Resurgence in Gauge Theory & Sigma Models

Gerald & Ricardo:

$$\mathcal{O}(\lambda) = \sum_n p_{n,0}\lambda^n + \sum_c e^{-\frac{S_c}{\lambda}}\sum_m\sum_n p_{n,c,m}\lambda^n \log(1/\lambda)^m$$

QM observables; matrix models; string theories.

semi-classical expansion.

general goal: use RHS to study LHS!

Natural questions: ① why should such expressions hold?

"Answer": Thimbles → talks by Yuya; also some comments here.

② What about QFT?

Especially realistic QFT, such as asymptotically-free, non-SUSY theories?

↳ main topic of my lectures.

As soon as we try to think about ②, need to deal with

practical
- ⓐ What is λ? It runs!
- ⓑ What are $e^{-S_c/\lambda}$ terms?
- ⓒ What are $\log(1/\lambda)$ terms?
- ⓓ How can we hope to say anything on paper, due to strong coupling?

e) What should we ask about?

↳ what $O(\lambda)$'s can we ask about?

Basic q's: - is a QFT gapped?
- does it confine?
- how are symmetries realized?
- what's the spectrum?

f) What can resurgence tell us about it?

↳ Start here, as a taste of what follows. Foreshadow/recapitulate other lectures.

$$O(\lambda) = \sum_n p_n \lambda^n + \ldots$$

Suppose $p_n \sim n! \left(\frac{1}{S}\right)^n$. What does it mean?

Well, λ runs according to $\lambda \sim \frac{1}{\beta_0 \log(Q/\Lambda)}$

$\beta_0 \sim \frac{11}{3}$ in YM.

external energy scale such as momentum

strong scale. ~Gap scale.

$p_n \sim n! \left(\frac{1}{S}\right)^n$ is called renormalon;

Key is for S to be N-independent. Then we get ambiguity $\sim \pm i e^{-\frac{S}{\lambda}}$

→

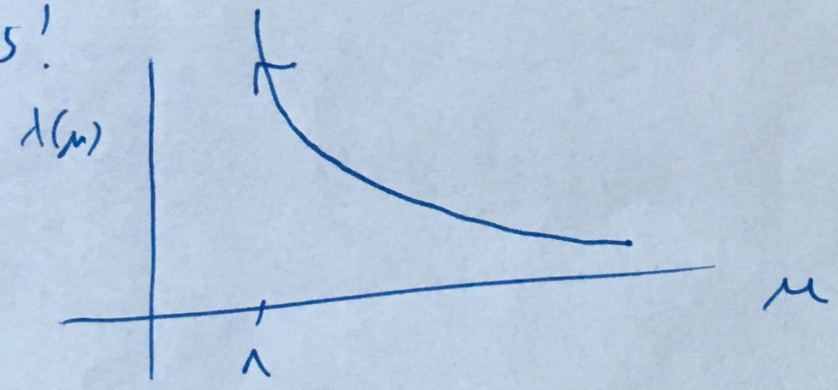
But then ambiguity $\sim \pm i e^{-S/\lambda} \sim \underline{a}_i$

$$\pm i \exp\left[\frac{S}{\beta_0}\log\left(\frac{\Lambda}{Q}\right)\right]$$

$$\pm i \left(\frac{\Lambda}{Q}\right)^{S/\beta_0} \qquad S/\beta_0 = \#.$$

So ambiguity is a "power correction", involving strong scale ~ mass gap. Understanding resurgent structure $\leftrightarrow$ understanding NP dynamics.

Before I can say more about resurgence, must answer (a) + (f); and along the way, explain how to tame AF QFTs!



<u>One option</u>: just explained above, use "OPE".

Lectures by Antonio Pineda

+ : very useful & interesting

− : can't get very far on paper, not easy to get info on confinement, gap, etc.

Another option: focus on SUSY QFTS, use localization etc ④

↳ see talk by Stefano

My focus: - adiabatic compactification technique due to Unsal + collaborators, last few years.

- works for generic non-SUSY AF QFTS, allows insights into confinement, symmetry breaking, mass gaps, resurgence, etc!

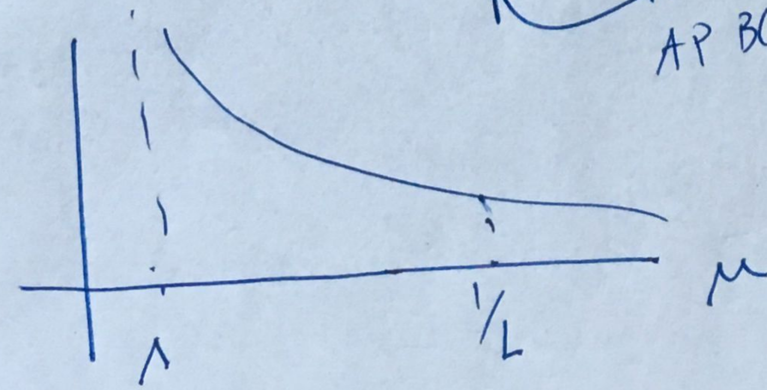
- "disadvantage": not easy to say what happens in $\mathbb{R}^4$ or $\mathbb{R}^2$ ← not such big embarassment
limit
no one has solved QCD.

Basic idea extremely simple.

1st try: $\mathbb{R}^4 \to \mathbb{R}^3 \times S^1_L$

PBCs for bosons
AP BCs for fermions.

$L = 1/T$
← temp



So could try to make ΛL small, then $\lambda(L^{-1})$ would be small, and we could calculate.

This FAILS. Several ~~ideas~~ issues:

- At small enough L, fermions decouple, AP BCs $\Rightarrow m_F \sim \frac{1}{L} \to \infty$.

  explain KK!

- Gauge sector can be reduced to KK zero mode:

$$S = \frac{1}{g^2}\int d^4x\ \mathrm{tr}\, F^2 \longrightarrow \frac{L}{g^2}\int d^3x\ \mathrm{tr}\, F^2 \equiv \frac{1}{g_{3D}^2}\int d^3x\ \mathrm{tr}\, F^2.$$

So long-distance dynamics = $\underline{3D}$ pure YM theory!

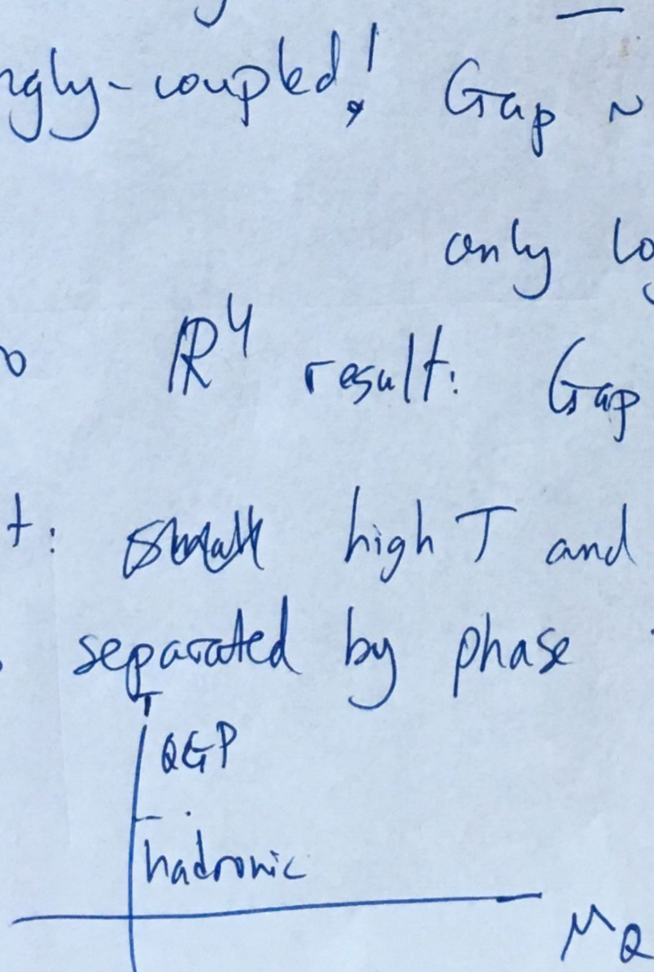
It's strongly-coupled! Gap $\sim \frac{\lambda_{4D}(1/L)}{L} \sim \frac{1}{L}\cdot\frac{1}{\log(\Lambda/L)}$,

only log dependence on Λ!

compare to $\mathbb{R}^4$ result: Gap $\sim \Lambda$

power-law on Λ.

- Worst part: ~~Small~~ high T and low T physics separated by phase transition



So the goal is to find compactification which is smooth – no phase transitions – which gives weakly coupled theory at small L.

Biggest danger: losing confinement!! How to tell?

Order parameter: center symmetry. Only defined on $M_3 \times S^1$, ob~~serve that~~ with gluon BCs

$$A_\mu(x_4 + L) = A_\mu(x_4)$$

Path integral invariant under gauge transformations that are aperiodic by an element of the center of $SU(N)$. [Center def.] [isomorphic to $\mathbb{Z}_N$ for $SU(N)$].

