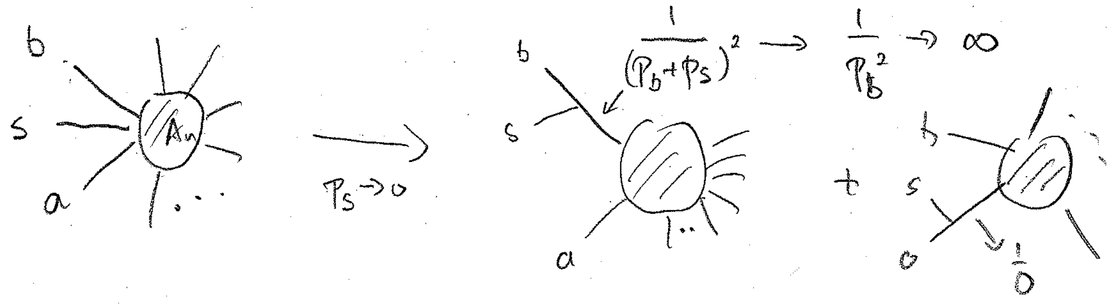
$p_i = z P$

$p_{i+1} = (1-z)\, P$

$P = p_i + p_{i+1}$

ii) Soft limit: 4-Momentum of single leg goes to zero:

$$p_s^{\mu} \to 0$$



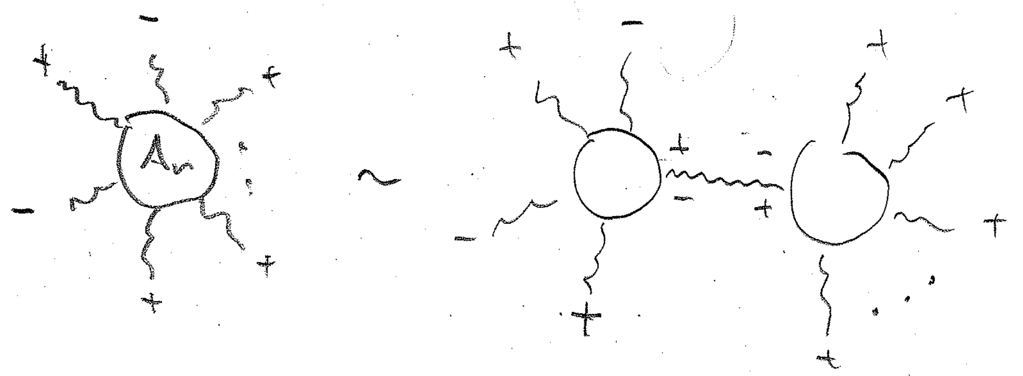
Tree (and loop-level) amplitudes possess important factorization properties in collinear and soft limits, with universal features.

Absence of multi-particle poles in MHV-amps

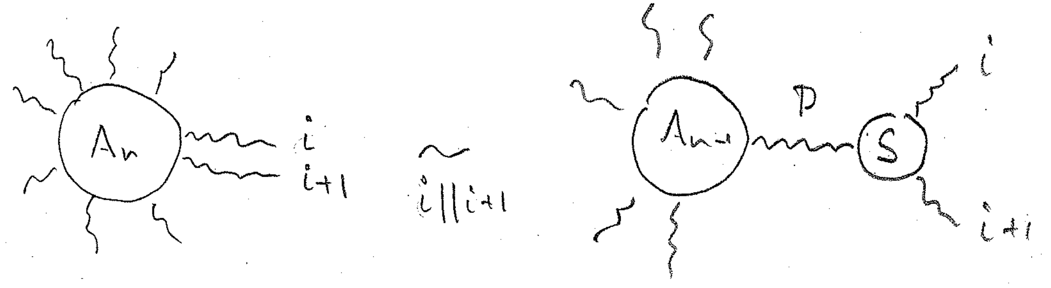
Multi-gluon amplitudes will in general have multi-particle poles, yet MHV-amplitudes are special:

Due to $A_n(1^\pm, 2^+, \dots, n^+) = 0$ MHV-amplitudes can factorise only over two-particle poles: A factorization of an MHV-amplitude will only 3 negative helicity legs distributed over two partial amplitudes.

This is always zero unless one partial amplitude is a 3-particle amp:



COLLINEAR SINGULARITIES



(II.7)

$$\boxed{A_n^{\text{tree}}(\dots, i^{\lambda_i}, i+1^{\lambda_{i+1}}, \dots) \xrightarrow[i \| i+1]{} \sum_{\lambda = \pm} \text{Split}^{\text{tree}}_{-\lambda}(z, i, i+1)\, A_{n-1}^{\text{tree}}(\dots, P^{\lambda}, \dots)}$$

The splitting amplitude $\mathrm{Split}^{\text{tree}}_{-\lambda}$ is universal: It does not depend on the momenta and helicities of the $(n-2)$-other legs beyond i & $i+1$. Will not prove this here (Berends, Giele '89; Mangano, Parke '91)

Tree-level gluon splitting functions can then be derived from 5-pt MHV-amplitude:

$$A_5^{\text{tree}}(1^-, 2^-, 3^+, \underset{\uparrow}{4^+}, \underset{\uparrow}{5^+}) = i \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

$$\xrightarrow[4\parallel 5]{} \frac{1}{\sqrt{z(1-z)}\,\langle 45\rangle} \times i \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 3P\rangle\langle P1\rangle}$$

$$= \mathrm{Split}^{\text{tree}}_{-}(z, 4^+, 5^+) \times A_4^{\text{tree}}(1^-, 2^-, 3^+, P^+)$$

$4\parallel 5$: $\lambda_4 = \sqrt{z}\,\lambda_P$, $\lambda_5 = \sqrt{1-z}\,\lambda_P$

$\tilde\lambda_4 = \sqrt{z}\,\tilde\lambda_P$, $\tilde\lambda_5 = \sqrt{1-z}\,\tilde\lambda_P$

$$A_5^{\text{tree}}(1^-, \underset{\uparrow}{2^-}, \underset{\uparrow}{3^+}, 4^+, 5^+) \xrightarrow[2\parallel 3]{} \frac{z^2}{\sqrt{z(1-z)}}\frac{1}{\langle 23\rangle} \; i \frac{\langle 1P\rangle^4}{\langle 1P\rangle\langle P4\rangle\langle 45\rangle\langle 51\rangle}$$

$$= \mathrm{Split}^{\text{tree}}_{+}(z, 2^-, 3^+) \times A_4^{\text{tree}}(1^-, P^-, 4^+, 5^+)$$

$$A_5^{\text{tree}}(\underset{\uparrow}{1^-}, \underset{\uparrow}{2^-}, 3^+, 4^+, 5^+) \xrightarrow[1\parallel 2]{} 0$$ consistent with (II.7) but no information on Split

$$A_5^{\text{tree}}(\overset{}{\underset{\uparrow}{1^-}},2^-,3^+,4^+,\underset{\uparrow}{5^+}) \xrightarrow[5 \parallel 1]{} \underbrace{\frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle 51\rangle}}_{\text{Split}_+(1-z,\,5^+,\,1^-)} \times A_4^{\text{tree}}(P^-,2^-,3^+,4^+)$$

We find: $P_i = z \cdot P$; $P_{i+1} = (1-z)\cdot P$

$$\text{Split}_-^{\text{tree}}(z,a^+,b^+) = \frac{1}{\sqrt{z(1-z)}}\,\frac{1}{\langle ab\rangle}$$

$$\text{Split}_+^{\text{tree}}(z,a^-,b^+) = \frac{z^2}{\sqrt{z(1-z)}}\,\frac{1}{\langle ab\rangle} = \text{Split}_+(1-z,b^+,a^-)$$

$$\text{Split}_+^{\text{tree}}(z,a^+,b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}}\,\frac{1}{\langle ab\rangle}$$

By looking at the collinear factorization of the 6-point MHV-amplitude $A(1^-,2^-,3^+,4^+,\underset{\uparrow}{5^+},\underset{\uparrow}{6^+}) \xrightarrow[5\parallel 6]{}$

$$\text{Split}_-^{\text{tree}}(z,5^+,6^+)\;A(1^-,2^-,3^+,4^+,P^+) + \text{Split}_+(z,5^+,6^+)$$
$$\times A(1^-,2^-,3^+,4^+,P^-)$$

One shows:

$$\text{Split}_+^{\text{tree}}(z,a^+,b^+) = 0$$

Via parity $\text{Split}_\lambda^{\text{tree}}(z,a^{\lambda_a},b^{\lambda_b}) = \left(\text{Split}_{\bar\lambda}^{\text{tree}}(z,a^{\bar\lambda_a},b^{\bar\lambda_b})\right)^*$

one derives the rest of the splitting functions:

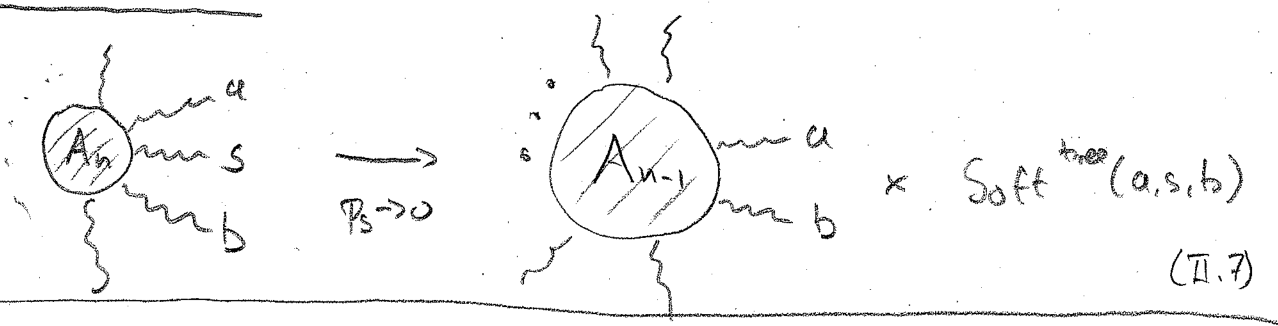
$$\text{Split}_+(z, a^-, b^-) = \frac{1}{\sqrt{z(1-z)}} \frac{1}{[ab]}$$

$$\text{Split}_-(z, a^-, b^-) = 0$$

$$\text{Split}_+(z, a^+, b^-) = \frac{z^2}{\sqrt{z(1-z)}} \frac{1}{[ab]}$$

$$\text{Split}_-(z, a^-, b^+) = \frac{(1-z)^2}{\sqrt{z(1-z)}} \frac{1}{[ab]}$$

SOFT LIMITS



(II.7)

$$\boxed{A_n^{\text{tree}}(\dots, a, s, b, \dots) \xrightarrow[p_s \to 0]{} \text{Soft}^{\text{tree}}(a,s,b) \times A_{n-1}^{\text{tree}}(\dots a, b, \dots)}$$

The soft behaviour is <u>universal</u> and independent of helicities of particles a, b or s !

From MHV-amplitudes one finds:

$$\boxed{\text{Soft}^{\text{tree}}(a,s,b) = \frac{\langle a,b \rangle}{\langle a,s \rangle \langle s,b \rangle}}$$

II.4 SYMMETRIES

OBVIOUS SYMMETRIES:

Scattering amplitudes should be invariant under Poincaré transformations. These are realised in the spinor helicity formulation of momentum space as

$$\boxed{\begin{aligned} &p^{\alpha\dot\alpha} = \sum_{i=1}^{n} \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha} && \text{(translation)} \\ &m_{\alpha\beta} = \sum_{i=1}^{n} \lambda_{i(\alpha}\,\partial_{i\beta)} \qquad \bar m_{\dot\alpha\dot\beta} = \sum_{i=1}^{n} \tilde\lambda_{i(\dot\alpha}\,\partial_{i\dot\beta)} && \text{(Lorentz)} \end{aligned}} \quad (\text{II}\,8)$$

where $\partial_{i\alpha} := \dfrac{\partial}{\partial\lambda_i^\alpha}$ $\qquad$ $\partial_{i\dot\alpha} := \dfrac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}$

and $T_{(\alpha\beta)} := \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right)$ symmetrization with unit weight

These generators obey:

$$\begin{cases} P^{\alpha\dot\alpha}\,A_n(\{\lambda_i,\tilde\lambda_i\}) = 0 & \text{(in distributional sense, i.e. } p\,\delta(p) = 0\,) \\[2ex] m_{\alpha\beta}\,A_n(\{\lambda_i,\hat\lambda_i\}) = 0 = \bar m_{\dot\alpha\dot\beta}(A_n\{\lambda_i,\hat\lambda_i\}) \end{cases}$$

Last line means all Weyl indices $\{\alpha,\dot\alpha\}$ are properly contracted.

LESS OBVIOUS SYMMETRIES:

(Classical) Yang-Mills theory is invariant under a larger symmetry group than Poincaré$_{d=4}$: Due to absence of any dimensionful parameter in the theory pure YM-theory and massless QCD is invariant under scale transformations:

$$X^\mu \to \kappa^{-1} \cdot X^\mu \quad \text{resp.} \quad p^\mu \to \kappa \cdot p^\mu$$

Scale transformations are generated by the <u>dilatation operator</u> d acting on amplitudes as

$$\boxed{d := \sum_{i=1}^{n} \left(\frac{1}{2} \lambda_i^\alpha \partial_{i\alpha} + \frac{1}{2} \tilde{\lambda}_i^{\dot\alpha} \partial_{i\dot\alpha} + 1 \right)} \qquad (\text{II}.9)$$

With $d \cdot A_n(\{\lambda_i, \tilde\lambda_i\}) = 0$.

☐ Check invariance of MHV-amplitudes:

$$A_n^{MHV} = g^{n-2}\, \delta^{(4)}\Big(\sum_i \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha} \Big) \frac{\langle \lambda_s, \lambda_t \rangle^4}{\langle 12 \rangle \ldots \langle n1 \rangle}$$

Operator d measures weight in momentum units plus # of legs

$$d \cdot \mathcal{O} = ([\mathcal{O}] + n)\, \mathcal{O}$$

$$[\delta^{(4)}] = -4 \qquad [\langle \lambda_s, \lambda_t \rangle^4] = 4 \qquad \left[\frac{1}{\langle 12 \rangle \ldots \langle n1 \rangle} \right] = -n$$

Hence:

$$d \circ A_n^{MHV} = \left(-4+4-n+n\right) A_n^{MHV} = 0 \quad \checkmark$$

Moreover, there is a farther symmetry of scale invariant theories: <u>Special conformal transformations</u>: $k_{\alpha\dot\alpha}$

Realised on amplitudes via the second order derivative operator

$$\boxed{k_{\alpha\dot\alpha} := \sum_{i=1}^{n} \partial_{i\alpha}\, \partial_{i\dot\alpha}} \qquad (\text{II}.10)$$

Together $\{p_{\dot\alpha\alpha}, k_{\alpha\dot\alpha}, m_{\alpha\beta}, \bar m_{\dot\alpha\dot\beta}, d\}$ form the conformal group in 4d $SO(2,4)$.

STANDARD REPRESENTATION OF CONFORMAL GROUP IN CONFIGURATION SPACE.

(GENERATORS)

$$\boxed{\begin{aligned} M_{\mu\nu} &= i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \qquad\qquad \partial_\mu = \frac{\partial}{\partial x^\mu} \\ P_\mu &= -i\,\partial_\mu \\ D &= -i\,x^\mu \partial_\mu \\ K_\mu &= i\left(x^2 \partial_\mu - 2\,x_\mu x^\nu \partial_\nu\right) \end{aligned}}$$

A Fouriertransform $\int d^4x\, e^{ip\cdot x}\, \mathcal{O}(x)$ brings this into momentum space, which in turn can be mapped to the helicity spinor representation of (II.8-II.10).

A finite special conformal transformation is given by:

$$x^\mu \to x'^\mu = \frac{x^\mu - a^\mu x^2}{1-2a\cdot x + a^2 x^2}$$

a^μ : transformation parameter

Ex: One shows that $K^\mu = I P^\mu I$ with I: inversion acting as $I \circ x^\mu = \frac{x^\mu}{x^2}$.

Commutation relations

$$[d, p_{\alpha\dot\alpha}] = p_{\alpha\dot\alpha} \qquad [d, k_{\alpha\dot\alpha}] = -k_{\alpha\dot\alpha}$$

$$[d, m_{\alpha\beta}] = 0 = [d, \bar m_{\dot\alpha\dot\beta}]$$

$$[k_{\alpha\dot\alpha}, p^{\beta\dot\beta}] = \delta_\alpha^\beta \delta_{\dot\alpha}^{\dot\beta} d + m_\alpha{}^\beta \delta_{\dot\alpha}^{\dot\beta} + \bar m_{\dot\alpha}{}^{\dot\beta} \delta_\alpha^\beta$$

Ex: Prove these relations

Check invariance of MHV amplitude:

$$\underline{k_{\alpha\dot\alpha} A_n^{MHV} \overset{?}{=} 0}$$

Invariance of MHV-amplitudes

$$A_n^{MHV} = g^{n-2}\,\delta^{(4)}\Big(\sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot\alpha}\Big)\;\frac{\langle \lambda_s, \lambda_t\rangle^4}{\langle 12\rangle \dots \langle n1\rangle}$$

① $\underline{\underline{d \circ A_n^{MHV}}}$: $\quad [\delta^{(4)}] = -4 \quad , \quad [\langle \lambda_s, \lambda_t\rangle] = 4$

$$\Rightarrow \quad [d, \delta^{(4)}(\textstyle\sum_i p_i)\,\langle s,t\rangle^4] = 0$$

$\dfrac{1}{\langle 12\rangle \dots \langle n1\rangle}$ is homogenous of degree -2 in all λ_i

$$\Rightarrow \quad d\,\frac{1}{\langle 12\rangle \dots \langle n1\rangle} = \sum_i \Big(\tfrac{1}{2}(-2) + 1\Big)\frac{1}{\langle 12\rangle \dots \langle n1\rangle} = 0$$

$$\Rightarrow \quad \boxed{d\,A_n^{MHV} = 0}$$

② $\underline{\underline{k_{\alpha\dot\alpha} A_n^{MHV}}} = \sum_i \dfrac{\partial^2}{\partial\lambda_{\alpha i}\,\partial\tilde{\lambda}_{i\dot\alpha}} \xrightarrow{} \delta^{(4)}(P)\,A_n(\{\lambda_i\})$

$$= \sum_i \frac{\partial}{\partial\lambda_{\alpha_i}}\;\underbrace{\frac{\partial P^{\beta\dot\beta}}{\partial\tilde{\lambda}_{i\dot\alpha}}}_{\lambda_i^\beta\,\delta_{\dot\alpha}^{\dot\beta}}\Big(\frac{\partial}{\partial p^{\beta\dot\beta}}\,\delta^{(4)}(P)\Big)\,A_n(\{\lambda_i\})$$

$$= \Big[\Big(n\,\frac{\partial}{\partial p^{\alpha\dot\alpha}} + P^{\beta\dot\beta}\,\frac{\partial}{\partial p^{\beta\dot\alpha}}\,\frac{\partial}{\partial p^{\alpha\dot\beta}}\Big)\,\delta^{(4)}(P)\Big]\,A_n(\lambda_i)$$

$$+ \Big(\frac{\partial}{\partial p^{\beta\dot\alpha}}\delta^4(P)\Big)\sum_{i=1}^{n} \lambda_i^\beta\,\frac{\partial}{\partial\lambda_{\alpha i}}\;A_n(\{\lambda_i\})$$

<u>Use:</u> $\sum_i \lambda_{i\alpha}\partial_{i\beta} = \sum_i \lambda_{i(\alpha}\partial_{i\beta)} + \frac{1}{2}\varepsilon_{\alpha\beta}\sum_i \lambda_i^{\gamma}\partial_{i\gamma}$

$$\left[\begin{array}{ll} \lambda_1\partial_2 = \frac{1}{2}(\lambda_1\partial_2+\lambda_2\partial_1) - (\lambda^1\partial_1+\lambda^2\partial_2)\frac{1}{2} & \lambda^{\alpha} = \varepsilon^{\alpha\beta}\lambda_{\beta} \\ \lambda_2\partial_1 = \frac{1}{2}(\lambda_1\partial_2+\lambda_2\partial_1) + (\lambda^1\partial_1+\lambda^2\partial_2)\frac{1}{2} & \lambda^1 = \lambda_2\,,\ \lambda^2 = -\lambda_1 \end{array}\right.$$

$$\Rightarrow \sum_i \lambda_i^{\gamma}\partial_{i\beta} = \sum_i \varepsilon^{\gamma\alpha}\lambda_{i(\alpha}\partial_{i\beta)} - \frac{1}{2}\varepsilon^{\gamma\alpha}\varepsilon_{\alpha\beta}\sum_i \lambda_i^{\gamma}\partial_{i\gamma}$$

$$= \varepsilon^{\gamma\alpha}\underbrace{\sum_i \lambda_{i(\alpha}\partial_{i\beta)}}_{m_{\alpha\beta}} + \frac{1}{2}\delta^{\gamma}_{\beta}\sum_i \lambda_i^{\delta}\partial_{i\delta}$$

Thus:

$$\sum_{i=1}^{n}\lambda_i^{\beta}\frac{\partial}{\partial\lambda_{i\alpha}}A_n = \frac{1}{2}\delta_{\alpha}^{\beta}\sum_i \lambda_i^{\delta}\partial_{i\delta}A_n = -(n-4)\,\delta_{\alpha}^{\beta}A_n$$

$$\Rightarrow k_{\alpha\dot\alpha}A_n^{MHV} = \left[4\frac{\partial}{\partial p^{\alpha\dot\alpha}}\delta^{(4)}(P) + P^{\beta\dot\beta}\frac{\partial}{\partial P^{\beta\dot\alpha}}\frac{\partial}{\partial P^{\alpha\dot\beta}}\delta^{(4)}(P)\right]$$

$$\times A_n^{MHV}(\{\lambda_i\})$$

☒ Now in a distributional sense $P^{\beta\dot\beta}\frac{\partial}{\partial p^{\alpha\dot\beta}}\frac{\partial}{\partial p^{\dot\alpha\beta}}\delta^{(4)}(P) = -4\frac{\partial}{\partial p^{\alpha\dot\alpha}}\delta^{(4)}(P)$.

<u>Proof:</u>

$$\int d^4p\; F(p)\; P^{\beta\dot\beta}\frac{\partial}{\partial p^{\beta\dot\alpha}}\frac{\partial}{\partial p^{\alpha\dot\beta}}\delta^{(4)}(P)$$

$$= +\int d^4p\left[\frac{\partial}{\partial p^{\beta\dot\alpha}}F(p)\right]2\cdot\delta^{\beta}_{\alpha} + \int d^4p\left[\frac{\partial}{\partial p^{\alpha\dot\beta}}F(p)\right]2\,\delta^{\dot\beta}_{\dot\alpha}$$

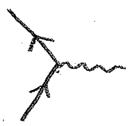
$$= \int d^4p\; 4\left[\frac{\partial}{\partial p^{\alpha\dot\alpha}}F(p)\right]\delta^{(4)}(P) = \int d^4p\, F(p)\left(-4\frac{\partial}{\partial p^{\alpha\dot\alpha}}\delta^{(4)}(P)\right) \square$$

II.5 N=4 super Yang-Mills theory: On-shell superspace and superamplitudes

So far we have discussed pure Yang-Mills theory or massless QCD. Hence our external states were either gluons $(h=\pm 1)$ or quarks $(h=\pm\frac{1}{2})$. A renormalizable QFT in $d=4$ could also contain scalar fields with helicity $h=0$ on the external legs.

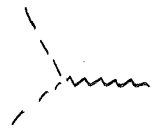
In *all* these gauge field theories with arbitrary fermion and scalar fields the *tree-level* gluon amplitudes are identical to the pure Yang-Mills case.

This is so as scalars or fermions coupled to gauge fields via interactions of the types



$\bar\psi A \psi$

$\bar\psi \psi \phi$

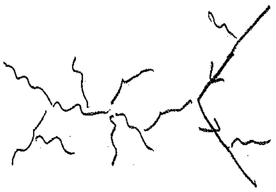


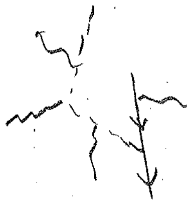
$\phi^2 \partial\cdot A$

$\phi^2 A\cdot A$

ϕ^4

Hence in a diagram with only external gluon legs scalars or fermions neither appear at tree level: They are always produced in pairs from gluon lines and thus have to exit the diagram at tree-level





A *very* *special* gauge theory surpassing all others in its remarkable properties is the *maximally supersymmetric* Yang-Mills theory, or $\mathcal{N}=4$ SYM.

Field content: A_μ^a : gluon $a=1,\dots,N^2-1$

SU(N) g.g.

$\psi^a_{\alpha A}$: 4 gluinos $\alpha=1,2$; $A=1,2,3,4$

$\bar{\psi}^{\dot{\alpha} a A}$: 4 anti-gluinos $\dot{\alpha}=1,2$; $A=1,2,3,4$

$\phi^{a,AB}$: 6 scalars antisymmetric in AB

Properties:

$$(\phi^{AB})^* = \phi_{AB} = \varepsilon_{ABCD}\,\phi^{CD}$$

$$(\psi^a_{\alpha A})^* = \bar{\psi}^{\dot{\alpha} a A}$$

All fields transform in the adjoint representation (opposed to quarks in QCD transforming in the fundamental representation).

Action: Uniquely fixed by supersymmetry!

$$\boxed{\begin{aligned} S = \frac{1}{g_{YM}^2}\int d^4x\ \mathrm{Tr}\Big(&-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(D_\mu\phi_{AB})D^\mu\phi^{AB} \\ &-\frac{1}{2}[\phi_{AB},\phi_{CD}][\phi^{AB},\phi^{CD}] \\ &+i\,\bar{\psi}^{A}_{\dot\alpha}\,\sigma_\mu^{\dot\alpha\alpha}D^\mu\psi_{\alpha A} - \frac{i}{2}\psi^{\alpha}_{A}[\phi^{AB},\psi_{B\alpha}] \\ &-\frac{i}{2}\bar{\psi}^{A}_{\dot\alpha}[\phi_{AB},\bar{\psi}^{\dot\alpha B}]\Big) \end{aligned}}$$

Two free parameters: g_{YM}^2 & N. Is quantum conformal field theory with $\beta_{g_{YM}}=0$ $\Rightarrow$ Conformal symmetry at tree-level survives the quantization and is present also at loop level

$N=4$ SYM is the interacting 4d QFT with highest degree of symmetry!

$\Rightarrow$ "H-atom of gauge theories"

We are interested in tree- and loop level color ordered (or partial) amplitudes in this theory!

☒ ON-SHELL STRUCTURE

8 bosonic and 8 fermionic states:

Bosons:	g_+	g_-	S_{AB}	**Fermions**:	Γ_A	$\bar{\Gamma}^A$
h	$+1$	-1	0		$+1/2$	$-1/2$
# d.o.f.	1	1	6		4	4
Name	gluon		scalar		gluino	anti-gluino
$SU(4)_R$ rep.:	singlet		a/sym (6)		fundamental (4)	anti-fund. ($\bar{4}$)

☒ $N=4$ SYM is unique as all on-shell states comprise a single PCT-self conjugate supermultiplet:

We can describe this multiplet via an on-shell superfield using the Grassman odd parameter η^A $(A=1,2,3,4)$:

☒ ON-SHELL STRUCTURE

8 bosonic and 8 fermionic states:

Bosons:	g_+	g_-	S_{AB}	Fermions:	$\tilde{g}_A$	$\tilde{\tilde{g}}^A$
h	$+1$	-1	0		$+1/2$	$-1/2$
# d.o.f.	1	1	6		4	4
Name	gluon		scalar		gluino	anti-gluino
$SU(4)_R$ rep.:	singlet		a/sym (6)		fundamental (4)	anti-fund. ($\bar{4}$)

☒ $N=4$ SYM is unique as all on-shell states comprise a single PCT-self conjugate supermultiplet:

We can describe this multiplet via an on-shell superfield using the Grassman odd parameter η^A $(A=1,2,3,4)$:

$$\boxed{\begin{aligned}\Phi(\eta) = g_+ + \eta^A \tilde{g}_A + \frac{1}{2!}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C \varepsilon_{ABCD}\tilde{\tilde{g}}^D \\ + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\varepsilon_{ABCD}\, g_-\end{aligned}}$$

(II.12)

We assign helicity $\frac{1}{2}$ to η^A then the superfield $\Phi(\eta)$ has uniform helicity 1:

$$\boxed{h = -\frac{1}{2}\left[-\lambda^\alpha \partial_\alpha + \tilde{\lambda}^{\dot\alpha} \partial_{\dot\alpha} + \eta^A \partial_A\right] \qquad \partial_A := \frac{\partial}{\partial \eta^A}} \qquad (\text{II}.13)$$

$$\Rightarrow \quad h \circ \Phi(\eta) = \Phi(\eta)$$

10 SUPERSYMMETRY

SUSY transformations are generated by $q^{\alpha A}$ and $\bar{q}^{\dot\alpha}{}_A$ with anti-commutator:

$$\{q^{\alpha A}, \bar{q}^{\dot\alpha}{}_B\} = P^{\alpha\dot\alpha}$$

As $P^{\alpha\dot\alpha} = \lambda^\alpha \tilde{\lambda}^{\dot\alpha}$ we have the natural representation:

$$\boxed{q^{\alpha A} = \lambda^\alpha \eta^A \qquad \bar{q}^{\dot\alpha}{}_A = \tilde{\lambda}^{\dot\alpha} \frac{\partial}{\partial \eta^A}} \qquad (\text{II}.14)$$

11 LORENTZ AND R-SYMMETRY

Next to the Lorentz symmetry generators one now also has R-symmetry generators refering to SU(4) rotations in the 'internal' η^A space:

$$\boxed{m_{\alpha\beta} = \lambda_{(\alpha}\partial_{\beta)}\,, \quad \bar{m}_{\dot\alpha\dot\beta} = \tilde{\lambda}_{(\dot\alpha}\partial_{\dot\beta)}\,, \quad r^A{}_B = \eta^A\partial_B - \frac{1}{4}\delta^A_B r^C\partial_C}$$

This enables us to read off the SUSY and R-symmetry transformations of the on-shell fields:

R-symmetry: Transformation parameter $\Lambda_A{}^B$.

$$[\Lambda_A{}^B r^A{}_B, \Phi(\eta)]$$

$$= \Lambda_A{}^B\Big(\eta^A\tilde{g}_B - \eta^A S_{AB} + \frac{1}{2!}\eta^A\eta^C\eta^D\varepsilon_{BCD}\bar{\tilde{g}}^D$$

$$+ \frac{1}{3!}\eta^A\eta^C\eta^D\eta^E\varepsilon_{BCDE}\,g_-\Big)$$

$$\Rightarrow \quad \delta_R g_+ = 0$$

$$\delta_R \tilde{g}_A = \Lambda_A{}^B\tilde{g}_B$$

$$\delta_R S_{AB} = -2\Lambda_A{}^C S_{BC}$$

$\vdots$

[illegible]

and similarly for SUSY-transformation:

$$+\lambda^\alpha\eta^A\eta^B\tilde{g}_B$$

$$[\xi_\alpha\Theta_A q^{A\alpha}, \Phi(\eta)] = \xi_{\alpha A}\Big[\lambda^\alpha\eta^A g_+ \checkmark + \frac{1}{2!}\lambda^\alpha\eta^A\eta^B\eta^C S_{BC}$$

$$+ \frac{1}{3!} \lambda^\alpha \eta^A \eta^B \eta^C \eta^D \varepsilon_{BCDE} \tilde{\bar{q}}^E \Big]$$

$$\overset{!}{=} \delta_q g_+ + \eta^A \delta_q \tilde{g}_A + \frac{1}{2} \eta^A \eta^B \delta_q S_{AB} + \dots$$

$$\Rightarrow \quad \delta_q g_+ = 0$$

$$\delta_q \tilde{g}_A = \langle \xi_A \lambda \rangle g_+$$

$$\delta_q S_{AB} = 2 \langle \xi_A \lambda \rangle \tilde{g}_B$$

$$\vdots$$

And similarly $\delta_q g_+ = [\hat{\lambda}, \xi^A] \tilde{g}_A$

$$\delta_q \tilde{g}_A = [\hat{\lambda}, \xi^B] S_{AB}$$

$$\vdots$$

III CONFORMAL AND SUPER-CONFORMAL SYMMETRY

The known conformal symmetry generator $\boxed{k_{\alpha\dot\alpha} = \partial_\alpha \partial_{\dot\alpha}}$ is augmented by two superconformal partners following from the commutators

$$[k_{\alpha\dot\alpha}, q^{\beta A}] = \delta_\alpha^\beta \bar{S}_{\dot\alpha}^A$$

$$[k_{\alpha\dot\alpha}, \bar{q}^{\dot\beta}{}_A] = \delta^{\dot\beta}_{\dot\alpha} S_{\alpha A}$$

$$\boxed{\begin{aligned} \bar{S}_{\dot\alpha}^A &= \eta^A \partial_{\dot\alpha} \\ S_{\alpha A} &= \partial_\alpha \partial_A \end{aligned}} \qquad (\text{II}.15)$$

The complete super-conformal symmetry algebra reads:

$$\{q^{\alpha A}, \bar{q}^{\dot\alpha}{}_B\} = \delta^A_B\, p^{\alpha\dot\alpha} \qquad \{s_{\alpha A}, \bar{s}_{\dot\alpha}{}^B\} = \delta^B_A\, r_{\alpha\dot\alpha}$$

$$\{q^{\alpha A}, s_{\beta B}\} = m^\alpha{}_\beta\, \delta^A_B + \delta^\alpha_\beta\, r^A{}_B + \frac{1}{2}\delta^\alpha_\beta\, \delta^A_B\, (d+c)$$

$$\{\bar{q}^{\dot\alpha}{}_A, \bar{s}_{\dot\beta}{}^B\} = \bar{m}^{\dot\alpha}{}_{\dot\beta}\, \delta^A_B - \delta^{\dot\alpha}_{\dot\beta}\, r^B{}_A + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\, \delta^B_A\, (d-c)$$

$$[p^{\alpha\dot\alpha}, s_{\beta A}] = \delta^\alpha_\beta\, \bar{q}^{\dot\alpha}{}_A \qquad [p^{\alpha\dot\alpha}, \bar{s}_{\dot\beta}{}^A] = \delta^{\dot\alpha}_{\dot\beta}\, q^{\alpha A}$$

with

$$d = \frac{1}{2}\left[\lambda^\alpha \partial_\alpha + \tilde{\lambda}^{\dot\alpha}\partial_{\dot\alpha} + 1\right] \qquad \text{dilatation}$$

$$c = 1 + \frac{1}{2}\left(\lambda^\alpha\partial_\alpha - \tilde{\lambda}^{\dot\alpha}\partial_{\dot\alpha} - \eta^A\partial_A\right) = 1 - h$$

N.B. $c \circ \Phi(\eta) = 0$.

Together with the bosonic commutation relations:

$$[r_{\alpha\dot\alpha}, p^{\beta\dot\beta}] = \delta_\alpha^\beta\, \delta_{\dot\alpha}^{\dot\beta}\, d + m_\alpha{}^\beta\, \delta^{\dot\beta}_{\dot\alpha} + \bar{m}_{\dot\alpha}{}^{\dot\beta}\, \delta^\beta_\alpha$$

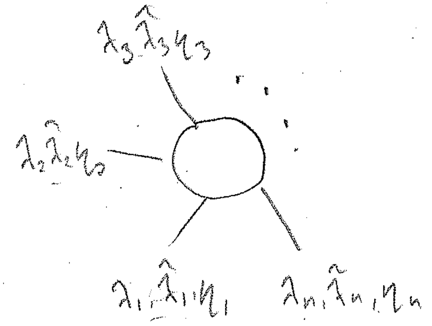
$$[d, p^{\alpha\dot\alpha}] = p^{\alpha\dot\alpha} \qquad [d, r_{\alpha\dot\alpha}] = -r_{\alpha\dot\alpha}$$

as well as $[c, *] = 0$

This is the super-conformal symmetry algebra $psu(2,2|4)$.