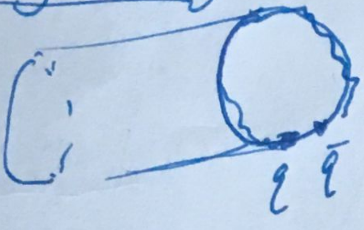
All local operators also invariant. But

$$\Omega = P\, e^{i\oint A_4 dx} \to \omega\, \Omega.$$

$\in \mathbb{Z}_N$, $\omega = e^{2\pi i k/N}$.

So $\langle \frac{1}{N} \mathrm{tr}\, \Omega \rangle$ is an order parameter for center symmetry.

Physically:



$$\Rightarrow \quad \langle \tfrac{1}{N} \operatorname{tr} \Omega \rangle \sim e^{-LF}$$

$F \sim$ free energy of heavy test quark!

confinement $\sim$ energy cost $\to \infty$, $F \to \infty$, $\langle \tfrac{1}{N} \operatorname{tr} \Omega \rangle = 0$

deconfinement $\sim$ $\langle \tfrac{1}{N} \operatorname{tr} \Omega \rangle \sim \mathcal{O}(1)$.

Another order parameter: free energy of full gauge theory $\to$ valid at large N.

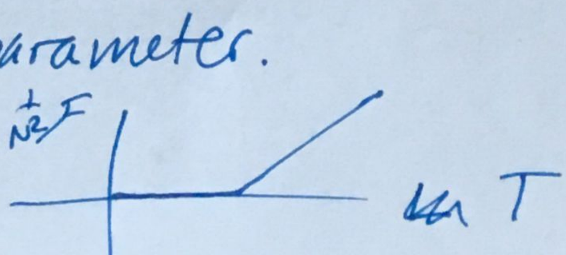
confined: degrees of freedom are glueballs & mesons
masses $\sim \mathcal{O}(N^0)$, degeneracies $\sim \mathcal{O}(N^0)$
interactions $\sim \mathcal{O}(1/N)$

$$\Rightarrow F \sim \mathcal{O}(N^0).$$

deconfined: quarks and gluons liberated
gluon dof's $(A_\mu)_{ij} \sim \mathcal{O}(N^2)$

$$\Rightarrow F \sim \mathcal{O}(N^2).$$

$\Rightarrow \quad \frac{1}{N^2} F$ is order parameter.



In fact $\frac{1}{N^2}F$ and $\langle \frac{1}{N} \mathrm{tr}\, \Omega \rangle$ order parameters are related, as we will see now.

Compactify pure YM on $\mathbb{R}^3 \times S^1_L$.
$\nwarrow x_4$.

Broken Lorentz $\Rightarrow$ $\langle A_4 \rangle \neq 0$ is <u>possible</u>.

$\langle A_4 \rangle$ is determined dynamically.

$$\int d^4x \, \frac{N}{\lambda} \cdot \frac{1}{2} \mathrm{tr}\, F_{MN} F^{MN} = \cancel{\int d^4x \frac{N}{2\lambda} \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}}$$

$$+$$

$$= \int d^4x \left[\frac{N}{2\lambda} \left(\mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} + 2\, \mathrm{tr}\, F_{\mu 4} F^{\mu 4} \right) \right].$$

Can always choose gauge where $\langle A_4 \rangle$ is diagonal.

$$F_{\mu 4} \sim D_\mu A_4 \sim \underbrace{\partial_\mu A_4}_{0} + \underbrace{[A_\mu, A_4]}_{0}.$$

$\Rightarrow$ $\langle A_4 \rangle$ has a classical <u>moduli space</u>.

free parameter <u>classically</u>.

$\Rightarrow$ quantum corrections generically lift it
$\sim$ generate potential. (exception: $\mathcal{N}=4$ SYM)

We have to compute 1-loop potential - this is a Coleman-Weinberg calculation, first done here by Gross, Pisarski, Yaffe: Write $A_4 = \langle A_4 \rangle + \tilde{A}_4$, then integrate out $\tilde{A}_4$, A_μ at one loop.

Result: $V_{eff} = V_{eff}(\langle A_4 \rangle)$.

Gauge invariance $\Rightarrow$ $V_{eff}(\langle A_4 \rangle) = V(\mathrm{tr}\,\Omega)$

$$\Omega = e^{i\oint A_4}$$

We are working @ 1-loop $\Rightarrow$

$$V_{eff}(\Omega) = \frac{1}{L^4} \sum_{n=1}^{\infty} c_n \, \mathrm{tr}_{Adj.} \Omega^n$$

dim-analysis

all fields adjoint

$$= \frac{1}{L^4} \sum_{n=1}^{\infty} c_n \left(|\mathrm{tr}_F \Omega^n|^2 - 1 \right)$$

pure numbers, have to be calculated.

Could do it directly. But won't! Eventually we want to minimize this w.r.t Ω. But suppose $\Omega = \mathbb{1}$.

Then $V_{eff} = \frac{1}{L^4} \sum_{n=1}^{\infty} c_n (N^2 - 1) = -F$

free energy density in massless theory, $L = 1/T$

physical # of d.o.f's: $2(N^2-1)$

polarizations

But we know that F for a single free massless boson is

$$F = \frac{2}{\pi^2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^4} \cdot T^4$$

$$\Rightarrow \quad c_n = \frac{2}{\pi^2} \cdot \frac{1}{n^4} .$$

$$\Rightarrow \quad V_{eff}^{YM}(\Omega) = -\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^4} \left(|tr_F \Omega^n|^2 - 1 \right)$$

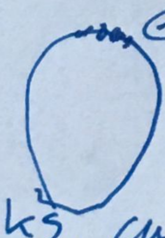
So, in pure YM, this is minimized when $\Omega = 1 \cdot \omega$; center is broken at small $L\Lambda$, where $\lambda(\frac{1}{L}) \to 0$.

Reproduces standard expectation: high $T \sim$ broken center.

$F \sim N^2 \to \frac{F}{N^2} \to \sim O(1)$ deconfinement

$\Omega \in SU(N) \Rightarrow$ diagonal basis, eigenvalues must lie on unit circle

$$\begin{pmatrix} e^{i\alpha_1} & & \\ & e^{i\alpha_2} & \\ & & \ddots \end{pmatrix}$$



$\leftarrow$ all clumped.

Without further tricks, can't proceed with pure YM.

What about <u>other</u> QFTs?

Try "adjoint QCD". SU(N) YM + N_f adjoint massless Majorana fermions.

$N_f = 1$ + massless $\Rightarrow$ $\underline{\mathcal{N}=1 \text{ SYM}}$.

Could do directly for V_{eff}; or do stat-mech again.
Or, use susy!
← easiest.

Suppose we preserve susy, $\mathbb{R}^3 \times S^1$ PBCs for everyone!
$N_f = 1$.

At 1-loop, bosonic & fermionic contributions must add.

$$V_{eff} = V_{bosonic} + V_{fermionic}.$$

$V_{bosonic}$: 2 d.o.f., from 1 loop

$V_{fermionic}$: susy: 2 fermionic d.o.f. from 1 loop.

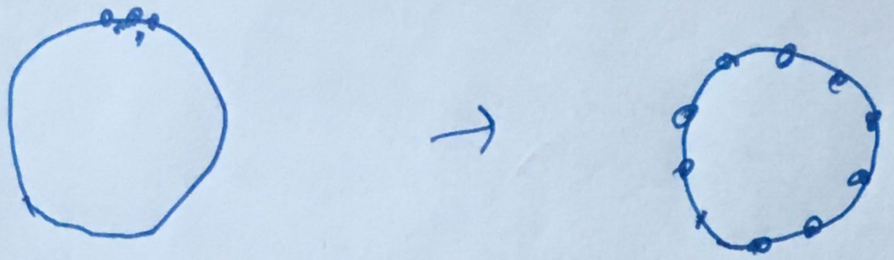
Fermion loop has (-1) sign. ⇒ 1-loop cancellation!, for $N_f = 1$.

⇒ generic N_f @ 1 loop

$$V_{eff}(\Omega) = (N_f - 1) \cdot \frac{2}{\pi^2} \cdot \frac{1}{L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \left(|\mathrm{tr}_F \Omega^n|^2 - 1 \right).$$

First focus on $\underline{N_f > 1}$. Before potential favored eigenvalue attraction. $N_f = 2$: sign flip!

Now, all eigenvalues repel.



smoothly distributed!

In terms of Ω, we get

$$\Omega = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix}, \quad \omega = e^{2\pi i/N} \text{ for odd } N.$$

<u>Exercise</u>: verify $\det \Omega = 1$, find fix for even N_f.

<u>Check</u>: $\mathrm{tr}\,\Omega = 1 + \underbrace{\omega + \cdots + \omega^{N-1}}_{\text{roots of unity}} = 0$!

$\mathrm{tr}\,\Omega^2 = 0, \ldots, \mathrm{tr}\,\Omega^n = 0 \;\forall\, n \neq Nk,\; k \in \mathbb{N}$.

<u>Consequence</u>: confinement is maintained even at very small L, center. sym. is good!

$$V_{eff}\Big|_{min} = \frac{2(N_f-1)}{\pi^2\beta^4} \sum_{n=1}^{\infty} \underbrace{\frac{1}{n^4}\left(|\mathrm{tr}\,\Omega^n|^2 - 1\right)}_{\left(\frac{1}{n^4}\right)(-1) + \mathcal{O}(1/N^2)}$$

$$= -\frac{2(N_f-1)}{\pi^2\beta^4}\zeta(4) + \mathcal{O}(1/N^2) = -F$$

So $\frac{F}{N^2} \sim \mathcal{O}(1/N^2) \to 0$ at large N.

two criteria for confinement agree.

What about spectrum? Is theory really weakly coupled? Is there a gap?

Translating $\Omega \to A_4$, $\mathrm{tr}\,\Omega = 0 \Rightarrow A_4 \neq 0$!.

But remember

$$\int d^4x \; \mathrm{tr}\, F_{MN} F^{MN} \Rightarrow L \int d^3x \left(F_{\mu\nu} F^{\mu\nu} + 2\, D_\mu A_4 \right)$$

A_4 acts like 3D adjoint scalar. $\langle A_4 \rangle \neq 0$

$\Rightarrow$ adjoint Higgs mechanism.

Let's focus on KK-zero modes; then e.g.

$$m_W^{m,n} = \underbrace{\frac{2\pi}{N_c L}}_{\text{eigenvalue spacing}} (i - j)$$

$i - j$ ← color labels.

Consequence: off-diagonal A's pick mass. $\sim \frac{1}{NL}$!

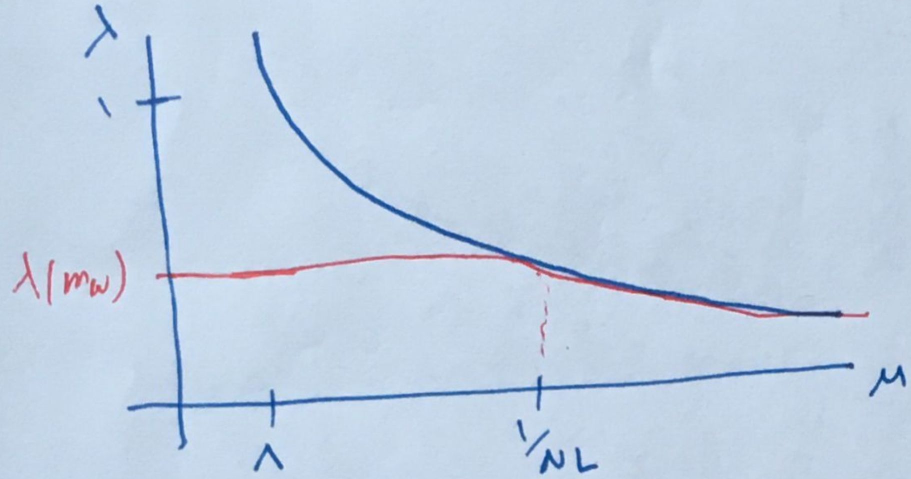
diagonal ones don't.

$SU(N) \to U(1)^{N-1}$ "Coulomb branch".

~~In fact we'll~~ Consequence:

All charged matter gets mass $\sim \frac{1}{NL}$.

$\Rightarrow$ coupling must stop running at $\sim \frac{1}{NL}$.



So if we take $\Lambda N L << 1$ we get a weakly coupled theory at all distances!

Abelianization key for this!

Long-distance theory in pert. theory:

$$S = \frac{L}{g^2} \int d^3x \sum_{n=1}^{N-1} F_{\mu\nu}^{(i)}$$ $\leftarrow$ Cartan gluons.

Already saw this at 1 loop. All-loop argument:

$$F_{\mu\nu}^{(i)} = \frac{g^2}{2\pi L} \epsilon_{\mu\nu\rho} \partial_\rho \sigma^{(i)}$$ $\leftarrow$ dual scalar.

3D Abelian duality!

Dual scalar has shift symm! $\sigma^{(i)} \to \sigma^{(i)} + c^{(i)}$.

Reason: associated conserved current $\Leftrightarrow$ abscence of magnetic monopoles!

Noether current: $\vec{J}_\mu = \partial_\mu \vec{\sigma}$.

But note $J_\mu = \epsilon_{\mu\nu\rho} F^{\nu\rho} =: B_\mu$.

magnetic field.

Then $\partial_\mu J^\mu = 0 \Rightarrow \partial_\mu B^\mu = 0$

This is condition for ~~abs~~ lack of magnetic monopoles.

$\vec{\sigma}$: neutral. W-bosons electrically charged

lightest $Q = g\alpha_i$

next lightest $g(\alpha_i + \alpha_{i+1})$ etc.

quarks are electrically charged.

NOTHING has magnetic charge... In pert. theory!

~~So what about~~ So long distance theory really is

$$S = \int d^3x \left(\frac{g}{2\pi}\right)^2 \cdot \frac{1}{L} \sum_{n=1}^{N-1} (\partial_\mu \sigma_n)^2 .$$

In perturbation theory, to all orders.

Gapless theory!? No. Have to look at NP effects.

Before we do, some comments.

Assumed $N_f \neq 1$. What about $N_f = 1$?

$V_{eff} = 0$ to all orders in p.t.

But, $\exists$ NP effects, generate potential

$$V_{eff} = e^{-16\pi^2/g^2N} \cdot \frac{\#}{\beta^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \left(|\mathrm{tr}\, \Omega^n|^2 - 1 \right)$$

stabilize center.

Hollowood, Khoze, Mattis 1997; ~~Sch~~ Poppitz, Schaefer, Unsal, 2012.

Our discussion then applies.

what about turning on fermion mass? ~~tom~~

For $N_f \geq 2$, $m_F \lesssim \frac{1}{NL}$, center still stabilized.

$N_f = 1$, subtle, see Poppitz et al.

$N_f = 0$? Can use double-trace deformations,

$$S_{YM} \to S_{YM} + S_d, \quad S_d = \int d^3x \sum_{n=1}^{\infty} a_n \left(|\mathrm{tr}\, \Omega^n| \right)^2$$

choose!

Small L: preserve center! Large L: negligible effect.

intermediate L: smooth, from lattice simulations.

Unsal + Yaffe, 2010.

Back to small L story. Focus on "dual scalars"

↳ universal.

Saw that they're gapless in PT, due to emergent topological symm, ($\Leftrightarrow$) absence of mag. monopoles.

At NP level, story changes!

$$SU(N) \to U(1)^{N-1} \qquad \pi_2\left(\frac{SU(N)}{U(1)^{N-1}}\right) = \mathbb{Z}^{N-1}.$$

$\Rightarrow \exists$ 't Hooft-Polyakov magnetic monopoles,

with magnetic charge

$c \in \mathbb{Z}$. $$\int_{S^2} d\Sigma \cdot F = \frac{2\pi}{g} \alpha_i c \text{ for type } i \text{ monopole.}$$

α_i = simple root of $su(N)$ algebra

$$\alpha_1 = (1,-1,0,\dots),\ \alpha_2 = (0,+1,-1,0\dots)$$

$$\alpha_{N-1} = \underbrace{(0,\dots,1,-1)}_{N\text{-dim basis}};\ \alpha_N = (-1,\dots,1) = -\sum_{j=1}^{N-1} \alpha_i$$

(α_N is <u>affine</u> root.

Using N-dim basis simplifies life; $\sim SU(N) \to U(N)$.

extra "diagonal" photon completely decouples.

$\mathrm{tr}\, t^a t^b = \delta^{ab}$.

<u>NB</u>: If Higgs field compact, $\exists$ N monopoles, <u>not</u> N−1.

Lee-Yi, Kraan-Van-Baal.

these monopoles are <u>also</u> instantons:

$$\int \frac{1}{16\pi^2} \operatorname{tr} F_{MN} \tilde{F}^{MN} \cdot d^4x = \frac{1}{N} k, \quad k \in \mathbb{Z}.$$

$\Rightarrow$ their weight in a path integral is

~~exp{~~

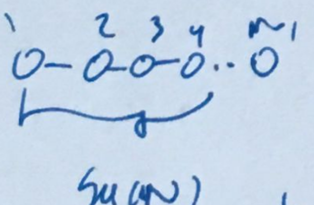
$$S = \int_{\mathbb{R}^3 \times S^1} \frac{1}{2g^2} \operatorname{tr} F_{MN}^2 \underset{\substack{\uparrow \\ \text{self-dual}}}{=} \left| \int_{\mathbb{R}^3 \times S^1} \frac{1}{2g^2} \operatorname{tr} F \wedge F \right| = \frac{8\pi^2}{g^2 N}$$

$\Rightarrow$ weight $= \exp\left[-\frac{8\pi^2}{g^2 N} \right]$ $\leftarrow$ 't-Hooft coupling!

<u>Note:</u> cyclic permutations of Ω eigenvalues is part of gauge sym.

~~that~~ cyclically permutes monopole-instantons

$\hookrightarrow$ only present for <u>compact</u> Higgsing



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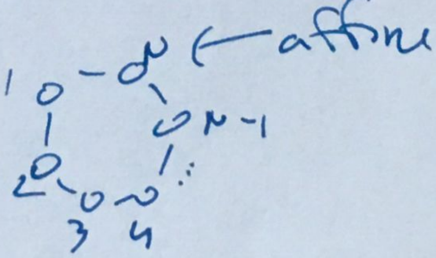


Dynkin diagrams.