SUPERAMPLITUDES

Now we consider color ordered superamplitudes:



$$\mathbb{A}_n(\{\lambda_i, \tilde\lambda_i, \eta_i\}) = \langle \Phi_1(\lambda_1, \tilde\lambda_1, \eta_1) \dots \Phi_n(\lambda_n, \tilde\lambda_n, \eta_n) \rangle$$

These package all possible component fields amplitudes involving gluons, gluinos and scalars into one object. The component level amplitudes may be extracted upon performing the η expansion of $\mathbb{A}_n$:

$\mathbb{A}_n(\{\lambda_i \tilde\lambda_i \eta_i\})$ is polynomial in η_i and contains terms as

$$(\eta_1)^4 (\eta_2)^4 \; A_n(-,-,+\dots+) \qquad \eta^4 := \frac{1}{4!} \epsilon_{ABCD} \eta^A \cdot \eta^D$$

$$(\eta_1)^4 \, \epsilon_{ACDE} \, \eta_2^C \eta_2^D \eta_2^E \; \eta_3^B \; A_n(-, \tilde g^A, \bar{\tilde g}_B, +, \dots +)$$

This follows from the η-expansion (II.12)

We also have

$$h_i \mathbb{A}_n(1, \dots n) = \mathbb{A}_n(1, \dots, n) \qquad \forall \, i \in \{1, \dots, N\}$$

☒ The superamplitudes of $N=4$ SYM are invariant under the superconformal symmetry algebra psu(2,2|4) discussed above.

As in the pure YM case the symmetry generators are simply the sum of the single-particle representations:

$$p^{\alpha\dot\alpha} := \sum_{i=1}^{n} \lambda_i^{\alpha}\tilde\lambda_i^{\dot\alpha} \qquad q^{\alpha A} := \sum_{i=1}^{n} \lambda_i^{\alpha}\eta_i^{A}$$

$$\bar q^{\dot\alpha}{}_{A} := \sum_{i=1}^{n} \tilde\lambda_i^{\dot\alpha}\,\partial_{iA} \qquad \text{etc.}$$

Note that only $p^{\alpha\dot\alpha}$ and $q^{\alpha A}$ act multiplicative, whereas $\{\bar q^{\dot\alpha}{}_A, s_{\alpha A}, \bar s_{\dot\alpha}{}^{A}, m_{\alpha\beta}, \bar m_{\dot\alpha\dot\beta}, d, r^{A}{}_{B}, h_i\}$ are first order differential operators while $k_{\alpha\dot\alpha}$ is a second order differential operator in $\{\lambda_i^{\alpha}, \tilde\lambda_i^{\dot\alpha}, \eta_i^{A}\}$.

☐ GENERAL FORM OF SUPERAMPLITUDES

Momentum and "super-momentum" conservation of superamplitudes

$$P^{\alpha\dot\alpha}\circ A_n(\{\lambda_i,\tilde\lambda_i,\eta_i\}) = 0$$

$$q^{\alpha A}\circ A_n(\{\lambda_i,\tilde\lambda_i,\eta_i\}) = 0$$

require a general form like:

$$\boxed{\mathbb{A}_n(\Phi_1,\dots,\Phi_n) = \frac{\delta^{(4)}(p)\ \delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}\ P_n(\lambda_i,\tilde{\lambda}_i,\eta_i)} \qquad (\text{II}.16)$$

N.B. Grassmann δ-functions

For θ Grassmann variable, we have the integration rules:

$$\int d\theta\ \theta = 1 \qquad \int d\theta\ 1 = 0 \qquad \text{"}\int d\theta = \frac{\partial}{\partial\theta}\text{"}$$

Hence $\delta(\theta) = \theta$ as

$$\int d\theta\ \delta(\theta-\theta_0)\ F(\theta) = \int d\theta\,(\theta-\theta_0)\ [F_0+\theta F_1]$$

$$= \int d\theta\left(-\theta_0 F_0 + \theta\left[F_0+\theta_0 F_1\right]\right) = F_0+\theta_0 F_1 = F(\theta_0).$$

Hence:

$$\delta^{(8)}(q) = \delta^{(8)}\Big(\sum_{i=1}^{n}\lambda_i^\alpha \eta_i^A\Big) = \prod_{\alpha=1}^{2}\prod_{A=1}^{4}\Big(\sum_{i=1}^{n}\lambda_i^\alpha\eta_i^A\Big) \sim \theta(\eta^8)$$

$\Rightarrow$ $\mathbb{A}_n$ has η-expansion starting at order η^8.

$\Rightarrow$ Large classes of component field amplitudes vanish.

In particular:

$$\mathbb{A}_n\big|_{\eta^0} = 0 \quad \Rightarrow \quad A_n^{\text{gluon}}(1^+, 2^+, \ldots, n^+) = 0$$

$$\mathbb{A}_n\big|_{\eta^4} = 0 \quad \Rightarrow \quad A_n^{\text{gluon}}(1^-, 2^+, \ldots, n^+) = 0$$

comes from $\varepsilon_{ABCD}\,\eta_1^A \eta_1^B \eta_1^C \eta_1^D$ term

in expansion of $\mathbb{A}_n$.

$$\Rightarrow \quad A_n^{\phi\phi g^{n-2}}(1_\phi, 2_\phi, 3^+, \ldots, n^+) = 0$$

from $\varepsilon_{ABCD}\,\eta_1^A \eta_1^B \eta_2^C \eta_2^D$ term.

$$\Rightarrow \quad A_n^{\tilde{g}\bar{\tilde{g}} g^{n-2}}(1^+_{\tilde{g}A}, 2^-_{\bar{\tilde{g}}A}, 3^+, \ldots, n^+) = 0$$

from $\varepsilon_{ABCD}\,\eta_1^A \eta_2^B \eta_2^C \eta_2^D$ term

Carries over to following statements also in massless QCD:

$$\boxed{\begin{aligned} &A^{QCD}_{g^n}(1^\pm, 2^+, 3^+, \ldots n^+) = 0 \\ &A^{QCD}_{q\bar{q}g^{n-2}}(1^+_q, 2^+, \ldots, i\text{-}1^+, i^-_{\bar{q}}, i\text{+}1^+, \ldots, n^+) = 0 \end{aligned}} \qquad (\text{II}.17)$$

"Secret (N=1) SUSY in QCD trees".

(Comment: For a single gluino line there can be no internal scalar exchange.)

Back to (II.16): The factor $P_n(\lambda,\tilde\lambda,\eta)$ has expansion in η's, by $SU(4)_R$ symmetry in (η^4):

$$\boxed{P_n(\lambda,\tilde\lambda,\eta) = \underset{\text{MHV}}{\underset{\updownarrow}{P_n^{(0)}}} + \underset{\text{NMHV}}{\underset{\updownarrow}{P_n^{(4)}}} + \underset{\text{N}^2\text{MHV}}{\underset{\updownarrow}{P_n^{(8)}}} + \dots + \underset{\overline{\text{MHV}}}{\underset{\updownarrow}{P_n^{(4n-16)}}}} \qquad (\text{II.18})$$

Where $P^{(n)} \sim \mathcal{O}(\eta^n)$ and $P_n^{(0)} = 1$.

II.6 SUPER BCFW-RECURSION

Want to construct a super version of the BCFW onshell recursion.

Guess: Need to augment the 2 line shifts

$$\boxed{\begin{aligned} \lambda_1 &\to \lambda_1 - z\lambda_n = \hat\lambda_1 \\ \tilde\lambda_n &\to \tilde\lambda_n + z\tilde\lambda_1 = \hat{\tilde\lambda}_n \end{aligned}} \qquad (\text{II.19a})$$

by shift in η_1 or η_n, but which?

Guideline: Conservation of super-momentum

Recall $P_1^{\alpha\dot\alpha} \to \hat P_1^{\alpha\dot\alpha} = \lambda_1\tilde\lambda_1 - z\lambda_n\tilde\lambda_1$ $\Rightarrow \hat P_1 + \hat P_n = P_1 + P_n$

$P_n^{\alpha\dot\alpha} \to \hat P_n^{\alpha\dot\alpha} = \lambda_n\tilde\lambda_n + z\lambda_n\tilde\lambda_1$

$q^{\alpha A} = \sum_{i=1}^{n} \lambda_i^\alpha \eta_i^A$ transforms under λ & $\hat{\lambda}$ shifts as

$$q_1^{\alpha A} \to \hat{q}_1^{\alpha A} = (\lambda_1 - z\lambda_n)\hat{\eta}_1 = q_1^{\alpha A} - z\lambda_n\eta_1$$

$$q_n^{\alpha A} \to \hat{q}_n^{\alpha A} = \lambda_n \hat{\eta}_n = q_n^{\alpha A} + z\lambda_n\eta_1$$

⇒ Natural η-shift then is:

$$\boxed{\begin{aligned}\hat{\eta}_n &= \eta_n + z\eta_1 \\ \hat{\eta}_1 &= \eta_1\end{aligned}} \qquad (\text{II}.19b)$$

Using these shifts one can derive a super version of the on-shell recursion which reads: (Shift legs $\hat{1}$ & $\hat{n}$):

$$\boxed{\begin{aligned}\mathbb{A}_n = \sum_{i=3}^{n-1} \int d^4\eta_{P_i}\ & \mathbb{A}_L(\hat{1}(z_{P_i}), 2, \ldots, i-1, -\hat{P}(z_{P_i})) \cdot \\ & \frac{1}{P_i^2}\ \mathbb{A}_R(\hat{P}(z_{P_i}), i, \ldots, n-1, \hat{n}(z_{P_i}))\end{aligned}} \qquad (\text{II}.20)$$

The sum over intermediate states ($\{+,-\}$ in pure gluon case) is performed in superamplitude formalism via the Grassmann integral $\int d^4\eta_{P_i}$.

Building blocks: 3-Point MHV superamplitude

MHV 3-pt kinematics : $[ij]=0$ but $\langle ij\rangle \neq 0 \quad \forall\, i,j \in \{1,2,3\}$

Then

$$\boxed{\mathbb{A}_3^{MHV} = \frac{\delta^{(4)}(p)\ \delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}} \qquad (\text{II}.21)$$

Unique result for $\mathbb{A}_3 \sim \delta^{(4)}(p)\ \delta^{(8)}(q)$ and uniform helicity requirement $h_i\, \mathbb{A}_3 = \mathbb{A}_3 \quad \forall\, i$.

3-point $\overline{\text{MHV}}$ superamplitude:

By parity:

$$\overline{A_3^{MHV}} = \frac{\delta^{(4)}(p)}{[12][23][31]}\ \delta^{(8)}\Big(\sum_{i=1}^{3} \tilde{\lambda}_i \bar{\eta}_i\Big)$$

As we are dealing with a chiral on-shell superspace initially: $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}$ need to Grassmann Fourier transform this result to η space:

$$\Phi(\eta) = \int d^4\bar{\eta}\ e^{i\eta\bar{\eta}}\ \bar{\Phi}(\bar{\eta})$$

Fourier transform $\bar{\eta}_i \to \eta_i$ for each leg.

Performing this Grassmann-Fourier-Transformation one finds

$$\boxed{\mathbb{A}_3^{\overline{MHV}} = \frac{\delta^{(4)}(p)\;\delta^{(4)}\left(\eta_1^A[23]+\eta_2^A[31]+\eta_3^A[12]\right)}{[12][23][31]}} \qquad (\text{II}.22)$$

→ Ex.

DECOMPOSITION OF THE SUPER BCFW-RECURSION

Decomposing the superamplitude into N^p MHV "(next-to)^p-MHV" subsectors one obtains the following recursive formula:

$$\boxed{\begin{aligned}\mathbb{A}_n^{N^pMHV} = &\int \frac{d^4\eta_P}{P^2}\,\mathbb{A}_3^{\overline{MHV}}(z_P)\,\mathbb{A}_{n-1}^{N^pMHV}(z_P)\\ &+\sum_{m=0}^{p-1}\sum_{i=4}^{n-1}\int\frac{d^4\eta_{P_i}}{P_i^2}\,\mathbb{A}_i^{N^mMHV}(z_{P_i})\,\mathbb{A}_{n-i+2}^{N^{(p-m-1)}MHV}\end{aligned}}$$

Reason: The η-count on the LHS has to equal η-count on RHS. Hence η-count of $\mathbb{A}_L$ and $\mathbb{A}_R$ together has to be 4-times longer than $\mathbb{A}_n$.

III. ONE LOOP STRUCTURE

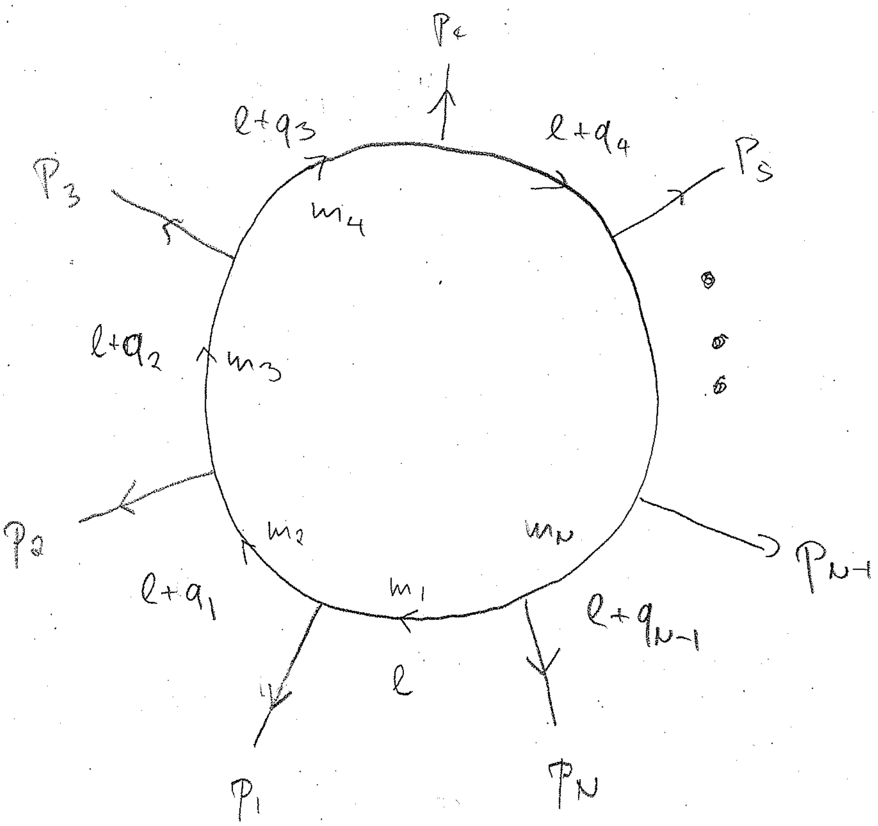
III.1 GENERAL REMARKS

We now turn to the discussion of one-loop graphs in gauge theories. One loop computations generically require the computation of integrals as

$$\boxed{I_N \sim \int \frac{d^4 l}{(2\pi)^4} \frac{N(l)}{\left[l^2 - m_1^2 + i\varepsilon\right]\left[(l+q_1)^2 - m_2^2 + i\varepsilon\right] \cdots \left[(l+q_{N-1})^2 - m_N^2 + i\varepsilon\right]}} \qquad (\text{III}.1)$$

N: # of external lines $\qquad q_j = \sum_{r=1}^{j} p_r \qquad p_j = q_j - q_{j-1}$

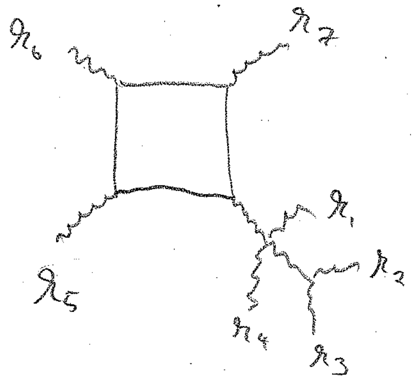
$\sum_{r=1}^{N} p_r = 0$, $N(l)$: Numerator function (polynomial in l^μ)



External lines can be massless or massive $p_i^2 = M_i^2$.

REMARKS

☒ Also for massless theories some or all of the p_i's can in general be massive:

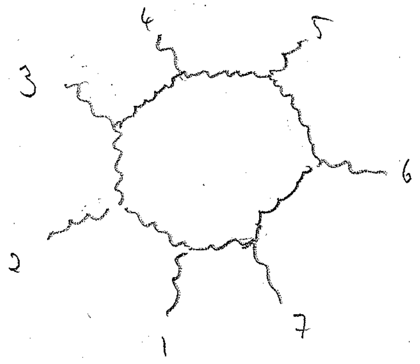


$$\Rightarrow \quad \begin{aligned} p_1 &= k_1+k_2+k_3+k_4 \\ p_2 &= k_5 \\ p_3 &= k_6 \\ p_4 &= k_7 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &p_1^2 \neq 0 \\ &p_2^2 = p_3^2 = p_4^2 = 0 \end{aligned}$$

☒ Also fermion loops yield the form (III.1):

$$\frac{1}{(\ell - m)} = \frac{\ell + m}{\ell^2 - m^2}$$

☒ The maximal degree of $N(\ell)$ ($\hat{=}$ rank of I_N) in a renormalizable QFT is N:



$$\sim \quad I_7 \sim \int \frac{d^4\ell}{(2\pi)^4} \frac{\mathcal{O}(\ell^7)}{\ell^2 (\ell+q_1)^2 \cdots (\ell+q_6)^2}$$

☐ I_N will be UV-divergent for high enough rank r:

I_N is UV-divergent when $r \geq 2N - 4$

by power counting. As $r \leq N$ we see that

UV-divergence appears for: $N \geq 2N-4 \Rightarrow N \leq 4$

$N=4$	$r=4$
$N=3$	$r=3,2$
$N=2$	$r=2,1,0$
$N=1$	$r=1,0$

UV-divergencies.

five and higher-point integrals are UV-finite.

▨ <u>UV-divergencies</u> are conventionally regulated by dimensional regularization: We set $\boxed{D=4-2\varepsilon}$ and let $\varepsilon\to 0$ at the end of the calculation ($\varepsilon>0$).