~~Thus~~ Since monopole-instantons are BPS, $\exists$ explicit solutions: (19)

For instance in $SU(2)$, $\Omega_{\text{center-sym}} = \begin{pmatrix} e^{iLv/2} & 0 \\ 0 & e^{-iLv/2} \end{pmatrix}$,

$v = \pi/L = \frac{1}{2}\langle A_4 \rangle$.

Then $A_4^a = \mp n_a v\, P(vr)$

$$A_i^a = \epsilon_{aij} n_j \frac{1 - A(vr)}{r}$$

$$P(x) = \coth x - 1/x \; ; \; A(x) = \frac{x}{\sinh x} \; ; \; n_a = \frac{r_a}{|\vec{r}|}$$

Note that they are independent of x_4!

In the $SU(2)$ case, can verify $S_E = \frac{4\pi L v}{g^2} = \frac{8\pi^2}{g^2 \cdot 2}$ ← N_c.

Fact that Ω is a compact variable

$\Leftrightarrow$ vacua with $\langle A_4^3 \rangle$ differing by $k \cdot \frac{2\pi}{L}$

can use are ~~equivalent, since what enters is $e^{gA_4} = \Omega$~~.

$\Rightarrow$ extra solutions which interpolate between vacua with different twists! Existence tied to 4D nature.

generates k-dep twist in x_4

Lowest action twisted monopole-instantons:

$$S_E = \frac{4\pi L}{g^2} \left| \frac{2\pi k}{L} + v \right| ; \text{ take } k = -1, v = \pi/L$$

$$S_E^{\text{center}} = \frac{8\pi^2}{g^2 \cdot 2} \quad \text{same as above}$$

Normal $\underbrace{\text{BPST instanton}}_{\text{4D beast!}}$ = sum of usual monopole-insts + $\underbrace{\text{KK one}}_{\text{vital}}$. All BPS!

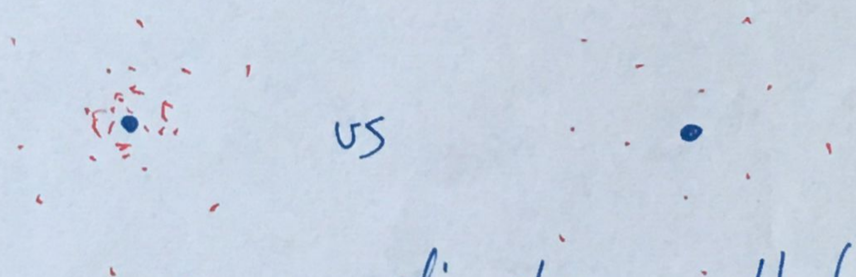
So what does the Euclidean vacuum look like?
Heuristically, we have a bunch of mag monopoles, interacting via

$$\underset{3D}{V_{\pm i, \pm j}(\vec{r})} = L\left(\frac{2\pi}{g}\right)^2 \underbrace{\frac{(\pm\vec{\alpha}_i)\cdot(\pm(\vec{\alpha}_j)}{4\pi\,|\vec{r}|}}_{\text{Coulomb!}} = \pm L\left(\frac{2\pi}{g}\right)^2 \frac{2\delta_{i,j} - \delta_{i,i+1} - \delta_{i,i-1}}{4\pi\,|\vec{r}|}.$$

multi-component classical plasma, nearest-neighbor interactions in "Dynkin-space".

$\Rightarrow$ Debye screening for <u>magnetic</u> charge

$\frac{1}{r} e^{-m_D r}$, for some m_D (needs calculating)



screening of magnetic charge ~ old dual superconductor picture of <u>confinement</u>.

How to calculate screening scale?

$\Rightarrow$ Understand $\vec{\sigma}$ interactions induced by monopole-instantons!

Remember that without monopoles,

$$S_\sigma = \int d^3x \left(\frac{g}{2\pi}\right)^2 \frac{1}{2} \sum_{n=1}^{N-1} (\partial_\mu \sigma_n)^2 .$$

writing $\sigma_n \to \underset{\substack{\wedge \\ N\text{-vec}}}{\vec{\sigma}}$ s.t. $\vec{\alpha}_n \cdot \vec{\sigma} = \sigma_n$, (added σ_0, $SU(N) \to \underset{\substack{\wedge \\ \text{extra free field}}}{U(N)}$)

$$S_\sigma = \int d^3x \left(\frac{g}{2\pi}\right)^2 \frac{1}{2} (\partial_\mu \vec{\sigma}) \cdot (\partial^\mu \vec{\sigma}) .$$

Now $\quad S_\sigma \to S_\sigma + \int d^3x \; e^{-S_0} f(\vec{\sigma})$

$\qquad\qquad\qquad\qquad\qquad\qquad \uparrow S_0 = \dfrac{8\pi^2}{g^2 N}$

$f(\vec{\sigma})$ not derivatively-coupled

$\rightsquigarrow$ monopoles have mg. charge, break $\vec{J}_\mu = \partial_\mu \vec{\sigma}$ symmetry.

Breaking <u>not</u> arbitrary: must respect $\vec{\sigma}$ periodicity!

comes from fact that $SU(N)$ is <u>compact</u>.

periodicity:

$$\vec{\sigma} = \vec{\sigma} + 2\pi \vec{\omega}_n , \quad n = 1, \ldots, N.$$

To figure this out, $\mathbb{R}^3 \times S^1$

$$\downarrow$$

$$\mathbb{R} \times \underset{\sim}{\mathbb{T}^2} \times S^1$$

magnetic fluxes live here

flux quantization $\Longrightarrow$ periodicity,

see e.g. Anber, Poppitz, Teeple 1406.1199.

To enforce periodicity, $f_i(\vec{\sigma}) = f(e^{i\vec{\alpha}_i^* \cdot \vec{\sigma}})$

$\vec{\alpha}_i^* \cdot w_j = \delta_{ij}$

co-root vectors

$\Rightarrow$ root vectors in $SU(N)$

To find actual $f(\vec{\sigma})$ need to actually calculate....
sum up fluctuations around monopole-instanton.
Gives 't Hooft-like "monopole-instanton vertex"

for type i $\rightarrow$ $$f_i(\vec{\sigma}) = \underbrace{A\, m_W^3 (g^2 N)^{-2}}_{\zeta} \left(e^{i\vec{\alpha}_i \cdot \vec{\sigma}} + h.c.\right)$$

. So, taking leading $\underline{NP}$ effects, the $\vec{\sigma}$ action is

$$S_\sigma = \int d^3x \left[\left(\frac{g}{2\pi}\right)^2 \cdot \frac{1}{L} \left(\partial_\mu \vec{\sigma}\right)^2 + \zeta \sum_{i=1}^{N} \cos(\vec{\alpha}_i \cdot \vec{\sigma}) \underbrace{+ \ldots}_{\text{higher-order effects}} \right]$$

This is a $\underline{\text{big deal}}$.

$\Rightarrow$ $\exists$ a mass gap for gauge fluctuations.

↳ solution of mass gap problem on $\mathbb{R}^3 \times S^1$.

↳ can also use it to show $\exists$ string tension.

Let's see the mass gap! Switch to canonical ~~coord~~ basis $\sigma \to \sqrt{L}\,\frac{2\pi}{g}\,\tilde{\sigma}$, expand around $\tilde{\sigma}_{min} = 0$,

$$V(\tilde{\sigma}_i) = \frac{1}{2} m_\gamma^2 \sum_{i=1}^{N} \left(\sigma_{i+1} - \sigma_i\right)^2 ,$$

$$m_\gamma = \cancel{\frac{(2\pi)^3}{g\lambda}}\ \frac{(2\pi)^2}{g^2 N} NL\zeta = A\, m_W^2 \left(\frac{2\pi}{\lambda}\right)^3 e^{-\frac{8\pi^2}{\lambda}} .$$

Not yet diagonal in label space.

$\Rightarrow$ Diagonalize via $\mathbb{Z}_N$ fourier transform:

$$\tilde{\sigma}_p = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{2\pi i \cdot pj}\, \sigma_j$$

$$\Rightarrow V(\tilde{\sigma}) = \frac{1}{2} \sum_{p=1}^{N-1} m_p^2 \left|\tilde{\sigma}_p\right|^2$$

$$m_p = m_\gamma \sin\left(\frac{\pi p}{N}\right), \quad p = N \text{ fictitious.}$$

$$m_p > 0 \ \forall\ p = 1, \dots, N-1$$

There's a gap!

<u>NB</u>: You might be worried about gauge-dep.