In particular the loop momentum l^μ becomes a D-dim vector and the integration measure in (III.1) is changed to:

$$\int \frac{d^4 l}{(2\pi)^4} \to \int \frac{d^D l}{(2\pi)^D}$$

▨ <u>IR-divergencies</u> arise for massless propagators ($m_i=0$) when a sufficient number of propagators in I_N go on mass-shell simultaneously introducing non-integrable singularities. These were classified by Landau in 1959. Most important examples are soft and colinear singularities:

soft: $l^\mu \to 0$ collinear: $l^\mu \parallel q_i^\mu$.

IR-divergencies are also completely regulated in dim. reg.

INTEGRAL REDUCTION:

It turns out that for D→4 I_N can be written as a linear combination of one-loop scalar integrals of four-, three-, two- and one point type and a remnant of dim. reg. known as the rational part $\mathcal{R}$:

$$\boxed{I_N = c_{4;j} I_{4;j} + c_{3;j} I_{3;j} + c_{2;j} I_{2;j} + \mathcal{R} + \mathcal{O}(\varepsilon)}$$

(III.2)

Coefficients $c_{N;j}$ are D=4 quantities, $I_{N;j}$ is scalar N pt. integral of type "j". Type "j" refers to distribution of p_i's on the N-legs of $I_{N;j}$.

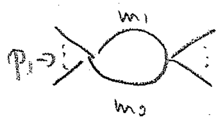
Central result for one-loop computations, will prove this in sequel.

Scalar integrals:

- todpoles:

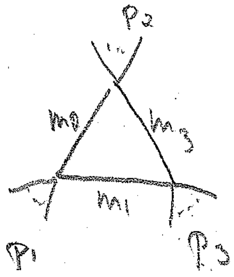
$$I_1(m_1^2) = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{d_1}$$

- bubbles:



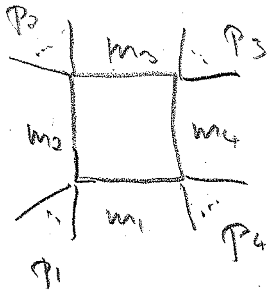
$$I_2(p_1^2, m_1^2, m_2^2) = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{d_1 d_2}$$

- triangles:



$$I_3(p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2) = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{d_1 d_2 d_3}$$

- boxes :



$$I_4(p_1^2,p_2^2,p_3^2,p_4^2;s_{12},s_{23};m_1^2,m_2^2,m_3^2,m_4^2) = \int\frac{d^Dl}{(2\pi)^D}\frac{1}{d_1d_2d_3d_4}$$

where $d_i = (l+q_{i-1})^2 - m_i^2 + i\varepsilon$, $q_n = \sum_{r=1}^{n} p_i$, $q_0 = 0$

$s_{ij} = (p_i+p_j)^2$.

All these integrals are known and tabulated, e.g. 't Hooft & Veltman 1979.

III.2 THE VAN NERVEN - VERMASEREN BASIS

The next goal is to show why the integral reduction formula (III.2) holds true. For this we want to

① Reduce tensor integrals $I_N = \int\frac{d^Dl}{(2\pi)^D}\frac{N(l)}{d_1\dots d_N}$ to scalar integrals $I_N = \int\frac{d^Dl}{(2\pi)^D}\frac{1}{d_1\dots d_N}$

② Reduce scalar integrals with $N>4$ to those with $N=0,1,2,3,4$ as (III.2) states.

① TENSOR INTEGRAL REDUCTION

IDEA: Expand loop momentum l^μ in a basis defined by external region momenta q_i (equivalently one could also take the inflowing momenta)

2d-example: q_1^ν & q_2^ν

In principle $l^\mu = c_1 q_1^\mu + c_2 q_2^\mu$ but $c_i \neq (l\cdot q_i)$ as $q_1 \cdot q_2 \neq 0$ (no ONB). An orthonormal basis is obtained by:

Schouten's identity: $l^\mu \varepsilon^{\nu_1 \nu_2} = l^{\nu_1} \varepsilon^{\mu \nu_2} + l^{\nu_2} \varepsilon^{\nu_1 \mu}$

$$\Rightarrow \quad l^\mu \varepsilon^{q_1 q_2} = (q_1 \cdot l)\, \varepsilon^{\mu q_2} + (q_2 \cdot l)\, \varepsilon^{q_1 \mu}$$

(with useful notation $\varepsilon^{\mu \nu_1} q_{2,\nu_1} = \varepsilon^{\mu q_2}$). Thus

$$\boxed{l^\mu = (q_1 \cdot l)\, v_1^\mu + (q_2 \cdot l)\, v_2^\mu} \qquad (\text{III}.3)$$

where
$$\boxed{v_1^\mu = \frac{\varepsilon^{\mu q_2}}{\varepsilon^{q_1 q_2}} \quad \& \quad v_2^\mu = \frac{\varepsilon^{q_1 \mu}}{\varepsilon^{q_1 q_2}}}$$

Now $(q_i \cdot v_j) = \delta_{ij}$ but $(q_i \cdot q_j) \neq \delta_{ij} \neq (v_i \cdot v_j)$

Importantly the coefficients $(\ell \cdot q_i)$ can be expressed by linear combinations of inverse propagators and external scalars:

$$\ell \cdot q_i = \frac{1}{2}\Big[\big((\ell+q_i)^2 - m_{i+1}^2 + i\varepsilon\big) - \big(\ell^2 - m_i^2 + i\varepsilon\big)\Big]$$

$$-(q_i^2 - m_{i+1}^2) - m_i^2$$

$$= \frac{1}{2}\left[d_{i+1} - d_i - (q_i^2 - m_{i+1}^2) + m_i^2\right]$$

This reduces any tensor integral to a scalar integral.

<u>Caveat</u>: The vector ℓ^μ is D-dimensional!

Rewrite (II.3) in a D-independent way:

$$v_1^\mu = \frac{\varepsilon_{q_1 q_2}\, \varepsilon^{\mu q_2}}{\varepsilon_{q_1 q_2}\, \varepsilon^{q_1 q_2}} \qquad v_2^\mu = \frac{\varepsilon_{q_1 q_2}\, \varepsilon^{q_1 \mu}}{\varepsilon_{q_1 q_2}\, \varepsilon^{q_1 q_2}}$$

with

$$\varepsilon^{\mu_1 \mu_2}\, \varepsilon_{\nu_1 \nu_2} = \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2} - \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1} = \det(\delta^\mu_\nu) =: \delta^{\mu_1 \mu_2}_{\nu_1 \nu_2}$$

we find

$$\boxed{\begin{array}{l} v_1^\mu = \dfrac{\delta^{\mu q_2}_{q_1 q_2}}{\Delta_2} \qquad v_2^\mu = \dfrac{\delta^{q_1 \mu}_{q_1 q_2}}{\Delta_2} \\[2ex] \Delta_2 = \delta^{q_1 q_2}_{q_1 q_2} = q_1^2 q_2^2 - (q_1 \cdot q_2)^2 \end{array}}$$

This may be embedded in a $D = 2 + d$ dim. space.

von Neerven - Vermaseren basis in general dims

Let $\boxed{D = d_p + d_t}$ and construct basis out of $q_1^\mu, q_2^\mu, \ldots, q_{d_p}^\mu$ vectors:

$$\boxed{v_i^{\mu}(q_1, \ldots, q_{d_p}) := \frac{\delta^{q_1 \cdots q_{i-1}\, \mu\, q_{i+1} \cdots q_{d_p}}_{q_1 \cdots q_{i-1}\, q_i\, q_{i+1} \cdots q_{d_p}}}{\Delta(q_1 \cdots q_{d_p})} \qquad i = 1, \ldots, d_p} \tag{III.4}$$

with the generalized Kronecker symbol:

$$\delta^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_n} = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \delta^{\mu_1}_{\nu_2} & \ldots & \delta^{\mu_1}_{\nu_n} \\ \delta^{\mu_2}_{\nu_1} & \delta^{\mu_2}_{\nu_2} & \ldots & \delta^{\mu_2}_{\nu_n} \\ \vdots & & & \\ \delta^{\mu_n}_{\nu_1} & \delta^{\mu_2}_{\nu_2} & \ldots & \delta^{\mu_n}_{\nu_n} \end{vmatrix}$$

(completely a/sym in $[\mu_1 \ldots \mu_n]$ and $[\nu_1 \ldots \nu_n]$).

and the usual notation $\delta^{p_1 \mu_2 \ldots \mu_n}_{\nu_1 q_2 \nu_3 \ldots \nu_n} := \delta^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_n} p_{1\mu_1} p_2^{\nu_2}$

This is a dual basis for the d_p dim. subspace:

$$v_i \cdot q_j = \delta_{ij} \qquad \text{for } i, j \in \{1, \ldots, d_p\}$$

Projection operator on transverse space of dim. d_t:

$$\Lambda_\mu{}^\nu(q_1, \ldots, q_{d_p}) := \frac{\delta^{q_1 \cdots q_{d_p} \nu}_{q_1 \cdots q_{d_p} \mu}}{\Delta(q_1 \cdots q_{d_p})}$$

obeys: $\Lambda_\mu^{\mu} = d_t \,, \quad \Lambda_\mu^{\nu} q_{i,\nu} = 0 \,, \quad \Lambda_\mu^{\nu} \Lambda_\nu^{\rho} = \Lambda_\mu^{\rho} \,.$

The transverse space is then spanned by $\boxed{n^\mu_\tau \,, \ \tau = 1, \dots, d_t}$ satisfying the orthonormality conditions:

$$\Lambda^{\mu\nu} = \sum_{\tau=1}^{d_t} n^\mu_\tau n^\nu_\tau \qquad n_\tau \cdot n_s = \delta_{\tau s} \,, \quad q_i \cdot n_\tau = 0 \,, \quad v_i \cdot n_\tau = 0 .$$

If the transverse space is one dimensional one has

$$n^\mu = \frac{\varepsilon^{q_1 \dots q_{d_p} \mu}}{\sqrt{\Delta(q_i)}} \,.$$

Hence we have the general expansion:

$$\boxed{\ell^\mu = \sum_{i=1}^{d_p} (\ell \cdot q_i)\, v_i^\mu + \sum_{\tau=1}^{d_t} (\ell \cdot n_\tau)\, n_\tau^\mu} \qquad (\text{III}.5)$$

with $\ell \cdot q_i = \frac{1}{2} \left[d_{i+1} - d_1 - (q_i^2 - m_{i+1}^2) - m_1^2 \right]$

III.3 INTEGRAND REDUCTION IN D-DIMENSIONS

A generic one-loop tensor integral of rank r and n-points has the integrand:

$$\boxed{\frac{\prod_{i=1}^{r} u_i \cdot \ell}{d_1 d_2 \dots d_n}} \qquad r \leq n \qquad (\text{III}.6)$$

u_i^μ are r 4d vectors made from external momenta and polarizations. In particular they lie in a 4d subspace of the general D-dim regulating space of l^μ.

i) $\underline{n \geq 5}$: $\underline{\text{Tensor}(r,n) \to \text{Scalar}(n' \leq n)}$

Use 4 linearly independent region momenta $q_1^\mu, q_2^\mu, q_3^\mu, q_4^\mu$ to span vN-V basis v_i^μ

$$l^\mu = \sum_{i=1}^{4} (l \cdot q_i)\, v_i^\mu + (l \cdot n_\varepsilon)\, n_\varepsilon^\mu$$

Since $u_i \cdot n_\varepsilon = 0$ (u_i's have no dep. on extra dims)

we have

$$(u_i \cdot l) = \sum_{j=1}^{4} (l \cdot q_j)(v_j \cdot u_i)$$

$$= \frac{1}{2} \sum_{j=1}^{4} \left[d_{j+1} - q_j^2 + m_{i+1}^2 \right] (v_j \cdot u_i)$$

$$- \frac{1}{2}(d_i + m_i^2) \sum_{j=1}^{4} (v_j \cdot u_i)$$

As u_i and v_i are independent of loop momentum l^μ we see that (III.6) can be completely reduced to sums of scalar integrands with $n' \leq n$ points.

ii) Scalar $(n>5)$ → Scalar $(n=5)$

Consider a scalar integrand $I_n = \prod_{i=1}^{n} \frac{1}{d_i}$ with $n>5$ and $d_i = (l+q_{i-1})^2 - m_i^2 + i\varepsilon$. Then there is a non-trivial solution to the five equations for the n α_i

$$\sum_{i=1}^{n} \alpha_i = 0 \qquad \sum_{i=1}^{n} \alpha_i q_{i-1}^{\mu} = 0$$

With this solution $$\sum_{i=1}^{n} \alpha_i d_i = \sum_{i=1}^{n} \alpha_i (l^2 + 2 l\cdot q_{i-1} + q_{i-1}^2 - m_i^2 + i\varepsilon)$$

$$= \sum_{i=1}^{n} \alpha_i (q_{i-1}^2 - m_i^2 + i\varepsilon)$$

$$\Rightarrow \quad 1 = \frac{\sum_{i=1}^{n} \alpha_i d_i}{\sum_{i=1}^{n} \alpha_i (q_{i-1}^2 - m_i^2 + i\varepsilon)}$$

N.B: α_i & denominator is l-independent.

$$\Rightarrow \quad I_n \cdot 1 = \sum_{i=1}^{n} \left(\prod_{j\neq i} \frac{1}{d_j} \right) \frac{1}{\sum_{r=1}^{n} \alpha_r (q_{r-1}^2 - m_r^2)} = \sum_{r} c_r I_{n-1,r}$$

Iterate this procedure to reduce any scalar $n \geq 6$ point integrand to the scalar pentagon graph.

Recap: Have shown that any one-loop integrand can be reduced to

$$I_N \sim I_5^{\text{Scalar}} + \sum_{N=1}^{4} I_N^{\text{tensor}} \, .$$

iii) Tensor reduction for I_N $(N \leq 4)$:

Let us start with the 4-point integral of maximal rank $r=4$:

$$I_{4,4}^{\text{tens}} = \int \frac{d^D l}{(2\pi)^D} \frac{N_4(l)}{d_1 d_2 d_3 d_4} \qquad N_4(l) = \prod_{i=1}^{4} (u_i \cdot l)$$

where the u_i depend on external 4d data (momenta, polarizations).

The physical space of this integral is $d_p = 3$ dimensional (spanned by region momenta $q_1, q_2 q_3$) while the transverse space is $d_t = 1 - 2\varepsilon$ dimensional:

$$\boxed{l^\mu = \sum_{i=1}^{3} v_i^\mu (l \cdot q_i) + n_4^\mu (l \cdot n_4) + (l \cdot n_\varepsilon) n_\varepsilon^\mu} \qquad (\text{III}.6)$$

$$\Rightarrow \quad u_i \cdot l = \sum_{j=1}^{3} (u_i \cdot v_j)(l \cdot q_j) + (u_i \cdot u_4)(l \cdot n_4)$$

As $2l \cdot q_j = d_j - d_i + \text{const}$ the first term above reduce either tensor or point rank of $I_{4,4}$ and may be considered reduced. Keeping only the 4-point integrals we reduce to

$$\left[\prod_{i=1}^{3} (u_i \cdot l)(u_4 \cdot l) \rightarrow \sum_{i=1}^{4} d_i \, (l \cdot n_4)^i + \mathcal{O}(d_1, \ldots d_4) \right]$$

Next we square (III.6) using $v_i \cdot n_j = 0$ & $n_4 \cdot n_\epsilon = 0$ and

$\ell^2 = d_i^2 + m_i^2$, $2\ell\cdot q_i = d_i - d_1 + \text{const}$ to find

$$(\ell\cdot n_4)^2 = -(\ell\cdot n_\epsilon)^2 + \mathcal{O}(d_1,\dots,d_4) + \text{const.}$$

Hence :

$$\boxed{\begin{aligned} N_4(\ell) = {} & \hat{d}_0 + \hat{d}_1(\ell\cdot n_4) + \hat{d}_0(\ell\cdot n_\epsilon)^2 + \hat{d}_3(\ell\cdot n_4)(\ell\cdot n_\epsilon)^2 \\ & + \hat{d}_4(\ell\cdot n_\epsilon)^4 \end{aligned}} \qquad (\text{III}.7)$$

☐ Similar considerations for the 3-point integrals using its $d_P = 2$ dimensional physical space basis v_1^μ & v_2^μ with

$$\ell^\mu = \sum_{i=1}^{2} v_i^\mu(\ell\cdot q_i) + n_3^\mu(\ell\cdot n_3) + n_4^\mu(\ell\cdot n_4) + n_\epsilon^\mu(\ell\cdot n_\epsilon)$$

and the dependence:

$$(\ell\cdot n_3)^2 + (\ell\cdot n_4)^2 = -(\ell\cdot n_\epsilon)^2 + \mathcal{O}(d_1,d_2,d_3) + \text{const}$$

allows one to reduce

$$\boxed{\begin{aligned} N_3(\ell) = \prod_{j=1}^{3}(w_j\cdot\ell) = {} & \hat{c}_0 + \tilde{c}_1(\ell\cdot n_3) + \tilde{c}_2(\ell\cdot n_3) \\ & + \hat{c}_3(\ell\cdot n_\epsilon)^2 + \tilde{c}_4\left((\ell\cdot n_3)^2 - \ell\cdot n_4)^2\right) \\ & + \hat{c}_5(\ell\cdot n_3)(\ell\cdot n_4) + \hat{c}_6(\ell\cdot n_3)^3 \\ & + \tilde{c}_7(\ell\cdot n_4)^3 + \tilde{c}_8(\ell\cdot n_4)^2(\ell\cdot n_\epsilon) + \tilde{c}_9(\ell\cdot n_3)^2(\ell\cdot n_\epsilon) \end{aligned}} \qquad (\text{III}.8)$$

Similar results hold for two-point and one-point integrals (see Kunszt, Ellis, Melnikov, Zanderighi, review ...)

III.4 HIGHER DIMENSIONAL LOOP-MOMENTUM INTEGRATION

Most terms in the expansions (III.7) and (III.8) do not contribute to the integrated I_N.

Consider the 4-point case:

$$I_4^{\text{tens.}} = \int \frac{d^{d_p+d_t}\ell}{(2\pi)^{d_p+d_t}} \frac{1}{d_1 d_2 d_3 d_4} \Big[\hat{d}_0 + \hat{d}_1 (\ell\cdot n_4) + \hat{d}_2 (\ell\cdot n_\varepsilon)^2 + \hat{d}_3 (\ell\cdot n_4)(\ell\cdot n_\varepsilon) + \hat{d}_4 (\ell\cdot n_\varepsilon)^4 \Big]$$

with
$$d_i = (\ell+q_i)^2 - m_{i+1}^2$$
$$= \ell_\perp^2 + (\ell_\parallel + q_i)^2 - m_{i+1}^2$$

where
$$\ell_\perp^\mu = n_4^\mu (\ell\cdot n_4) + n_\varepsilon^\mu (\ell\cdot n_\varepsilon)$$
$$\ell_\parallel^\mu = \sum_{i=1}^{3} v_i^\mu (\ell\cdot q_i)$$

as $q_i\cdot n_j = 0 = q_i\cdot n_\varepsilon$.

Now note that $N_4(\ell) = N_4(\ell_\perp)$, which is a general property also holding for the lower point tensor integrals.

But then the integration over the transverse space simplifies according to:

$$I_n = \int \frac{d^{d_t} l_\perp \, d^{d_p} l_{\parallel}}{(2\pi)^D} \; F(l_\perp^2 ; l_{\parallel}^{\mu}) \; N_n(l_\perp^{\mu})$$

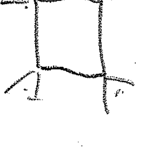
by rotational symmetry in the transverse space we have:

$$\int d^{d_t} l_\perp \; F(l_\perp^2 ; l_{\parallel}^{\mu}) \begin{pmatrix} l_\perp^{\mu_1} \\ l_\perp^{\mu_1} l_\perp^{\mu_2} \\ l_\perp^{\mu_1} l_\perp^{\mu_2} l_\perp^{\mu_3} \\ l_\perp^{\mu_1} \dots l_\perp^{\mu_4} \end{pmatrix} =$$

$$\int d^{d_t} l_\perp \; F(l_\perp^2 ; l_{\parallel}^{\mu}) \begin{pmatrix} 0 \\ l_\perp^2 \cdot \eta_\perp^{\mu_1 \nu_2} \cdot c_1 \\ 0 \\ (l_\perp^2)^2 \big(\eta_\perp^{\mu_1 \mu_2} \eta_\perp^{\mu_3 \mu_4} + \eta_\perp^{\mu_1 \mu_3} \eta_\perp^{\mu_2 \mu_4} \\ \quad + \eta_\perp^{\mu_1 \mu_4} \eta_\perp^{\mu_2 \mu_3} \big) c_2 \end{pmatrix}$$

Now as $n_4 \cdot n_\varepsilon = 0$ we obtain for the 4-point tensor integral: the reduction to the 'box' and a contribution to $\mathcal{R}$:

$$I_4^{\text{tensor}} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{d_1 d_2 d_3 d_4} \Big[\hat{d}_0 + n_\varepsilon^2 c_1 \hat{d}_2 \, l_\perp^2 + \\ + (n_\varepsilon^2)^2 c_2 \hat{d}_4 (l_\perp^2)^2 \Big]$$

$= \hat{d}_0 \cdot$  $+ \; \varepsilon \cdot \text{Const} \cdot \hat{d}_2 \int \frac{d^D l}{(2\pi)^D} \frac{l_\perp^2}{d_1 d_2 d_3 d_4}$

$$+ \; \varepsilon^2 \cdot \text{Const} \cdot \hat{d}_4 \int \frac{d^D l}{(2\pi)^D} \frac{(l_\perp^2)^2}{d_1 d_2 d_3 d_4}$$

Similarly:

$$I_3^{\text{tensor}} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{d_1 d_2 d_3} \left[\tilde{c}_0 + \tilde{c}_3\, c_1\, n_\varepsilon^2\, l_\perp^2 \right]$$

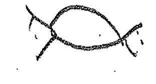
$= \tilde{c}_0 \cdot$  $+ \varepsilon \cdot \text{const} \cdot \tilde{c}_3 \int \frac{d^D l}{(2\pi)^D} \frac{l_\perp^2}{d_1 d_2 d_3}$

and

$$I_2^{\text{tensor}} = \int \frac{d^D l}{(2\pi)^D} \frac{1}{d_1 d_2} \Big[\hat{b}_0 + (\text{terms linear in } l)$$

$$\left.\begin{array}{l} + \hat{b}_4 \left[(l\cdot n_2)^2 - (l\cdot n_4)^2 \right] \\ + \hat{b}_5 \left[(l\cdot n_3)^2 - (l\cdot n_4)^2 \right] \\ + \hat{b}_6 \, (l\cdot n_2)(l\cdot n_3) \\ + \hat{b}_7 \, (l\cdot n_3)(l\cdot n_4) \\ + \hat{b}_8 \, (l\cdot n_2)(l\cdot n_4) \end{array}\right\} \text{traceless}$$

$$+ \hat{b}_9 \, (l\cdot n_\varepsilon)^2$$

$= \hat{b}_0 \cdot$  $+ \varepsilon \cdot \text{const} \cdot \hat{b}_9 \cdot \int \frac{d^D l}{(2\pi)^D} \frac{l_\perp^2}{d_1 d_2}$

The $O(\varepsilon)$ terms contribute to the aforementioned rational part: Are naively zero for $\varepsilon \to 0$ unless the remaining integrals are divergent.

□ Similar arguments allow one to show that

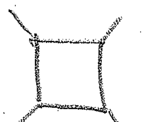
$$I_5^{\text{scalar}} = \sum_{i=1}^{4} c_i \, I_{4,i}^{\text{scalar}} + \mathcal{O}(\epsilon) \qquad \rightarrow \quad \text{III.17.5}$$

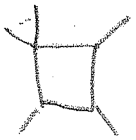
→ not shown here due to time. See in original work: Vermaseren & van Neerven, PLB 137 (1984) 241.

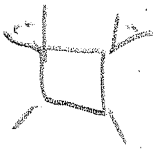
SUMMARY:

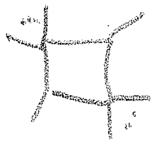
Have shown that a general one-loop amplitude is expandable in terms of bubbles, triangles, boxes and a rational part. Massive theories also contain tadpoles.

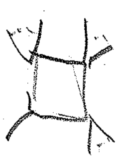
Types of integrals:

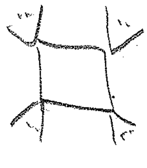


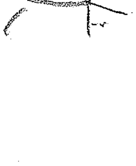












<u>Explicitly:</u> $E_{01234} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{d_0 d_1 \cdots d_4}$ $\quad D_{1234} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{d_1 \cdots d_4}$

$$d_i = (l + q_i)^2 - m_i^2 + i\varepsilon$$

$$r_i = q_i^2 - m_i^2 + m_0^2 \qquad w^\mu = \sum_{i=1}^{4} r_i v_i^\mu$$

$$\Delta_4 = \delta^{q_1 q_2 q_3 q_4}_{q_1 q_2 q_3 q_4} \qquad v_1^\mu = \varepsilon^{\mu q_2 q_3 q_4} \text{ etc.}$$

$$\Rightarrow E_{01234} = \frac{1}{w^2 - 4\Delta_4 m_0^2} \Big[D_{1234} \left(2\Delta_4 - w\cdot(v_1 + v_2 + v_3 + v_4) \right)$$

$$+ D_{0234} v_1 \cdot w + D_{0134} v_2 \cdot w + D_{0124} v_3 \cdot w$$

$$+ D_{0123} v_4 \cdot w \Big]$$

van Neerven + Vermaseren '84.

III. 5 UNITARITY

Optical theorem in QFT: Unitarity of the S-Matrix is a fundamental aspect of any QFT $\Leftrightarrow$ conservation of probability.

$$\hat{S}\hat{S}^{+} = \mathbb{1} \qquad \text{with} \qquad \hat{S} = \mathbb{1} + i\,\hat{T}$$

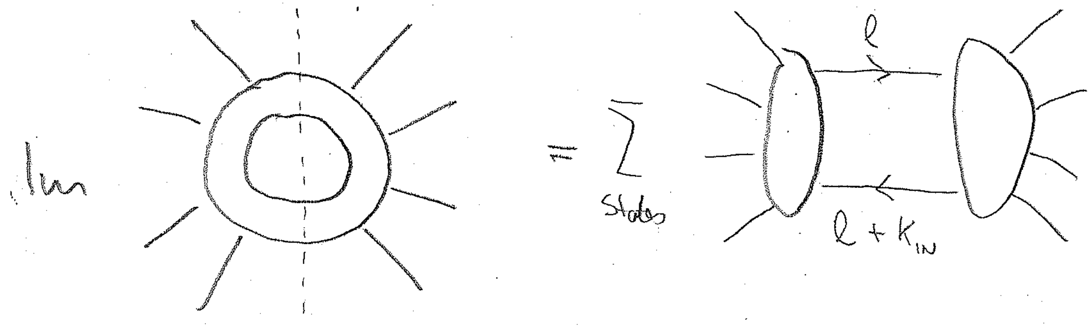
$\hat{T}$: Non-forward part of the scattering matrix, perturbatively evaluated

$$\hat{T} = g\,\hat{T}^{(\text{tree})} + g^2\,\hat{T}^{(\text{1-loop})} + \ldots$$

$$\hat{S}\hat{S}^{+} = \mathbb{1} \Rightarrow \boxed{-i(\hat{T} - \hat{T}^{+}) = \hat{T}\hat{T}^{+}} \qquad (\text{IV}.9)$$

$$\Rightarrow \quad \text{Im}\,\hat{T}^{(\text{1-LOOP})} = \hat{T}^{(\text{tree})} \cdot \hat{T}^{(\text{tree})}$$

$$\text{Im}\,\langle \text{OUT}|\,\hat{T}^{(\text{1-loop})}\,|\text{IN}\rangle = \sum_n \langle \text{OUT}|\,\hat{T}^{(\text{tree})}\,|n\rangle\langle n|\,T^{(\text{tree})}\,|\text{IN}\rangle$$



Want to use this relation to determine coefficients c_i in

$$A_{\text{1-Loop}} = \sum_i c_i\,I_i \qquad I_i\text{: Loop basis integrals}$$

COMMENTS:

☐ LHS corresponds to discontinuity in the scattering amplitude: a branch cut in the complex plane

☐ RHS may be obtained from one-loop amplitude by <u>cutting</u> two propagators in a given channel and putting legs on-shell.

$$\frac{1}{p^2+i\varepsilon} \to 2\pi\,\delta(p^2)$$

☐ Use this relation to find the coefficients c_j in

$$\operatorname{Im} A_n^{1\text{-loop}} = \sum_{j\in B} c_j \operatorname{Im} I_j$$

by fusing trees.

EXAMPLE

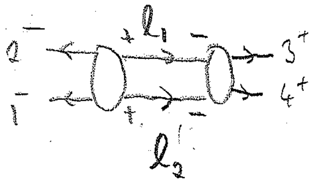
Let us look at the simplest example: $A_4^{\text{one-loop}}(1^-,2^-,3^+,4^+)$ in $N=4$ SYM theory.

Generally we can write this amplitude as

$$A_4^{\text{one-loop}} = A_4^{\text{tree}} \cdot c \cdot \int \frac{d^Dl}{(2\pi)^D} \frac{1}{l^2(l-p_1)^2(l-p_1-p_2)^2(l+p_4)^2}$$

Example:

$A(1^-,2^-,3^+,4^+)_{1\text{-loop}}$ cut in (1,2) channel



$$l_2 + l_1 = p_3 + p_4 = -p_1 - p_2$$

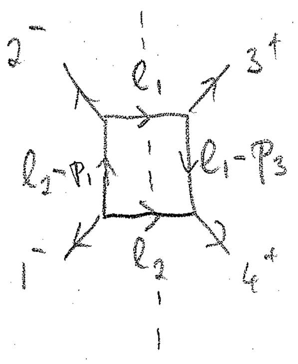
$$\text{Im}\, A_{1\text{-loop}} = \int \frac{d^4 l_1}{(2\pi)^4}\, \delta(l_1^2)\, \delta((l_1+K)^2)\; A_4^{\text{tree}}(1^-, 2^-, l_1^+, l_2^+)$$
$$A_4^{\text{tree}}(-l_1^-, 3^+, 4^+, -l_2^-)$$

Integrand:

$$\cancel{i}^2 \;\frac{\langle 1,2\rangle^4}{\langle 12\rangle\langle 2l_1\rangle\cancel{\langle l_1 l_2\rangle}\langle l_2 1\rangle}\;\frac{\langle l_1 l_2\rangle^{\cancel{4}\,2}}{\langle l_1 3\rangle\langle 34\rangle\langle 4,l_2\rangle\cancel{\langle l_2 l_1\rangle}} \qquad (\text{III}.10)$$

$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\;\frac{\langle 23\rangle\langle 41\rangle\;\langle l_1 l_2\rangle^2}{\langle 2 l_1\rangle\langle l_2 1\rangle\langle l_1 3\rangle\langle 4 l_2\rangle}$$

Relation to cut box: What do we expect?



$$\sim \frac{1}{(l_2-p_1)^2}\,\frac{1}{(l_1-p_3)^2} = \frac{1}{\langle l_2 1\rangle[1 l_2]}\;\frac{1}{\langle l_1 3\rangle[3 l_1]}$$

Thus:

$$\frac{\langle 23\rangle\langle 41\rangle\langle l_1 l_2\rangle^2}{\langle 2l_1\rangle\underline{\langle l_2 1\rangle}\underline{\langle l_1 3\rangle}\langle 4 l_2\rangle}\;\frac{[1 l_2][3 l_1]}{\underline{[1 l_2]}\;\underline{[3 l_1]}} = \frac{\langle 23\rangle\langle 41\rangle\;\overset{p_3+p_4-l_2}{\overset{/\!/}{[3 l_1]}}\langle l_1 l_2\rangle\;\overset{-l_1-p_1-p_2}{\overset{/\!/}{[1 l_2]}}\langle l_2 l_1\rangle}{\langle 2 l_1\rangle\langle 4 l_2\rangle\;(l_2-p_1)^2\,(l_1-p_3)^2}$$

$$= \frac{\langle 23\rangle \overbrace{\langle 41\rangle [34]\langle 4\cancel{\ell_2}\rangle [12]}^{-p_2-p_3-p_4} \langle 2\cancel{\ell_1}\rangle (-)}{\langle 2\cancel{\ell_1}\rangle \langle 4\cancel{\ell_2}\rangle \, (\ell_2-p_1)^2 (\ell_1-p_3)^2} = + \frac{\langle 23\rangle\langle 43\rangle [34][32]}{(\ell_2-p_1)^2(\ell_1-p_3)^2}$$

$$= \frac{(p_3+p_4)^2 \, (p_2+p_3)^2}{(\ell_2-p_1)^2 \, (\ell_1-p_3)^2}$$

Hence:

$$\mathrm{Im}\, A_{1\text{-Loop}}\Big|_{s\text{-channel}} = A^{tree} \cdot (p_1+p_2)^2 (p_1+p_4)^2 \cdot \int \frac{d^D\ell}{(2\pi)^D} \, \delta(\ell^2) \, \delta((\ell-p_3-p_4)^2) \frac{1}{(\ell-p_3)^2} \frac{1}{(\ell-p_3-p_4-p_1)^2}$$

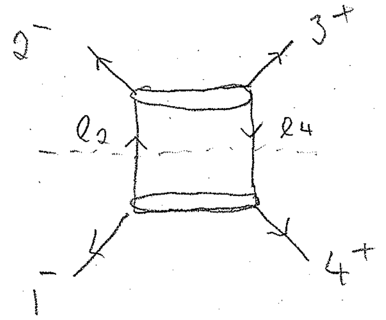
$$= A^{tree} \, (p_1+p_2)^2 (p_1+p_4)^2 \, I_4(s,t)\Big|_{1,2\text{-cut}}$$

Lifting the cut we deduce:

$$A^{1\text{-Loop}}_{(4)} = A_4^{tree} \cdot (p_1+p_2)^2 (p_1+p_4)^2 \, I_4(s,t) + \text{triangles} + \text{bubbles}.$$

True for any gauge theory 4-gluon amplitudes.