σ_p modes have gauge-invariant expression:

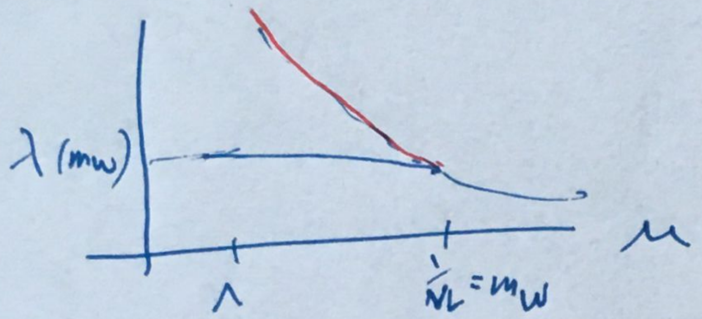
Note that $\underbrace{\frac{1}{N}\sum_{n=0}^{N-1} \omega^{-kn}\,\Omega^n}_{P_k}\Big|_{\text{center-symm}} = \begin{pmatrix} 0 & & & 0 \\ 0 & \ddots & 1 & \\ 0 & & 0 & \ddots \\ & & & 0 \end{pmatrix} \leftarrow k$

$$\Rightarrow F_{\mu\nu}^k = \frac{1}{N} \mathrm{tr}\left(P_k F_{\mu\nu}\right) \approx \epsilon_{\mu\nu\rho}\, \partial^\rho \sigma^k$$

We can write gap in terms of Λ_{QCD}:

$$\Lambda^{b_0} = \mu^{b_0} \left(N g^2(\mu)\right)^{-b_1/b_0} e^{-\frac{8\pi^2}{g^2(\mu) N}}$$

Dimensional transmutation



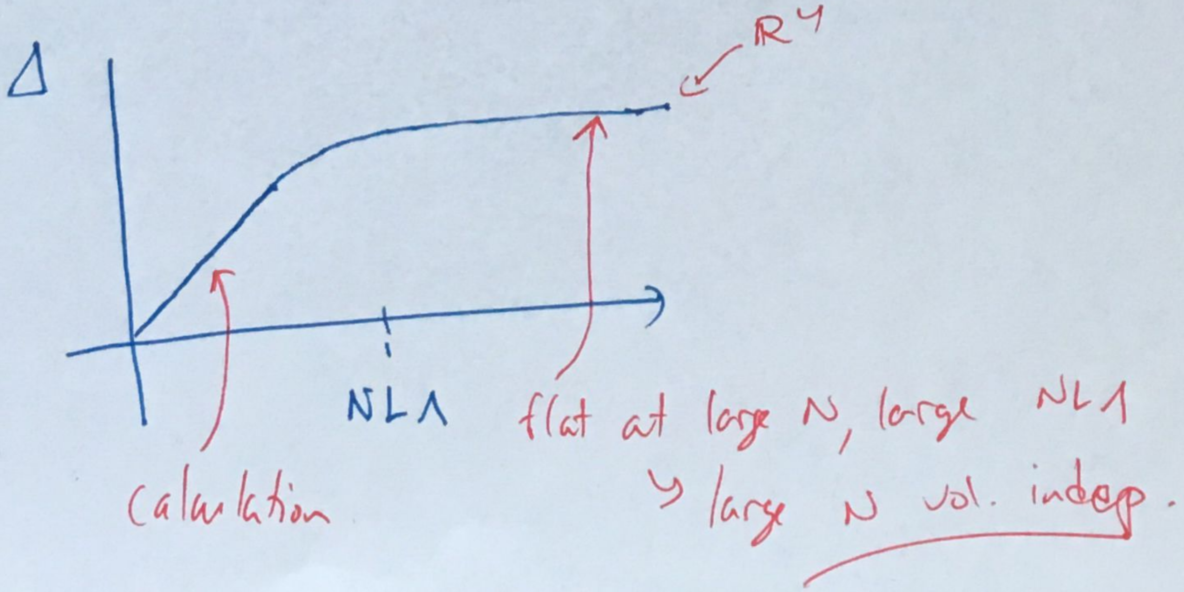
$m_W \gg \Lambda$, can take $\mu = m_W$.

Then $m_\gamma = \tilde{A}\, \Lambda \cdot (\Lambda NL)^{5/6} \left|\ln NL\Lambda\right|^{9/11}$.

since for pure YM $b_0 = 11/3$, $b_1 = \frac{17}{3}$.

$\Rightarrow$ Dependence on Λ is power-like, not purely logarithmic as at high T.

$\rightarrow$ Long-distance-dynamics is weakly coupled.



Before diving into corrections to this picture, and renormalons, comment on QFTs with <u>massless fermions</u>:

monopole-instantons carry (mag, top) charge ~~(top,~~

$$(\int F \, , \int F\tilde{F})$$

BPS : $(+1, +1/N)$ $\quad$ $\overline{\text{BPS}} = (-1, -1/N)$ ← N-1 types.

KK : $(-1, +1/N)$ $\quad$ $\overline{\text{KK}} = (+1, -1/N)$

Suppose we add N_f massless adjoint fermions.

Index theorem ⇒ charge Q config has

$2 N_f N_c Q$ fermion zero modes.

Here $2N_f N_c Q = 2N_f$ for any given object.

Adjoint Higgsing gives "mass" to everything except <u>Cartan</u> fermions $\vec{\Psi}_a$ ← flavor.

N-vector, like $\vec{\sigma}$, obeying $\alpha_k \cdot \vec{\Psi}_a = \underbrace{\Psi_{k,a}}_{\text{Cartan}}$.

For simplicity, take $N_f = 1$. Then monopole-inst amplitude is
$N_c =$

$$\sim e^{-\frac{8\pi^2}{g^2 N}} \psi_k \psi_k e^{i \alpha_k \cdot \vec{\sigma}}$$

$\Rightarrow$ interaction term is $\int d^3x \; g \sum_k \cos(\vec{\alpha}_k \cdot \vec{\sigma}) \psi_k \psi_k$

This is not an interaction of σ with itself!

NOT mass term for $\vec{\sigma}$!

$\Rightarrow$ Monopole-instantons contributes to fermionic potential, NOT bosonic one.

What if we were in a purely 3D theory?

$\Rightarrow$ NO KK-monopole-inst's.

Any combination of BPS KK M_n's

either has ① (mag, top) = (0, 0) BPS + $\overline{\text{BPS}}$

OR ② $(\#, \pm\frac{k}{N})$ BPS + BPS
$k \in \mathbb{N}$.

But

$$\partial_M J_5^M = \frac{g^2 N}{32\pi^2} \cdot 2N_f \operatorname{tr} F_{MN} \tilde{F}_{MN}$$

$\Rightarrow$ $\Delta Q_5 \neq 0$ so long as $Q = \int d^4x \operatorname{tr} F \tilde{F} \neq 0$.

$\Rightarrow$ ② has fermion zero modes!
can't contribute.

BPS + $\overline{\text{BPS}}$ has no zero modes, but also has no magnetic charge!

$\Rightarrow$ Can't break $\vec{\sigma}$ shift symmetries.

$\Rightarrow$ Famous result: 3D SU(N) + ~~Adj~~ Higgs + ~~Adj~~ ~~fermion~~ massless fermions

has no gap! Affleck, Harvey, Witten 1982.

But we're working with locally 4D theory, and we do have KK-monopole-instantons!

SU(2): BPS + $\overline{\text{KK}}$ = (2, 0) = (mag, top).
$(1, 1/2)$ $\quad (1, -1/2)$

- No top charge $\Rightarrow$ no overall zero modes.
- Non-zero mag. charge $\Rightarrow$ will generate mass gap for $\vec{\sigma}$ fields.
- No top charge $\Rightarrow$ NON-BPS; called "magnetic bion".

No closed-form exact solution in 4D case.

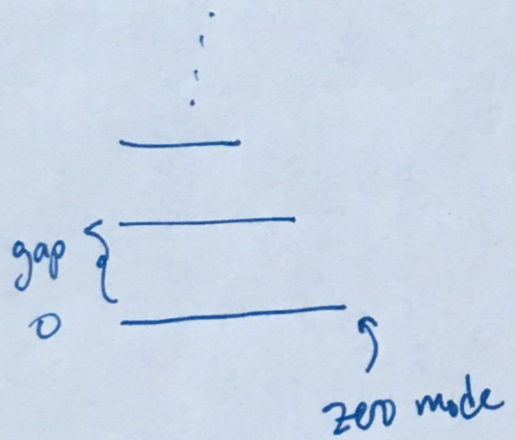
~~with~~ Discussing magnetic bions will bring us to resurgence.

Topological Molecules

Suppose we have finite action configuration;

$$Z \ni e^{-S} \cdot \Sigma \text{ fluctuations}$$

What do fluctuations look like?



Origin of zero modes: symmetries!

3 translation zero modes

Monopole-instanton + bunch of color-orientation ones.

- The zero mode integrals are always trivially doable, give factors like $(S_I)^n$ for various n depending on # of zero modes.
- Rest of modes can be integrated out in Gaussian approx.

Exception :

← non-zero, but very close!

This is called a quasi-zero mode. Has to be treated separately from other modes, integral needs to be done exactly.

Where would QZM's come from?

$$\underset{BPS}{\circ} \xleftrightarrow{\tau \to QZM} \underset{\overline{BPS}}{\circ} \qquad \underline{\text{NOT}} \text{ solution to EoM.}$$

$\overset{\leftrightarrow}{\bot}$ zero-mode

$$\text{Action} = 2S_0 - c\,e^{-\tau/m_\omega}$$

action <u>decreases</u> for $\tau \to m_\omega$.