Now consider the t-channel cut:



Now we can have all particles (gluons, gluinos & scalars) crossing the cut. The relevant tree amps are:

$$A_4^{\text{tree}}(-\ell_2^{-h}, 2^-, 3^+, \ell_4^h) = i \, \frac{\langle -\ell_2 2\rangle^{2+2h} \langle \ell_4 2\rangle^{2-2h}}{\langle -\ell_2 2\rangle\langle 23\rangle\langle 3\ell_4\rangle\langle \ell_4, -\ell_2\rangle}$$

$$A_4^{\text{tree}}(-\ell_4^{-h}, 4^+, 1^-, \ell_2^h) = i \, \frac{\langle -\ell_4 1\rangle^{2+2h} \langle \ell_2 1\rangle^{2-2h}}{\langle -\ell_4 4\rangle\langle 41\rangle\langle 1\ell_2\rangle\langle \ell_2 \ell_4\rangle}$$

In this channel h can take the values $\{-1, -1/2, 0, 1/2, 1\}$ and work in a prescription $|-p\rangle = |p\rangle$, $|-p] = -|p]$, $-|p]\langle p| = |-p]\langle p|$:

$$A_4 \otimes A_4 = \frac{-i}{\langle \ell_2 2\rangle\langle 23\rangle\langle 3\ell_4\rangle\langle \ell_4\ell_2\rangle} \, \frac{i}{\langle \ell_4 4\rangle\langle 41\rangle\langle 1\ell_2\rangle\langle \ell_2\ell_4\rangle} \cdot N$$

$$\begin{aligned} N = {} & \langle \ell_2 2\rangle^4 \langle \ell_4 1\rangle^4 + \textcircled{4}\langle \ell_2 2\rangle^3\langle \ell_4 2\rangle\langle \ell_4 1\rangle^3\langle \ell_2 1\rangle \\ & + \textcircled{6}\langle \ell_2 2\rangle^2\langle \ell_4 2\rangle^2\langle \ell_4 1\rangle^2\langle \ell_2 1\rangle^2 + \textcircled{4}\langle \ell_2 2\rangle\langle \ell_4 2\rangle^3\langle \ell_4 1\rangle\langle \ell_2 1\rangle^3 \\ & + \langle \ell_4 2\rangle^4\langle \ell_2 1\rangle^4 \end{aligned}$$

$\bigcirc$: $N=4$ SYM field content.

$$= \left(\langle \ell_2 2\rangle\langle \ell_4 1\rangle - \langle \ell_4 2\rangle\langle \ell_2 1\rangle\right)^4 \underset{\text{Schouten}}{=} \langle 12\rangle^4\langle \ell_4\ell_2\rangle^4$$

But this is identical to the result (III.10)

From this we learn:

$$\mathrm{Im}\, A_4^{\text{1-loop}}\Big|_{t\text{-channel}} = A_4^{\text{tree}} \cdot s \cdot t \cdot I_4(s,t)\Big|_{t\text{-channel}}$$

Hence there are no triangles or bubbles in the full answer, as these would have been detected in the t-channel: We have proven that:

$$\boxed{A_4^{\text{1-loop}}(1^-,2^-,3^+,4^+) = A_4^{\text{tree}}(1^-,2^-,3^+,4^+) \cdot s t \cdot I_4(s,t)}$$

$$s=(p_1+p_2)^2 \;, \quad t=(p_2+p_3)^2 .$$

In $\underline{\text{QCD}}$ the t-channel analysis would have differed by the terms

$$A_4 \otimes A_4\Big|_{t\text{-chann}} = (N=4 \text{ SYM result})$$

$$- [(4-n_f)\left(\langle l_2 2\rangle^2 \langle l_4 1\rangle^2 + \langle l_4 2\rangle^2 \langle l_2 1\rangle^2\right) + 6 \langle l_2 2\rangle \langle l_4 2\rangle$$

$$\langle l_4 1\rangle \langle l_2 1\rangle] \langle l_2 2\rangle \langle l_4 2\rangle \langle l_4 1\rangle \langle l_2 1\rangle]$$

$$\times \frac{(-1)}{\langle l_2 2\rangle\langle 23\rangle\langle 3 l_4\rangle\langle l_4 l_2\rangle\langle l_4 4\rangle\langle 41\rangle\langle 1 l_2\rangle\langle l_2 l_4\rangle}$$

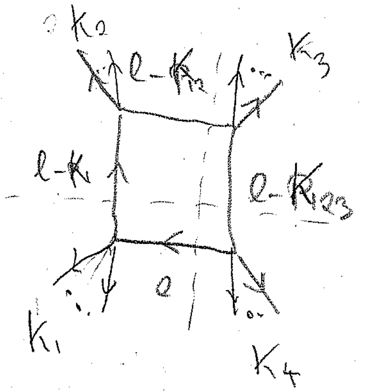
These remaining terms then give rise to cut triangles and bubbles in the t-channel.

III.6 GENERALIZED UNITARITY

Here the idea is to go beyond double cuts and consider quadruple and triple cuts in amplitudes to isolate the expansion coefficients:

$$A^{1\text{-loop}} = \underset{\text{boxes}}{\sum} \underset{\substack{\uparrow \\ \text{from quadruple} \\ \text{cuts}}}{c_i I_{4,i}} + \underset{\text{triangles}}{\sum} \underset{\substack{\uparrow \\ \text{contribute to} \\ \text{triple cuts}}}{d_i I_{3,i}} + \underset{\text{bubbles}}{\sum} \underset{\substack{\uparrow \\ \text{have double} \\ \text{cuts.}}}{e I_{2,i}}$$

In particular the quadruple cut completely fixes the c_i coefficients. For this one takes the cut momenta to be strictly four-dimensional & the four δ-functions determine the loop momentum entirely to the number of solutions to the simultaneous on-shell conditions.



$$\int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 + i\varepsilon)\,[(\ell - \not{K}_1)^2 + i\varepsilon]\,[(\ell - \not{K}_{12})^2 + i\varepsilon]\,[(\ell - \not{K}_{123})^2 + i\varepsilon]}\Bigg|_{\text{quadruple cut}}$$

$$\to \int \frac{d^4 \ell}{(2\pi)^4}\, \delta^{(+)}(\ell^2)\ \delta^{(+)}[(\ell - \not{K}_1)^2)\ \delta^{(+)}[(\ell - \not{K}_{12})^2]\ \delta^{(+)}[(\ell - \not{K}_{123})^2]$$

$$= \frac{1}{2}\ \# \,\text{solution}\,(\ell^{(m)})$$

It turns out that there are 2 solution of the 4 on-shell conditions →

Then c_j reads:

$$c_j = \frac{1}{2} \sum_{m=1,2} A^{\text{tree}}_{n_1}(\ldots, -\ell^{(m)}, \ell^{(m)} - K_1, \ldots)$$

$$\times A^{\text{tree}}_{n_2}(\ldots, -\ell^{(m)} + K_1, \ell^{(m)} - K_{12}, \ldots)$$

$$\times A^{\text{tree}}_{n_3}(\ldots, -\ell^{(m)} + K_{12}, \ell^{(m)} - K_{123}, \ldots)$$

$$\times A^{\text{tree}}_{n_4}(\ldots, -\ell^{(m)} + K_{123}, \ell^{(m)}, \ldots)$$

with $\ell^{(m)}$ denoting the 2 solutions of the on-shell condition.

EXAMPLE:

$A_5^{\text{1-loop}}(1^-, 2^-, 3^+, 4^+, 5^+)$ in pure YM theory.

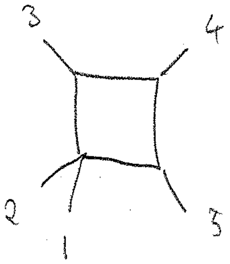
As $n=5$ the boxes will have one massive and 3 massless legs.

We have

$$A_5^{1\text{-loop}} = C_{12}\cdot I(K_{12}) + C_{23}\cdot I(K_{23}) + C_{34}\cdot I(K_{34}) + C_{45}\, I(K_{45}) + C_{51}\cdot I(K_{51})$$

where

$I(K_{12}) =$



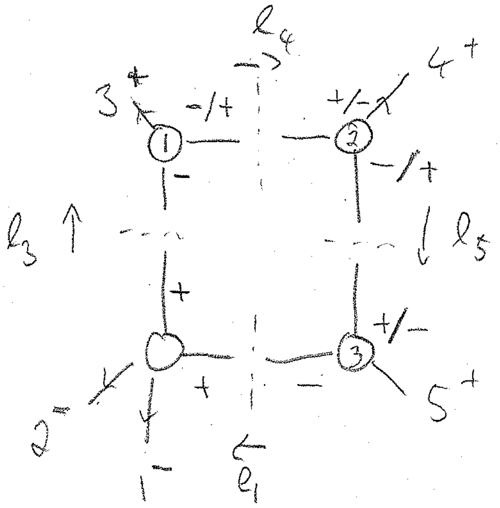
etc.

note that $A_5^{1\text{-loop}}(1^-,2^-,3^+,4^+,5^+) = - A_5^{1\text{-loop}}(2^-,1^-,5^+,4^+,3^+)$

which relates C_{51} to C_{23} and C_{45} to C_{34}.

Hence we only need to determine 3 coefficients $\{C_{12}, C_{23}, C_{34}\}$.

$\underline{C_{12}}$:



two channels

①/②

On-shell conditions on vertex ① require:

①: $\tilde{\lambda}_3 \parallel \tilde{\lambda}_{\ell_4}$ (MHV) or $\lambda_3 \parallel \lambda_{\ell_4}$ ($\overline{\text{MHV}}$)

②: $\tilde{\lambda}_{\ell_4} \parallel \tilde{\lambda}_4$ (MHV) or $\lambda_{\ell_4} \parallel \lambda_4$ ($\overline{\text{MHV}}$)

As $\lambda_3 \nparallel \lambda_4$ or $\tilde{\lambda}_3 \nparallel \tilde{\lambda}_4$ as $p_3 \cdot p_4 \neq 0$ generically this demands to either have the

MHV − $\overline{\text{MHV}}$ $\tilde\lambda_3 \parallel \tilde\lambda_{\ell_4}$ and $\lambda_{\ell_4} \parallel \lambda_4$

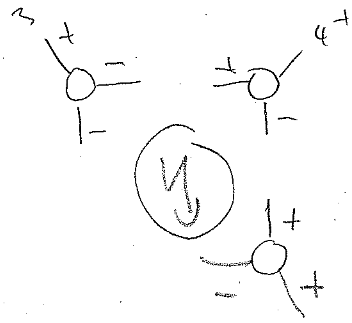
or

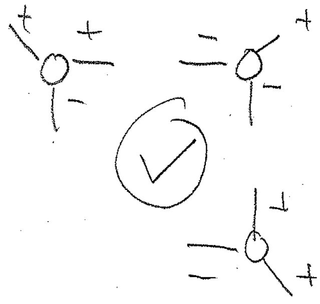
$\overline{\text{MHV}}$ − MHV $\lambda_3 \parallel \lambda_{\ell_4}$ and $\tilde\lambda_{\ell_4} \parallel \tilde\lambda_4$

combination: In general one needs alternating sequences of

MHV − $\overline{\text{MHV}}$ − MHV or $\overline{\text{MHV}}$ − MHV − $\overline{\text{MHV}}$

① ② ③ ① ② ③





Hence coefficient is given by:

$$C_{12} = \frac{1}{2} A_4^{\text{tree}}(-\ell_1^+, 1^-, 2^-, \ell_3^+)\, A_3^{\text{tree}}(-\ell_3^-, 3^+, \ell_4^+)$$

$$A_3^{\text{tree}}(-\ell_4^-, 4^+, \ell_5^-)\ A_3^{\text{tree}}(-\ell_5^+, 5^+, \ell_1^-)$$

$$= \frac{1}{2} \frac{\langle 12\rangle^3}{\langle 2\ell_3\rangle\langle \ell_3 \ell_1\rangle\langle \ell_1 1\rangle} \frac{[3\ell_4]^3}{[\ell_4\ell_3][\ell_3 3]} \frac{\langle \ell_5\ell_4\rangle^3}{\langle 4\ell_5\rangle\langle \ell_4 4\rangle}$$

$$\times \frac{-[\ell_5 5]^3}{[5\ell_1][\ell_1\ell_5]} = \frac{1}{2} \frac{\langle 12\rangle^3\ [3|\ell_4\ell_5|5]^3}{\langle 2|\ell_3|3]\ \langle 1|\ell_1\ell_5|4\rangle\ \langle 4|\ell_4\ell_3\ell_1|5]}$$

$= \frac{1}{2}$ ⟨12

Need to find a solution for a loop momentum (say l_4) to the on-shell constraints: 3 of them:

$$l_4^2 = 0 \;;\; l_3^2 = (l_4+p_3)^2 = 2l_4\cdot p_3 = 0 \;;\; l_5^2 = (l_4-p_4)^2$$
$$= 2l_4\cdot p_4 = 0$$

are solved with

$$\boxed{l_4^\mu = \frac{1}{2}\xi_4 \langle 3|\gamma^\mu|4]}$$

as $l_4^2 = \frac{1}{2}\xi_4^2 \langle 33\rangle\langle 44\rangle = 0$

$$2l_4\cdot p_3 \sim \langle 3|3\rangle[4|4] = 0 \qquad 2l_4\cdot p_4 \sim \langle 3|4]\langle 4|4] = 0$$

The coefficient ξ_4 follows from the last on-shell condition:

$$l_1^2 = (l_4 - K_{45})^2 = -2l_4\cdot(p_4+p_5) + 2p_4\cdot p_5$$
$$= -\xi_4 \langle 3|5|4] + s_{45} = 0$$

$$\Rightarrow \boxed{\xi_4 = \frac{s_{45}}{\langle 3|5|4]}} = \frac{\langle 45\rangle[54]}{\langle 35\rangle[54]} = \frac{\langle 45\rangle}{\langle 35\rangle}$$

Back to C_{12}:

$$-\frac{s_{34}s_{44}}{2\langle 35\rangle}$$
$$\shortparallel$$
$$-\frac{\langle 45\rangle}{2\langle 35\rangle}[3|4]\langle 34\rangle[45]$$
$$\shortparallel$$
$$[3|l_4 l_5|5] = [3|l_4(l_4-4)|5] = -[3|l_4|4\rangle[45]$$

$$\langle 2|l_3|3] = \langle 2|l_4+p_3|3] = \langle 2|l_4|3] = \frac{\langle 45\rangle}{2\langle 35\rangle}$$

IV,29

$$\langle 1|l_1 l_5|4\rangle = \langle 1|(l_4 - K_{45})(l_4 - p_4)|4\rangle$$

$$= -\langle 1|5\, l_4|4\rangle = -\langle 15\rangle[5|l_4|4\rangle = -\frac{\langle 15\rangle\langle 45\rangle}{2\langle 35\rangle}[54]\cdot\langle 34\rangle$$

$$\langle 4|l_4 l_3 l_1|5] = \langle 4|l_4(l_4 + p_3)(l_4 - p_4)|5]$$

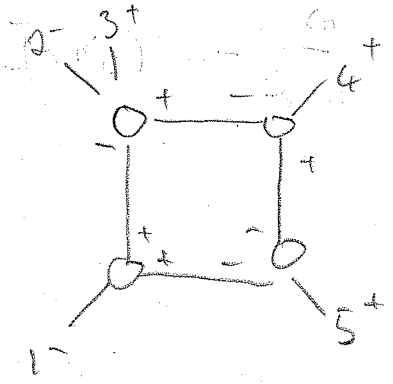
$$= \langle 4|l_4|3]\big(\langle 3|l_4|5] - \langle 34\rangle[45]\big)$$

$$= \langle 4|$$

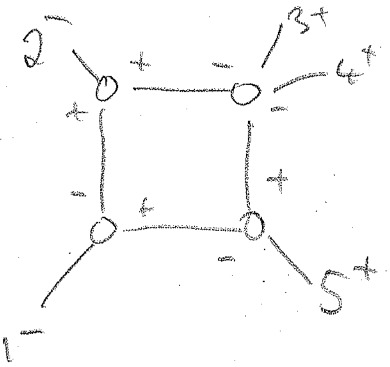
$$\vdots$$

$$\boxed{C_{12} = -\frac{1}{2}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\cdot s_{34}s_{45} = \frac{i}{2}s_{34}s_{45}A_5^{\text{tree}}(1^-,2^-,3^+4^+5^+)}$$

The other coeffs may be found with similar techniques:



$I(K_{23})$



$I(K_{34})$

Symmetries and Dualities of Scattering amplitudes

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Lecture 5

Parma International School of Theoretical Physics

$\mathcal{N} = 4$ super Yang Mills: The simplest interacting 4d QFT

- Field content: All fields in adjoint of $SU(N)$, $N \times N$ matrices
  - Gluons: A_μ, $\mu = 0, 1, 2, 3$, $\Delta = 1$
  - 6 real scalars: Φ_I, $I = 1, \ldots, 6$, $\Delta = 1$
  - $4 \times$ **4** real fermions: $\Psi_{\alpha\, A}$, $\bar{\Psi}^{\dot{\alpha}}_{A}$,$\alpha, \dot{\alpha} = 1, 2$. $A = 1, 2, 3, 4$, $\Delta = 3/2$
  - Covariant derivative: $\mathcal{D}_\mu = \partial_\mu - i[A_\mu, *]$, $\Delta = 1$
- Action: Unique model completely fixed by SUSY

$$S = \frac{1}{{g_{\mathrm{YM}}}^2} \int d^4x \, \mathrm{Tr} \Big[\tfrac{1}{4} F^2_{\mu\nu} + \tfrac{1}{2} (D_\mu \Phi_I)^2 - \tfrac{1}{4} [\Phi_I, \Phi_J][\Phi_I, \Phi_J] + \bar{\Psi}^A_{\dot{\alpha}} \sigma^{\dot{\alpha}\beta}_{\mu} \mathcal{D}^\mu \Psi_{\beta\, A} - \tfrac{i}{2} \Psi_{\alpha\, A} \sigma^{AB}_{I} \epsilon^{\alpha\beta} \left[\Phi^I, \Psi_{\beta\, B}\right] - \tfrac{i}{2} \bar{\Psi}_{\dot{\alpha}\, A} \sigma^{AB}_{I} \epsilon^{\dot{\alpha}\dot{\beta}} \left[\Phi^I, \bar{\Psi}_{\dot{\beta}\, B}\right] \Big]$$

- $\boxed{\beta_{g_{\mathrm{YM}}} = 0}$: Quantum Conformal Field Theory, 2 parameters: N & $\lambda = {g_{\mathrm{YM}}}^2 N$
- Shall consider 't Hooft planar limit: $N \to \infty$ with λ fixed.
- Is the 4d interacting QFT with highest degree of symmetry!