But for large τ, dependence small, and $\tau \to \infty$, dependence vanishes.

$\Rightarrow$ BPS - $\overline{\text{BPS}}$ is a <u>quasi-solution</u>, or, "solution at infinity", with action $2S_0$, and

① zero modes
② quasi-zero modes
③ Regular gaussian modes.

View ∞-sep config as <u>the</u> solution, all else as <u>fluctuations</u>.

Remember our original interest in resurgence:

$$\langle \sigma \rangle = \sum_m p_m \lambda^m + \sum_c e^{-S_c} \sum_{n,k} \lambda^n \log(1/\lambda)^k\, p_{n,k}$$

fluctuations around $S_c = 0$ saddle point

fluctuations around $S_c \neq 0$ saddle!

So what does sum/integral over QZM look like?

BPS type i $\overline{\text{BPS}}$ j interact via $\begin{cases} \sigma\text{-exchange} \\ \psi\text{-exchange}. \end{cases}$

[in SUSY, extra massless mode]

$\longleftrightarrow$ r

Compute action, S, write

$$S - 2S_0 = V_{eff}(r)$$

$$V^{ij}_{eff}(r) = -(\vec{\alpha}_i \cdot \vec{\alpha}_j)\, \frac{2\pi}{g^2} \cdot \frac{L}{r} + 4 n_f \log(r)$$

coeff is 4π in SUSY case

"σ-exchange"
depends on
α_i, α_j,
could be attractive,
or repulsive

"ψ-exchange"
__always__
attractive.

This is the potential entering Euclidean monopole plasma picture!

So what does the bion amplitude look like?

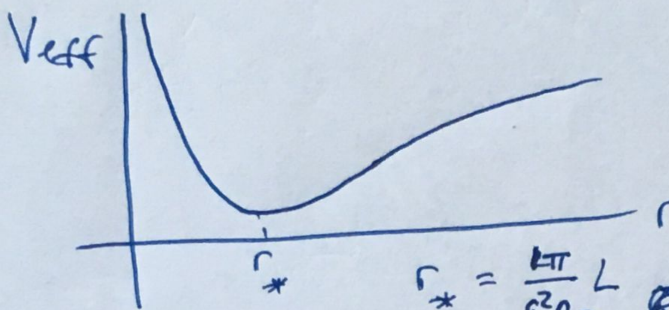
- $e^{-2S_0} \, e^{i(\vec{\alpha}_i - \vec{\alpha}_j)\cdot\vec{\sigma}} \cdot (\text{fluc})$

$$\text{fluct} \approx (\text{monopole fluc})^2 \int d^3 r \, e^{-V_{\text{eff}}^{ij}(r)}.$$

Suppose $\alpha_i \cdot \alpha_j < 0$; happens for nearest-neighbors since

$$\alpha_i \cdot \alpha_j = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}$$

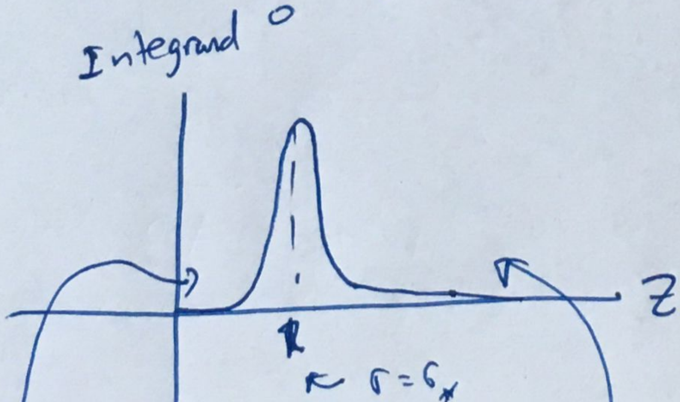
Then



$$r_* = \frac{4\pi}{g^2 n_f} L \approx \frac{m_W^{-1}}{\lambda n_f}$$

$$\int d^3 r \, e^{-V_{\text{eff}}^{ij}(r)} \quad \cancel{\int dz \, z}$$

$$= \int_0^\infty \left(\frac{\pi L}{g^2 n_f}\right)^{3-4n_f} e^{-\frac{4 n_f}{z}} z^{2-4n_f} \, dz$$



"Coulomb blockade"
Can't get on top of each other.

zero-mode suppression!
individual monopoles forbidden.

This is good news!

$$r_M \sim m_W^{-1} \ll r_b \sim \frac{m_W^{-1}}{\lambda} \ll m_W e^{+S_0/3} \ll m_W e^{2S_0/3}$$

$\# \text{ of directions}$

$$\approx r_{M-m} \qquad r_{b-b}$$

makes sense to think of bions as "molecules".

$$I(g^2) = \int_0^\infty dz\, e^{-\frac{4n_f}{z}} z^{2-4n_f} \left(\frac{\pi L}{g^2 N_F}\right)^{3-4n_f} \sim \left(\frac{1}{g^2}\right)^{3-4n_f} \Gamma(4n_f - 3).$$

So, after all that, we get magnetic-bion induced potential for σ:

$$S_\sigma = \int d^3x \left(\frac{g}{2\pi}\right)^2 \frac{1}{L} (\partial_\mu \vec{\sigma})^2 + e^{-2S_0} \sum_{i=1}^{N} \cos((\vec{\alpha}_i - \vec{\alpha}_{i+1}) \cdot \vec{\sigma})$$

Can again expand around vacuum, diagonalize,

result: $\quad m_p^2 \sim m_W^2 e^{-2S_0} \sin^2\left(\frac{\pi p}{N}\right)$

$$p = 1, \dots, N-1.$$

So again, $M_{min} > 0$. Mass gap!

But exponentially smaller than in the bosonic case, or systems with massive fermions.

But what happens if we look at $BPS_i - \overline{BPS}_i$ composites?

Then $$V_{eff}^{ii} = -\vec{\alpha}_i \cdot \vec{\alpha}_i \frac{2\pi L}{g^2 r} + 4 n_f \log r$$

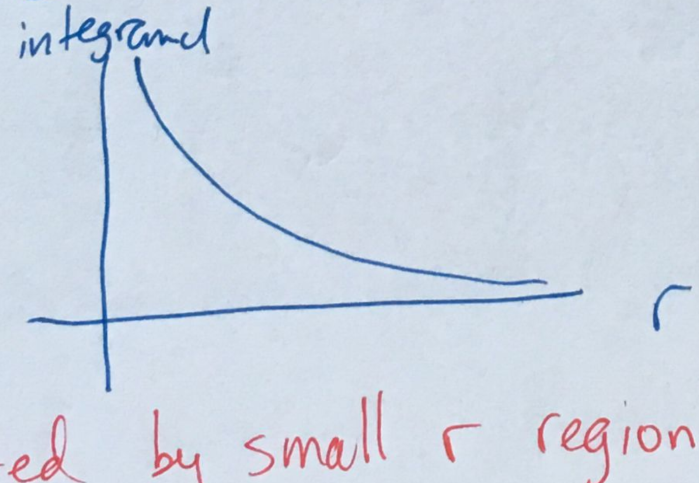
$$\int d^3 r\, e^{-V_{eff}^{ii}} = \int_0^\infty dr\, r^2 \exp\left[+\frac{4\pi L}{g^2 r} - 4 n_f \log r\right]$$

$$= \int_0^\infty dr\, \exp\left[+\frac{4\pi L}{g^2 r} - (4 n_f - 2) \log r\right].$$

Before proceeding, note that such events should contribute to e.g. effective potential for $\langle \Omega \rangle$, to vacuum energy, etc.

$\hookrightarrow$ $\exists$ perturbative contributions.

The "neutral bion" QZM integral deeply problematic



- Dominated by small r region!
- Integral diverges!

This is a disaster, on its face.

$\Rightarrow$ "neutral bion" indistinguishable from pert vacuum.

$\Rightarrow$ dilute gas approximation is not valid.

$\Rightarrow$ Same rules gave nice result for magnetic bion, and mass gap, give garbage for neutral bion. Why trust any of it?

- We will see that the disaster is blessing in disguise $\rightsquigarrow$ untangling reveals the semi-classical avatar of renormalons!

To explain this, will give two stories.

- Old: "Bogomolny - Zinn-Justin Prescription" 1980s, from QM.
  works, but ad hoc.
- New: Lefshetz Thimbles.
  - much more systematic, gives extra insights.

Lefshetz thimbles

Go back to the beginning. Why transseries?

$$\langle \mathcal{O}(\lambda) \rangle = \frac{1}{Z} \int d[\phi] \underbrace{e^{-\frac{1}{\lambda} S[\phi]} \mathcal{O}}_{\exp[-\frac{1}{\lambda} S(\phi) + \log \mathcal{O}]}.$$

Why should

$$\langle \mathcal{O} \rangle = \sum p_n \lambda^n + \sum e^{-S_c/\lambda} \sum b_n \lambda^n \;?$$

Ans: Each group = expansion around saddles with action $\frac{S_c}{\lambda}$.

Saddle-point expansion = steepest descent expansion.

Easy in 1D integrals, where steepest descent = ~~constant~~ stationary phase.

$\exists$ generalization to many dimensions.

steepest descent integration "cycles" = Lefshetz Thimbles.

& Each "attached" to a critical point

$$\underbrace{\sum p_n \lambda^n}_{\substack{\text{"perturbative"} \\ \text{thimble} \\ \text{integral}}} + \underbrace{\underbrace{e^{-\frac{2S}{\lambda}}}_{\substack{\text{saddle} \\ \text{action}}} \underbrace{\sum b_n \lambda^n}_{\text{fluctuations}}}_{\text{NP Thimble integral}}$$

General fact: ~~Fram~~

To find thimble contours, complexify field space

① $\mathbb{R}^N \to \mathbb{C}^N$

② Find saddle points in $\mathbb{C}^N$. (Not just on $\mathbb{R}^N$.)

↳ not just minima!

③ For each saddle, thimble is half-dim ~~contour~~ manifold in $\mathbb{C}^N$ obeying

$$* \quad \frac{dz_i}{du} = \overline{\left(\frac{\delta S(z)}{\delta z_i}\right)}, \qquad z_i(u\to-\infty) = \underbrace{z^*_{i(c)}}_{\text{saddle}}$$

flow time

can show $\dfrac{\partial \, \mathrm{Im}[S]}{\partial u} = 0$ if $*$ is obeyed.

Call thimble associated to saddle z^*_c $\mathcal{J}_c$.

Then integral on $\mathcal{I}$ = integral on $\sum_c \sigma_c \mathcal{J}_c$ ← integers,

~~Work~~ For this to work, important to avoid degenerate points in parameter space where steepest descent would connect two or more critical points ← Stokes lines.

↳ In QFT, almost always sitting on Stokes line!

Warning: Theory reasonably developed in $d=0$, finite-dim integrals.

· At formal level, ∃ thimble theory for QM + practically, strong evidence from direct calculation.

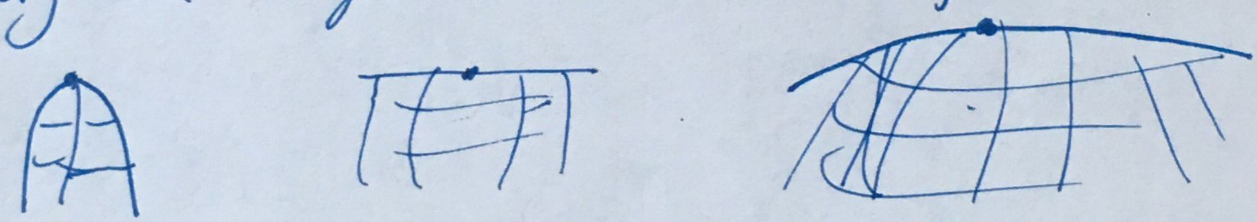
- Application to ∞-dim QFT path integrals still at an early stage.
- Will take perspective that success of lattice field theory $\Rightarrow$ nothing too wild happens in ∞-dim limit or $V \to \infty$ limit, and thimble ideas should still be useful.

<u>Application</u>: · $BPS_i - \overline{BPS_i}$ is a saddle at <u>infinity</u>.

- Want to sum over fluctuations $= \begin{cases} \text{zero modes} \\ \text{QZM's} \\ \text{Gaussian modes.} \end{cases}$

This is a thimble integral.

- Very hard in general: ∞-dim integration.



no QZM, zero modes all "Gaussian"

zero mode + Gaussian

QZM + Gaussian.

Justification $\downarrow$ weak coupling

- Idea: · The "gaussian" part is usual perturbative modes $\to$ can be done with usual pert. theory.
  · The ZM's and QZMs are the subtle part.

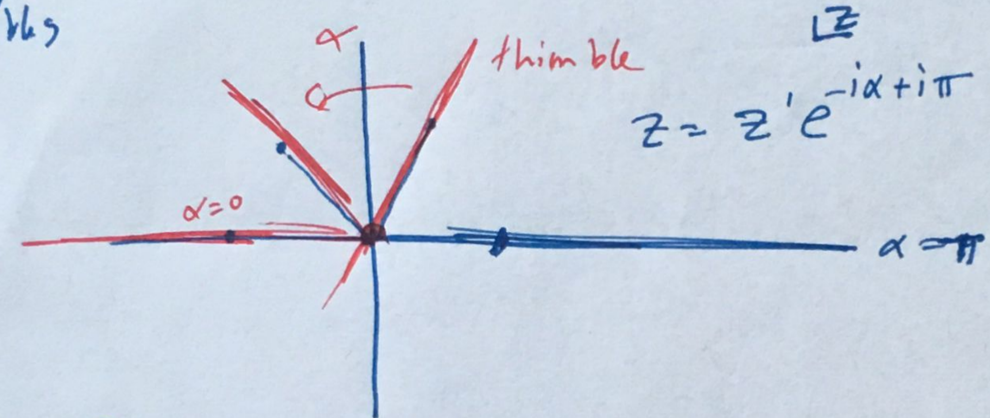
In fact, QZM are real issue, so, focus on them, and use Lefschetz-thimble idea:

$$I(g^2, n_f) = \int_0^\infty dr\, r^2 \exp\left[+ \frac{4\pi L}{g^2 r} - 4 n_f \log r \right].$$

$$g^2 = G e^{i\alpha}$$

Saddle: $r = r_* = \dfrac{-e^{-i\alpha}}{2G(2n_f - 1)}$

change variables $r \to z = \frac{r}{r_*}$



$\mathbb{Z}$

$z = z' e^{-i\alpha + i\pi}$

$$I = \int_0^\infty dr\,(\cdots) \to \int_J dz\,(\cdots)$$

Result:

$$I = e^{-i(\pi - \alpha)(4n_f - 3)}\, G^{-3+4n_f}\, \Gamma(4n_f - 3)$$

valid for any α, & G!

We wanted $\alpha = 0$, then

$$I_{\text{neutral}} = -\, G^{4n_f - 3}\, \Gamma(4n_f - 3)$$

integer n_F!

$$I_{\text{magnetic}} \underset{\alpha = \pi}{=} +\, G^{4n_f - 3}\, \Gamma(4n_f - 3).$$

Note that for e.g. $n_f = 1$, this leads to cancellation of sum of neutral & magnetic bion amplitudes.

$n_f = 1$ SUSY theory; Behtash, Sulejmanpasic, Unsal 2015: this happens in NP contributions to vacuum energy for $\mathcal{N}=1$ SYM!

Minus sign coming from thimbles vital for avoiding NP development of vacuum energy $\Rightarrow$ avoiding SUSY breaking.

What about resurgence?

Let's think of bosonic theory as $n_f \to 0$ limit of QCD(Adj).

$$\int dA \, e^{-S(A) + n_f \, \mathrm{tr} \log(\not{D})}$$

view $n_f = \epsilon$ as a "regulator" - role will be clear shortly!

Focus on $\epsilon \sim 0$: purely bosonic case; implicit: double-tr to stabilize center.

$$I = (-\underbrace{Ge^{i\alpha}}_{-g^2})^{4n_f-3}\,\Gamma(4n_f-3) \quad \text{near } \alpha = 0, \; n_f = \epsilon \to 0$$

$$= \frac{1}{24(g^2)^3}\left[\frac{1}{\epsilon} - \frac{2}{3}\left(-11 + 6\left[\gamma_E + \log\left(\frac{-1}{g^2}\right)\right]\right)\right] + \mathcal{O}(\epsilon)$$

divergence comes from uncorrelated $\mathcal{M}\,\bar{\mathcal{M}}$ event!

$M \sim \mathcal{V}$

$M\bar{M} \sim \mathcal{V}^2 + V$

uncorrelated

molecular part, correlated.

Dropping divergent (unambiguous) part, we get

$$[M_i \bar{M}_i] \approx \frac{-2}{24 g^6}\left[6\left(\gamma_E + \log\left(\frac{-1}{g^2}\right)\right)\right] e^{-2S_0}.$$

$$[M_i \bar{M}_i] \sim +\frac{2}{3}\cdot\frac{1}{(2g^2)^6}\left[6\gamma_E + \log\left(\frac{1}{g^2}\right)\right] e^{-2S_0}$$

$$\log\left(\frac{-1}{g^2}\right) = \log\left(\frac{1}{g^2}\right) \pm i\pi$$

$$\sim [M_i \bar{M}_i] \sim \ldots \pm i\pi\, e^{-2S_0}$$

But remember $e^{-2S_0} \sim e^{-2\frac{8\pi^2}{g^2 N}} = e^{-2 \cdot 8\pi^2/\lambda}$

This is an infrared renormalon!