$\Rightarrow$ **"H-atom of gauge theories"**

Superconformal symmetry

- Symmetry: $\mathfrak{so}(2,4) \otimes \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$

Poincaré: $p^{\alpha\dot{\alpha}} = p_\mu \, (\sigma^\mu)^{\dot{\alpha}\beta}, \quad m_{\alpha\beta}, \quad \bar{m}_{\dot{\alpha}\dot{\beta}}$

Conformal: $k_{\alpha\dot{\alpha}}, \quad d \qquad (c$: central charge$)$

R-symmetry: r_{AB}

Poncaré Susy: $q^{\alpha\, A}, \bar{q}^{\dot{\alpha}}_A$ Conformal Susy: $s_{\alpha\, A}, \bar{s}^A_{\dot{\alpha}}$

- $4+4$ Supermatrix notation $\bar{A} = (\alpha, \dot{\alpha}|A)$

$$J^{\bar{A}}{}_{\bar{B}} = \begin{pmatrix} m^\alpha{}_\beta - \frac{1}{2}\,\delta^\alpha_\beta\,(d + \frac{1}{2}c) & k^\alpha{}_{\dot{\beta}} & s^\alpha{}_B \\ p^{\dot{\alpha}}{}_\beta & \overline{m}^{\dot{\alpha}}{}_{\dot{\beta}} + \frac{1}{2}\,\delta^{\dot{\alpha}}_{\dot{\beta}}\,(d - \frac{1}{2}c) & \bar{q}^{\dot{\alpha}}{}_B \\ q^A{}_\beta & \bar{s}^A{}_{\dot{\beta}} & -r^A{}_B - \frac{1}{4}\delta^A_B\, c \end{pmatrix}$$

- Algebra:

$$[J^{\bar{A}}{}_{\bar{B}}\,,\, J^{\bar{C}}{}_{\bar{D}}\} = \delta^{\bar{C}}_{\bar{B}}\, J^{\bar{A}}{}_{\bar{D}} - (-1)^{(|\bar{A}|+|\bar{B}|)(|\bar{C}|+|\bar{D}|)}\delta^{\bar{A}}_{\bar{D}}\, J^{\bar{C}}{}_{\bar{B}}$$

Gluon Amplitudes and Helicity Classification

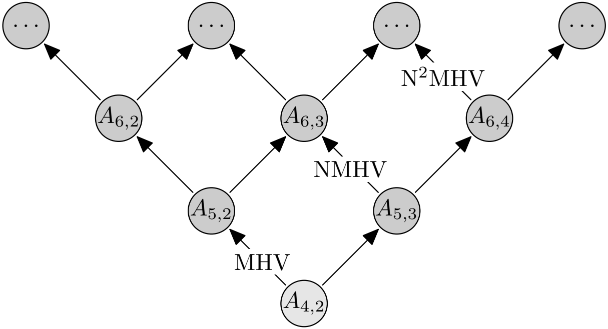
Classify gluon amplitudes by # of helicity flips

- By SUSY Ward identities: $\mathcal{A}_n(1^+, 2^+, \ldots, n^+) = 0 = \mathcal{A}_n(1^-, 2^+, \ldots, n^+)$ true to all loops
- Maximally helicity violating (MHV) amplitudes

$$\mathcal{A}_n(1^+, \ldots, i^-, \ldots, j^-, \ldots n^+) = \delta^{(4)}(\sum_i p_i) \frac{\langle i, j\rangle^4}{\langle 1, 2\rangle \langle 2, 3\rangle \ldots \langle n, 1\rangle}$$

[Parke, Taylor]

- Next-to-maximally helicity amplitudes (N^{k}MHV) have more involved structure!



$$A_{n,m} : g_+^{n-m} g_-^m$$

[Picture from T. McLoughlin]

On-shell superspace

- Augment λ_i^α and $\tilde\lambda_i^{\dot\alpha}$ by Grassmann variables η_i^A $\quad A = 1, 2, 3, 4$ [Nair]
- On-shell superspace $(\lambda_i^\alpha, \tilde\lambda^{\dot\alpha}, \eta_i^A)$ with on-shell superfield:

$$\Phi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar\Gamma^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \,\epsilon_{ABCD} G^-(p)$$

- Superamplitudes: $\Big\langle \Phi(\lambda_1, \tilde\lambda_1, \eta_1)\, \Phi(\lambda_2, \tilde\lambda_2, \eta_2) \ldots \Phi(\lambda_n, \tilde\lambda_n, \eta_n) \Big\rangle$

  Packages all n-parton gluon$^\pm$-gluino$^{\pm 1/2}$-scalar amplitudes

- General form of **tree superamplitudes**:

$$\boxed{\mathbb{A}_n = \frac{\delta^{(4)}(\sum_i \lambda_i \tilde\lambda_i)\, \delta^{(8)}(\sum_i \lambda_i \eta_i)}{\langle 1,2\rangle \langle 2,3\rangle \ldots \langle n,1\rangle}\, \mathcal{P}_n(\{\lambda_i, \tilde\lambda_i, \eta_i\})}$$

  Conservation of super-momentum: $\quad \delta^{(8)}(\sum_i \lambda^\alpha \eta_i^A) = (\sum_i \lambda^\alpha \eta_i^A)^8$

- η-expansion of $\mathcal{P}_n$ yields N^{k}MHV-classification of superamps as $h(\eta) = -1/2$

$$\boxed{\mathcal{P}_n = \mathcal{P}_n^{\text{MHV}} + \eta^4\, \mathcal{P}_n^{\text{NMHV}} + \eta^8\, \mathcal{P}_n^{\text{NNMHV}} + \ldots + \eta^{4n-16}\, \mathcal{P}_n^{\overline{\text{MHV}}}}$$

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  Packages all n-parton gluon$^\pm$-gluino$^{\pm 1/2}$-scalar amplitudes

- General form of **tree superamplitudes**:

$$\boxed{\mathbb{A}_n = \frac{\delta^{(4)}(\sum_i \lambda_i\, \tilde{\lambda}_i)\, \delta^{(8)}(\sum_i \lambda_i\, \eta_i)}{\langle 1, 2\rangle\, \langle 2, 3\rangle \ldots \langle n, 1\rangle}\, \mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})}$$

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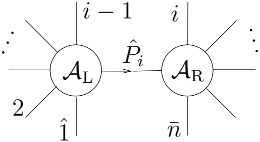
Super BCFW-recursion

- Efficient way of constructing tree-level amplitudes via BCFW recursion using an on-shell superspace via shift in $(\lambda_i, \tilde{\lambda})$ and η_i [Elvang et al, Arkani-Hamed et al, Brandhuber et al]

$$\mathbb{A}_n = \sum_i \int d^{4\eta_P}\, \mathbb{A}^L_{i+1}\, \frac{1}{P_i^2}\, \mathbb{A}^R_{n-i+1}$$

$\sum$ 

- Reformulation of recursion relations in terms of functions $\mathcal{P}_n(1, 2, \dots, n)$:

$$\mathcal{P}_n = \mathcal{P}_{n-1}(\hat{P}, 3, \dots, \hat{n}) + \sum_{i=4}^{n-1} R_{n;2,i}\, \mathcal{P}_i(\hat{1}, 2, \dots, -\hat{P}_i)\, \mathcal{P}_{n-i+2}(\hat{P}_i, i, \dots, \hat{n})$$

- Is much simpler and can be solved analytically!

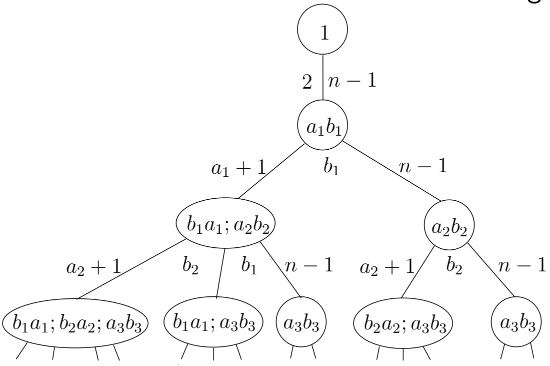
$\Rightarrow$ $\boxed{\mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) \text{ known in closed analytical form at tree-level}}$ [Drummond,Henn]

The Drummond-Henn solution

$\mathcal{P}_n$ expressed as sums over R-invariants determined by paths on rooted tree

$$\mathcal{P}_n^{\text{N}^k\text{MHV}} = \sum_{\substack{\text{all paths}\\ \text{of length } k}} 1 \cdot R_{n,a_1b_1} \cdot R_{n,\{I_2\},a_2b_2}^{\{L_2\};\{U_2\}} \cdot \ldots \cdot R_{n,\{I_p\},a_pb_p}^{\{L_p\};\{U_p\}}$$

E.g.



$$\mathcal{P}^{\text{NMHV}} = \sum_{1<a_1,b_1<n} R_{n,a_1b_1}$$

$$\mathcal{P}_n^{\text{N}^2\text{MHV}} = \sum_{1<a_1,b_1<n} R_{n;a_1b_1} \times \left[\sum_{a_1<a_2,b_2\le b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2} + \sum_{b_1\le a_2,b_2<n} R^{a_1b_1;0}_{n;a_2b_2} \right]$$

$$R_{n;b_1a_1;b_2a_2;\ldots;b_ra_r;ab} = \frac{\langle a\ a-1\rangle\langle b\ b-1\rangle\delta^{(4)}(\langle\xi|x_{a_ra}x_{ab}|\theta_{ba_r}\rangle + \langle\xi|x_{a_rb}x_{ba}|\theta_{aa_r}\rangle)}{x_{ab}^2\langle\xi|x_{a_ra}x_{ab}|b\rangle\langle\xi|x_{a_ra}x_{ab}|b-1\rangle\langle\xi|x_{a_rb}x_{ba}|a\rangle\langle\xi|x_{a_rb}x_{ba}|a-1\rangle},$$

with

$$\langle\xi| = \langle n|x_{nb_1}x_{b_1a_1}x_{a_1b_2}x_{b_2a_2}\ldots x_{b_ra_r}\,.$$

Dual Superconformal symmetry

- Introduce dual on-shell superspace [Drummond, Henn, Korchemsky, Sokatchev]

$$(x_i - x_{i+1})^{\alpha\dot\alpha} = \lambda_i^\alpha\,\tilde\lambda_i^{\dot\alpha} \qquad (\theta_i - \theta_{i+1})^{\alpha\,A} = \lambda_i^\alpha\,\eta_i^A$$

- Transformation properties under inversions $I[\ldots]$ in dual x-space

$$I[\langle i\,i+1\rangle] = \frac{\langle i\,i+1\rangle}{x_i^2} \quad I[\delta^4(p)\delta^8(q)] = \delta^4(p)\delta^8(q)$$

$$I[\langle n|x_{na}x_{ab}|b\rangle] = \frac{\langle n|x_{na}x_{ab}|b\rangle}{x_n^2x_a^2x_b^2}\,, \quad I[\langle n|x_{na}x_{ab}|b-1\rangle] = \frac{\langle n|x_{na}x_{ab}|b-1\rangle}{x_n^2x_a^2x_{b-1}^2}$$

- One shows that $I[R_{n;b_1a_1;\ldots;b_ra_r;ab}] = R_{n;b_1a_1;\ldots;b_ra_r;ab}$ as all weights cancel!
- Simple proof of dual conformal symmetry: $R_{n,st}$ is I-invariant, assume $\mathcal{P}_{k<n}$ are I-invariant. Then RHS of recursion relation is invariant too, thus $\mathcal{P}_n$ also I-invariant.
- Hence:

$$I[\mathbb{A}_n] = x_1^2\,x_2^2\ldots x_n^2\,\mathbb{A}_n$$

Infinitesimal form of dual superconformal symmetry

- Infinitesimally one has: $K^{\alpha\dot\alpha} = \sum_i x_i^{\alpha\dot\beta}\, x_i^{\dot\alpha\beta}\, \dfrac{\partial}{\partial x_i^{\beta\dot\beta}} + x_i^{\dot\alpha\beta}\, \theta_i^{\alpha\, B}\, \dfrac{\partial}{\partial\theta_i^{\beta\, B}}$.

  Bosonic part derives from $K_\mu = x^2\,\partial_\mu - 2x_\mu\, x\cdot\partial$.

- Indeed: Trees are dual superconformal covariant:

$$K^{\alpha\dot\alpha}\,\mathbb{A}_n^{\text{tree}} = -\sum_{i=1}^{n} x_i^{\alpha\dot\alpha}\,\mathbb{A}_n^{\text{tree}} \qquad S^{\alpha A}\,\mathbb{A}_n^{\text{tree}} = -\sum_{i=1}^{n} \theta_i^{\alpha A}\,\mathbb{A}_n^{\text{tree}}$$

  $\Rightarrow$ $\boxed{\tilde K = K + \sum_i x_i \text{ and } \tilde S = S + \sum_i \theta_i}$ annihilate the amplitude.

- Extend dual superconformal generators so that they commute with constraints

$$(x_i - x_{i+1})^{\alpha\dot\alpha} = \lambda_i^\alpha\,\tilde\lambda_i^{\dot\alpha} \qquad (\theta_i - \theta_{i+1})^{\alpha\, A} = \lambda_i^\alpha\,\eta_i^A$$

  leads to expression for $K^{\alpha\dot\alpha}$ acting in joint super-space $\{\lambda_i, \tilde\lambda_i, \eta_i; x_i, \theta_i\}$

$$K^{\alpha\dot\alpha} = \sum_i x_i^{\alpha\dot\beta}\, x_i^{\dot\alpha\beta}\, \frac{\partial}{\partial x_i^{\beta\dot\beta}} + x_i^{\dot\alpha\beta}\, \theta_i^{\alpha\, B}\, \frac{\partial}{\partial\theta_i^{\beta\, B}}$$

$$+x_{i\dot\alpha}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot\beta}\tilde\lambda_{i\dot\alpha}\partial_{i\dot\beta} + \tilde\lambda_{i\dot\alpha}\theta^{B}_{i+1\,\alpha}\partial_{iB}$$

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$$K^{\alpha\dot\alpha} = \sum_i x_i^{\alpha\dot\beta}\, x_i^{\dot\alpha\beta}\, \frac{\partial}{\partial x_i^{\beta\dot\beta}} + x_i^{\dot\alpha\beta}\, \theta_i^{\alpha\, B}\, \frac{\partial}{\partial \theta_i^{\beta\, B}}$$

$$+x_{i\dot\alpha}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot\beta}\tilde\lambda_{i\dot\alpha}\partial_{i\dot\beta} + \tilde\lambda_{i\dot\alpha}\theta^{B}_{i+1\,\alpha}\partial_{iB}$$

The natural question

Q: What algebraic structure emerges when one commutes conformal with dual conformal generators? [Drummond,Henn,Plefka]

First Task: Tranform dual superconformal generators expressed in dual space (x_i, θ_i) into original on-shell superspace $(\lambda_i, \tilde{\lambda}_i, \eta_i)$!

1. Open chain by droping $x_{n+1} = x_1$ and $\theta_{n+1} = \theta_1$ conditions, implemented via δ-fcts: $\delta^{(4)}(p)\,\delta^{(8)}(q) = \delta^{(4)}(x_1 - x_{n+1})\,\delta^{(8)}(\theta_1 - \theta_{n+1})$
2. Express dual variables via “non-local’ relations:

$$x_i^{\alpha\dot{\alpha}} = x_1^{\alpha\dot{\alpha}} + \sum_{j<i} \lambda_j^{\alpha}\,\tilde{\lambda}_j^{\dot{\alpha}} \qquad \theta_i^{\alpha A} = \theta_1^{\alpha A} + \sum_{j<i} \lambda_j^{\alpha}\,\eta_j^{A}$$

   Now set $x_1 = \theta_1 = 0$ by dual translation P and Poincare Susy Q.

3. Can now drop all x_1 and θ_i derivatives in dual superconformal generators.

The natural question

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Dual $\mathfrak{psu}(2,2|4)$ generators

- Dual superconformal generators acting in standard on-shell superspace $(\lambda, \tilde\lambda, \eta)$:

$$P_{\alpha\dot\alpha} = 0\,, \qquad Q_{\alpha A} = 0\,, \qquad \overline{Q}^{A}_{\dot\alpha} = \sum_i \eta_i^A \partial_{i\dot\alpha} = \bar{s}^{A}_{\dot\alpha}$$

$$M_{\alpha\beta} = \sum_i \lambda_{i(\alpha}\partial_{i\beta)} = \bar{m}_{\dot\alpha\dot\beta}\,, \qquad \overline{M}_{\dot\alpha\dot\beta} = \sum_i \tilde\lambda_{i(\dot\alpha}\partial_{i\dot\beta)} = m_{\alpha\beta}\,,$$

$$R^{A}{}_{B} = \sum_i \eta_i^A\,\partial_{iB} - \tfrac{1}{4}\delta^A_B \eta_i^C\,\partial_{iC} = -r^{A}{}_{B}\,,$$

$$D = \sum_i -\tfrac{1}{2}\lambda_i^\alpha \partial_{i\alpha} - \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} = -d\,,$$

$$C = \sum_i -\tfrac{1}{2}\lambda_i^\alpha \partial_{i\alpha} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + \tfrac{1}{2}\eta_i^A \partial_{iA} = 1 - c\,,$$

$$S^A_\alpha = \sum_i \lambda_{i\alpha}\theta_i^{\gamma A}\partial_{i\gamma} + x_{i+1\,\alpha}{}^{\dot\beta}\eta_i^A\partial_{i\dot\beta} - \theta^B_{i+1\,\alpha}\eta_i^A\partial_{iB}\,,$$

$$\overline{S}_{\dot\alpha A} = \sum_i \tilde\lambda_{i\dot\alpha}\partial_{iA} = \bar{q}_{\dot\alpha A}\,,$$

$$K_{\alpha\dot\alpha} = \sum_i x_{i\dot\alpha}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot\beta}\tilde\lambda_{i\dot\alpha}\partial_{i\dot\beta} + \tilde\lambda_{i\dot\alpha}\theta^B_{i+1\,\alpha}\partial_{iB}$$

Nonlocal structure of dual K and S

- We are left with the dual generators K and S, all others trivially related to standard superconformal generators.

$$\tilde{K}^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} x_i^{\dot{\alpha}\beta}\,\lambda_i^{\alpha}\,\frac{\partial}{\partial\lambda_i^{\beta}} + x_{i+1}^{\alpha\dot{\beta}}\,\tilde{\lambda}_i^{\dot{\alpha}}\,\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\beta}}} + \tilde{\lambda}_i^{\dot{\alpha}}\,\theta_{i+1}^{\alpha B}\,\frac{\partial}{\partial\eta_i^{B}} + x_i^{\alpha\dot{\alpha}}$$

$$x_i^{\alpha\dot{\alpha}} = \sum_{j=1}^{i-1}\lambda_j^{\alpha}\,\tilde{\lambda}_j^{\dot{\alpha}} \qquad \theta_{i+1}^{\alpha A} = \sum_{j=1}^{i}\lambda_j^{\alpha}\,\eta_j^{A}$$

Nonlocal structure!

Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

- Can show that dual superconformal generators K and S may be lifted to level 1 generators of a Yangian algebra $Y[\mathfrak{psu}(2,2|4)]$:

$$[J_a^{(0)}, J_b^{(0)}\} = f_{ab}{}^{c}\, J_c^{(0)} \qquad \text{conventional superconformal symmetry}$$

$$[J_a^{(1)}, J_b^{(0)}\} = f_{ab}{}^{c}\, J_c^{(1)} \qquad \text{from dual conformal symmetry}$$

with nonlocal generators

$$\boxed{J_a^{(1)} = f^{cb}{}_{a} \sum_{1<j<i<n} J_{i,b}^{(0)}\, J_{j,c}^{(0)}}$$

and super Serre relations (representation dependent). [Dolan,Nappi,Witten]

$$[J_a^{(1)}, [J_b^{(1)}, J_c^{(0)}\}\} + (-1)^{|a|(|b|+|c|)}[J_b^{(1)}, [J_c^{(1)}, J_a^{(0)}\}\} + (-1)^{|c|(|a|+|b|)}[J_c^{(1)}, [J_a^{(1)}, J_b^{(0)}\}\}$$
$$= h(-1)^{|r||m|+|t||n|}\{J_l^{(0)}, J_m^{(0)}, J_n^{(0)}] f_{ar}{}^{l} f_{bs}{}^{m} f_{ct}{}^{n} f^{rst}.$$

Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

- Bosonic invariance $\boxed{p^{(1)}_{\alpha\dot\alpha}\,\mathbb{A}_n = 0}$ with

$$p^{(1)}_{\alpha\dot\alpha} = \tilde K_{\alpha\dot\alpha} + \Delta K_{\alpha\dot\alpha} = \frac{1}{2}\sum_{i<j}(m_{i,\alpha}{}^{\gamma}\delta^{\dot\gamma}_{\dot\alpha} + \bar m_{i,\dot\alpha}{}^{\dot\gamma}\delta^{\gamma}_{\alpha} - d_i\,\delta^{\gamma}_{\alpha}\delta^{\dot\gamma}_{\dot\alpha})\,p_{j,\gamma\dot\gamma} + \bar q_{i,\dot\alpha C}\,q^{C}_{j,\alpha} - (i \leftrightarrow j)$$

- In supermatrix notation: $\bar A = (\alpha, \dot\alpha | A)$

$$J^{\bar A}{}_{\bar B} = \begin{pmatrix} m^{\alpha}{}_{\beta} - \frac{1}{2}\,\delta^{\alpha}_{\beta}\,(d + \frac{1}{2}c) & k^{\alpha}{}_{\dot\beta} & s^{\alpha}{}_{B} \\ p^{\dot\alpha}{}_{\beta} & \overline{m}^{\dot\alpha}{}_{\dot\beta} + \frac{1}{2}\,\delta^{\dot\alpha}_{\dot\beta}\,(d - \frac{1}{2}c) & \bar q^{\dot\alpha}{}_{B} \\ q^{A}{}_{\beta} & \bar s^{A}{}_{\dot\beta} & -r^{A}{}_{B} - \frac{1}{4}\delta^{A}_{B}\,c \end{pmatrix}$$

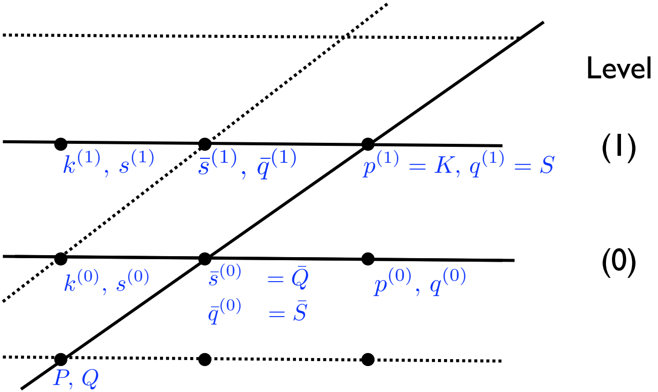
and $$\boxed{J^{(1)\,\bar A}{}_{\bar B} := -\sum_{i>j}(-1)^{|\bar C|}(J_i^{\bar A}{}_{\bar C}\,J_j^{\bar C}{}_{\bar B} - J_j^{\bar A}{}_{\bar C}\,J_i^{\bar C}{}_{\bar B})}$$

- Integrable spin chain picture also for colour ordered scattering amplitudes!
- Implies an infinite-dimensional symmetry algebra for $\mathcal{N} = 4$ SYM scattering amplitudes!

Summary of Yangian Structure

- Combination of standard and dual superconformal symmetry lifts to Yangian $Y[\mathfrak{psu}(2,2|4)]$

[Picture: Beisert]



- Tree level superamplitudes invariant: $\boxed{\mathcal{J} \circ \mathbb{A}_n^{\text{tree}} = 0}$ for $\mathcal{J} \in Y[\mathfrak{psu}(2,2|4)]$.

Dual conformal symmetry at loop level

- 4-point MHV-amplitude at 1-loop: $(a = \lambda/8\pi^2)$

$$\mathbb{A}_4^{\text{MHV, 1-loop}} = \mathbb{A}_4^{\text{MHV, tree}} \cdot \frac{a}{2}\, st \cdot I(s,t)$$

Scalar box integral: $I(s,t) = \int \frac{d^4k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2}$

No bubbles or triangles!

- Transform to dual coordinates: $x_{ij} = x_i - x_j$

$$p_1 = x_{12} \quad p_2 = x_{23} \quad p_3 = x_{34} \quad p_4 = x_{41} \quad k = x_1 - x_5$$

then $\boxed{I(s,t) = \int \frac{d^4x_5}{x_{15}^2\, x_{25}^2\, x_{35}^2\, x_{45}^2}}$ which is (naively) dual conformal invariant

$$I[\frac{d^4x_5}{x_{15}^2\, x_{25}^2\, x_{35}^2\, x_{45}^2}] = x_1^2 x_2^2 x_3^2 x_4^2\, \frac{d^4x_5}{x_{15}^2\, x_{25}^2\, x_{35}^2\, x_{45}^2}$$

- Note $st = (2p_1 \cdot p_2)(2p_1 \cdot p_3) = x_{13}^2\, x_{24}^2$, hence $st\, I(s,t)$ is dual conformal inv.

Pseudo conformal invariance at loop level

- One-loop box is only "pseudo-conformal" invariant as $I(s,t)$ is IR-divergent and needs to be regularized: $\boxed{d^4x_5 \to d^{4-2\epsilon}x_5}$. This breaks dual conformal invariance.
- Indeedexact dual conformal invariance would imply $st\,I(s,t)=0$ as there are no conformal invariant cross-ratios for 4 light-like seperated points:

$$\text{Dual conformal cross-ratios:} \quad R(i,j,k,l) = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

- Indeed one finds a non-vanishing result

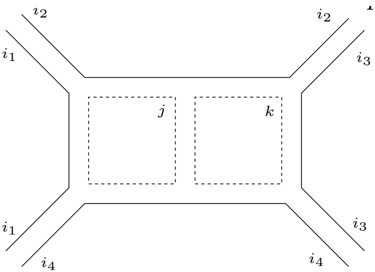
$$\mu^{2\epsilon} e^{-\epsilon\gamma_E}\, st\,I(s,t) = \frac{2}{\epsilon^2}\left[(\frac{\mu^2}{s})^{\epsilon} + (\frac{\mu^2}{t})^{\epsilon}\right] - \log^2(s/t) - \frac{4\pi^2}{3}$$

$\Rightarrow$ dual conformal anomaly

- "Pseudo" dual conformal invariance still a very useful concept as it constrains the possible scalar-integrals appearing at higher loops.

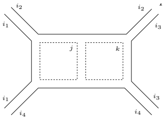
Dual conformal invariance at higher loops

- E.g. at 2 loops: Only one integral is allowed by dual conformal symmetry:



  Similar restrictions at higher loops.

- One observes exponentiation: [Bern,Dixon,Smirnov]



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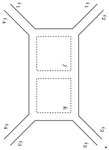
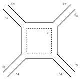


$= \exp\Big[\Gamma_{\text{cusp}}(\lambda)$



$\Big]\Big|_{\lambda^2}$

What about higher loops?

- Spezialize to MHV for simplicity: $\boxed{\mathcal{A}_n^{\text{MHV}} = \mathcal{A}_{n,0}^{\text{MHV}} \, \mathcal{M}_n^{\text{MHV}}(p_i \cdot p_j; \lambda)}$
- All loop planar amplitudes can be split into IR divergent and finite parts:

$$\ln \mathcal{M}_n^{\text{MHV}} = D_n + F_n + \mathcal{O}(\epsilon)$$

IR divergencies exponentiate in any gauge theory $(a = \lambda/8\pi^2)$ [Mueller,Collins,Sterman,. . .]

$$D_n = -\frac{1}{2} \sum_{l=1}^{\infty} a^l \left(\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{G^{(l)}}{l\epsilon} \right) \sum_{i=1}^{n} (2p_i \cdot p_j)^{l\epsilon}$$

$$\Gamma_{\text{cusp}}(a) = \sum_l a^l \Gamma_{\text{cusp}}^{(l)} \,, \quad \text{cusp anomalous dimension}$$

$$G(a) = \sum_l a^l G^{(l)} \,, \quad \text{colinear anomalous dimension}$$

- IR divergencies break $\{s, \bar{s}, k, K, S, \bar{Q}\}$ but leave $\{p, q, \bar{q}, P, Q, \bar{S}\}$ intact.

[Korchemsky,Sokatchev]

Dual conformal anomaly

- Breaking of K_μ is under control and proportional to $\Gamma_{\text{cusp}}(g)$ for MHV amplitudes. From dual Wilson loop picture: UV anomaly due to cusps for finite piece F_n

$$K_\mu F_n = \sum_{i=1}^{n} \left[2x_{i\mu} x_i^\nu \frac{\partial}{\partial x_i^\nu} - x_i^2 \frac{\partial}{\partial x_i^\mu} \right] F_n = \frac{1}{2} \Gamma_{\mathrm{cusp}}(a) \sum_{i=1}^{n} \left[x_{i,i+1}^\mu \, \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} \right] F_n$$

- Conjecture: Dual superconformal 'anomaly' is the same for MHV and non-MHV amplitudes [Drummond,Henn,Korchemsky,Sokatchev '08]
- ‘Anomaly’ fixes the MHV 4 & 5 gluon amplitudes completely ⇔ BDS-ansatz. Nontrivial structure starts with $n = 6$.
- ⇒ Remainder function, non-trivial function of dual conformal invariants
- **Q:** Can the other broken Yangian symmetry be repaired at loop level?

  ⇒ Does this constrain the answers?

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From $\mathcal{N}=4$ SYM trees to massless QCD

Goal: Project onto component field amplitudes [Dixon, Henn, Plefka, Schuster]

$$x_i - x_{i+1} = p_i \qquad x_{ij} := x_i - x_j \overset{i<j}{=} p_i + p_{i+1} + \cdots + p_{j-1}$$

- All amplitudes expressed via momentum invariants x_{ij}^2 and the scalar quantities:

$$\begin{aligned} \langle n a_1 a_2 \ldots a_k | a \rangle &:= \langle n | x_{na_1} x_{a_1 a_2} \ldots x_{a_{k-1} a_k} | a \rangle \\ &= \lambda_n^{\alpha} (x_{na_1})_{\alpha\dot{\beta}} (x_{a_1 a_2})^{\dot{\beta}\gamma} \ldots (x_{a_{k-1} a_k})^{\dot{\delta}\rho} \lambda_{a\,\rho} \end{aligned}$$

- Building blocks for amps: $\tilde{R}$ invariants and path matrix Ξ_n^{path}

$$\tilde{R}_{n;\{I\};ab} := \frac{1}{x_{ab}^2} \frac{\langle a(a-1) \rangle}{\langle n\,\{I\}\,ba|a\rangle \langle n\,\{I\}\,ba|a-1\rangle} \frac{\langle b(b-1) \rangle}{\langle n\,\{I\}\,ab|b\rangle \langle n\,\{I\}\,ab|b-1\rangle} ;$$

$$\Xi_n^{\text{path}} := \begin{pmatrix} \langle nc_0 \rangle & \langle nc_1 \rangle & \ldots & \langle nc_p \rangle \\ (\Xi_n)^{c_0}_{a_1 b_1} & (\Xi_n)^{c_1}_{a_1 b_1} & \ldots & (\Xi_n)^{c_p}_{a_1 b_1} \\ (\Xi_n)^{c_0}_{\{I_2\};a_2 b_2} & (\Xi_n)^{c_1}_{\{I_2\};a_2 b_2} & \ldots & (\Xi_n)^{c_p}_{\{I_2\};a_2 b_2} \\ \vdots & \vdots & & \vdots \\ (\Xi_n)^{c_0} & (\Xi_n)^{c_1} & \ldots & (\Xi_n)^{c_p} \end{pmatrix}$$

All gluon-gluino trees in $\mathcal{N} = 4$ SYM [Dixon, Henn, Plefka, Schuster]

- MHV gluon amplitudes [Parke, Taylor]

$$A_n^{\text{MHV}}(c_0^-, c_1^-) = \delta^{(4)}(p)\frac{\langle c_0\, c_1\rangle^4}{\langle 1\ 2\rangle\langle 2\ 3\rangle \dots \langle n\ 1\rangle}$$

- N$^{\text{p}}$MHV gluon amplitudes:

$$A_n^{\text{N}^{\text{p}}\text{MHV}}(c_0^-, \dots, c_{p+1}^-) = \frac{\delta^{(4)}(p)}{\langle 1\ 2\rangle \dots \langle n\ 1\rangle} \sum_{\substack{\text{all paths}\\ \text{of length } p}} \left(\prod_{i=1}^{p} \tilde{R}^{L_i;R_i}_{n;\{I_i\};a_i b_i}\right) (\det \Xi)^4$$

- MHV gluon-gluino amplitudes (single flavor)

$$A_n^{\text{MHV}}(a^-, b_q, c_{\bar q}) = \delta^{(4)}(p)\frac{\langle a\ c\rangle^3 \langle a\ b\rangle}{\langle 1\ 2\rangle \dots \langle n\ 1\rangle}$$

- N$^{\text{p}}$MHV gluon-gluino amplitudes:

$$A_{(q\bar q)^k,n}^{\text{N}^{\text{p}}\text{MHV}}(\dots, c_k^-, \dots, (c_{\alpha_i})_q, \dots, (c_{\bar\beta_j})_{\bar q}, \dots) =$$

$$\frac{\delta^{(4)}(p)\text{sign}(\tau)}{\langle 1\ 2\rangle\langle 2\ 3\rangle \dots \langle n\ 1\rangle} \times \sum_{\substack{\text{all paths}\\ \text{of length } p}} \left(\prod_{i=1}^{p} \tilde{R}^{L_i;R_i}_{n;\{I_i\};a_i b_i}\right) \left(\det \Xi\big|_q\right)^3 \det \Xi(q \leftrightarrow \bar q)\big|_{\bar q}$$

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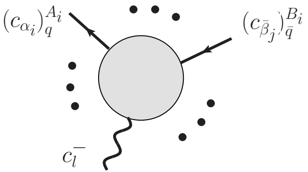

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From $\mathcal{N}=4$ to massless QCD trees

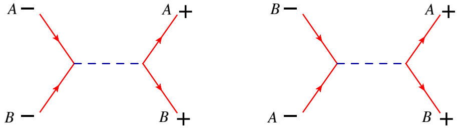
- Differences in color: SU(N) vs. SU(3); Fermions: adjoint vs. fundamental Irrelevant for color ordered amplitudes, as color d.o.f. stripped off anyway. E.g. single quark-anti-quark pair

$$\mathcal{A}_n^{\text{tree}}(1_{\bar{q}}, 2_q, 3, \ldots, n) = g^{n-2} \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma(3)}} \ldots T^{a_{\sigma(n)}})_{i_2}^{\;\bar{i}_1} \, A_n^{\text{tree}}(1_{\bar{q}}, 2_q, \sigma(3), \ldots, \sigma(n))$$

  Color ordered $A_n^{\text{tree}}(1_{\bar{q}}, 2_q, 3, \ldots, n)$ from two-gluino-$(n-2)$-gluon amplitude.

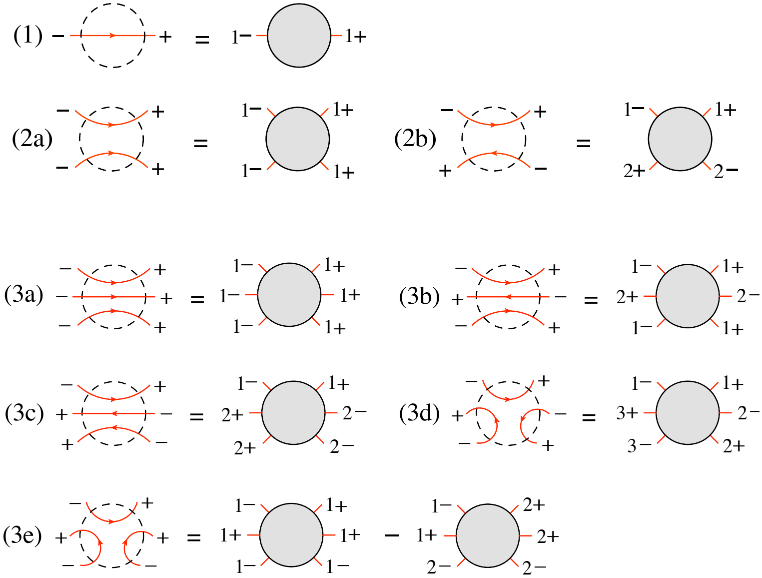
- For more than one quark-anti-quark pair needs to accomplish:

  (1) Avoid internal scalar exchanges (due to Yukawa coupling)



From $\mathcal{N} = 4$ to massless QCD trees

(2) Allow all fermion lines present to be of different flavor



From $\mathcal{N} = 4$ to massless QCD trees

- Also worked out explicitly for 4 quark-anti-quark pairs.
- Conclusion: Obtained all (massless) QCD trees from the $\mathcal{N} = 4$ SYM trees
- Comparison of numerical efficiency to Berends-Giele recursion: Analytical formulae faster for MHV and NMHV case, competitive for NNMHV

  [Biedermann, Uwer, Schuster, Plefka, Hackl]