Compare magnitude to mass gap:

$$\Delta^2 \sim e^{-S_0} \qquad \text{pure YM}^*$$

$$\Rightarrow \Delta \sim e^{-S_0/2}$$

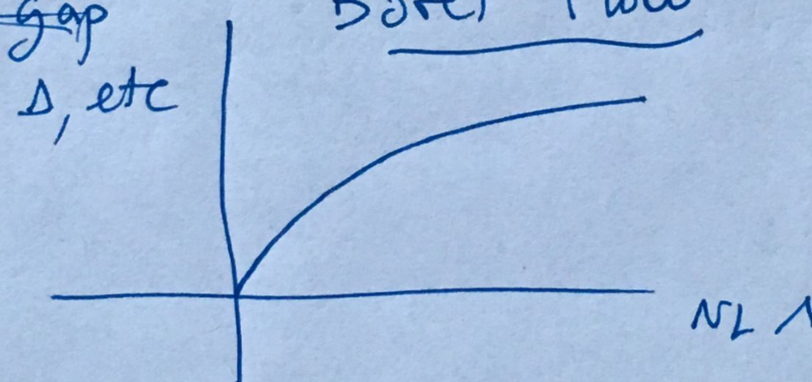
$$\Rightarrow \text{renorm} \sim \Delta^4 = e^{-2S_0}.$$

Same as pattern in QCD on $\mathbb{R}^4$: gap $\sim \Lambda$

renorm: $\langle \mathrm{tr} F^2 \rangle \sim \Lambda^4$.

~~We get gap~~

Borel flow



Computing the "Borel flow" = solving theory on $\mathbb{R}^4$!

Can not show cancellations:

Even in abelianized $\mathbb{R}^3$ theory, diagrams too hard!

2D Sigma Models

Abandon 4D, look at 2D QFTs, using same trick! $\mathbb{R}^2 \to \mathbb{R}\times S^1$. Why?

Small L EFT = QM! (0+1 d QFT)

$\Rightarrow$ know much more about pert. theory in QM!

Can actually show the leading resurgent cancellation.

Start with "CP^{N-1} model". (Follow Dunne-Unsal, 2012).

$$S = \frac{2}{g^2}\int d^2x \, (D_\mu \vec{n})^\dagger (D_\mu \vec{n}) \qquad \vec{n}\cdot\vec{n} = 1$$

N-vector, complex.

$D_\mu = \partial_\mu + iA_\mu \leftarrow$ constraint.

Note, "$SU(N)$" Global symmetry, $n \to Un$.

target $\left\{ \frac{U(N)}{U(N-1)\times U(1)} \simeq CP^{N-1} \right.$ $\qquad \beta_0 = N$.

$$\mathbb{R}^2 \Rightarrow CP^{N-1}.$$

- asymp. free!
- has mass gap, $\sim \Lambda \sim \mu e^{-4\pi/\lambda}$ $\leftarrow$ cutoff.
- $\pi_2(CP^{N-1}) = \mathbb{Z}$, has instantons.
- $Q = \frac{1}{2\pi}\int d^2x \, \epsilon_{\mu\nu} (D_\mu n)^\dagger (D^\nu n)$.

- Affleck, 1980s: On $\mathbb{R}^{\beta} \times S^1$ ← thermal

$\exists$ phase transition, at $N \to \infty$

heuristic $Z(\beta) \sim 1 + N e^{-m\beta}$ ← free ptcls

↑ fund of $SU(N)$.

$\beta_{crit} \sim \log(N)\, m^{-1}$ $\mathbb{R}^2$ and $\mathbb{R} \times S^1_{small}$ not smoothly connected.

~~Can we get around it?~~

- $\mathbb{R}^2$ theory strongly coupled ~ no hope!
  - ~ instanton gas makes no sense.
  - ~ PT can't be done.

Need analog of "center symmetric" compactification!

Dunne-Unsal: Just do it. We have $SU(N)$ global.

$$\mathbb{R}^2 \to \mathbb{R} \times S^1_{\text{twisted}}$$

$$\vec{n}(x+L) = \Omega\, n(x)$$

$\Omega \in SU(N)$.

"background holonomy".
"twisted boundary condition".
"imaginary flavor chemical potentials".

$\Omega = \mathbb{1} \Rightarrow$ thermal, no good.

Could there be "better" Ω choice?

it's up to us — we aren't trying to do finite T, want to minimize finite volume effects.

Ans: YES. $\Omega = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \omega^2 & \\ & & & \ddots \end{pmatrix}$, $\omega = e^{2\pi i/N}$.

why? Several answers....

- Large L theory, no dependence on BCs ← due to gap!
- Can't maintain that at small L. But could ask at least $\frac{\partial}{\partial(BC)} (\log Z) = 0$.

$\Rightarrow$ motivates looking at $\log Z[BCs]$.

$\rightarrow$ Fun calculation, result: one-loop

$$-\log Z = V(\Omega) = -\frac{2}{\pi^2 L^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(|\mathrm{tr}\, \Omega^n|^2 - 1 \right).$$

↳
- "two" extrema, $\Omega = 1$ BAD
- the other is Ω = center-sym might be good.

thermal extremum is N fold degenerate

↳ try.

Other perspective:

twisted ~~Pfaff~~ partition function

$$\tilde{Z} = \mathrm{tr}\, \hat{Q}\, e^{-\beta H}$$

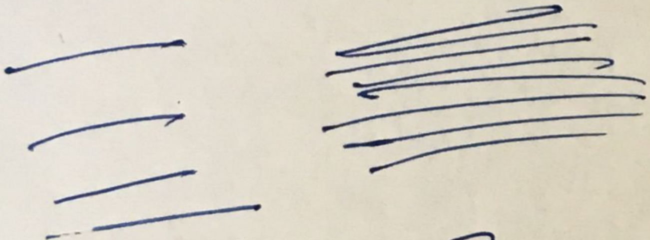
kills "fundamental" contribution.....

(AC+ Yaffe, Sen, Wagman) to appear.

Try it and see!

Consequences of twist:

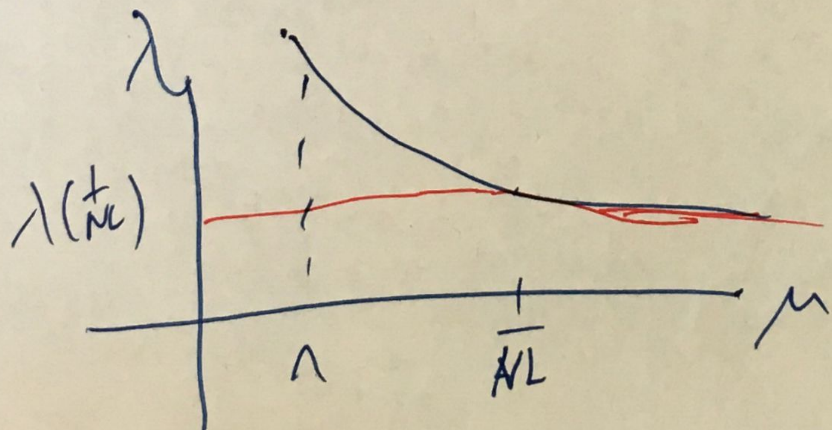
KK spectrum $\sim \frac{2\pi}{L} k \Longrightarrow \underbrace{\frac{2\pi}{NL}}_{\text{like gauge theory}} k$



thermal $\qquad \mathbb{Z}_N$

$N \to \infty$, $NL\Lambda \gg 1$, vol. ind

$NL\Lambda \ll 1$, control.



Integrate out KK modes, focus on zero mode.

Write $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} e^{-i\phi/2} \\ \sin\theta/2 \end{pmatrix}$ $\phi \in (0, 2\pi)$

CP^1 $\theta \in (0, \pi)$.

$$S = \frac{2}{g^2} \int (D_\mu n)^2 d^2x = \frac{2}{g^2} \int dx\, dt \cdot (\partial_\mu \theta)^2 + \sin^2\theta \, (\partial_\mu \phi + \xi \delta_{\mu,2})^2$$

twist.

$$\boxed{\xi = \pi/L \text{ for } CP^1}$$

$$\Downarrow$$

$$\frac{2L}{g^2} \int dt \left((\partial_t \theta)^2 + \sin^2\theta (\partial_t \phi)^2 + \xi^2 \sin^2\theta \right)$$

potential

This is QM.

What can we do with it?

Two directions: - Perturbative

- Non-perturbative.

Start with perturbative!

The small L EFT is a 0+1 dim. QFT

= QM!

Can use Hamiltonian Methods!

$$\mathcal{L} \approx \frac{1}{g^2}\left(\dot\theta^2 + \xi^2 \sin^2\theta + \sin^2\theta\, \dot\phi^2\right)$$

$$\frac{\delta \mathcal{L}}{\delta \theta}, \quad \frac{\delta \mathcal{L}}{\delta \dot\theta}, \quad \frac{\delta \mathcal{L}}{\delta \phi}, \quad \frac{\delta \mathcal{L}}{\delta \dot\phi}$$

momenta

$$H = \underbrace{p_\theta \dot\theta + p_\phi \dot\phi}_{\text{rewrite in terms of } p_\theta,\, p_\phi.} - \mathcal{L}$$

$$H = \frac{g^2}{2} p_\theta^2 + \frac{\xi^2}{2g^2}\sin^2\theta + \frac{g^2}{2\sin^2\theta} p_\phi^2$$

Note, ϕ is a cyclic coordinate.

- Emergent quantum number: p_ϕ momentum like "angular momentum" on internal space.

- Also have $P: \theta \to \pi - \theta$ symmetry quantum #: $+1, -1$.

Recall, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$.

So, let's pick an observable:
ground state energy.
↳ could also look at energy differences.

What are quantum #'s of ground state?

Answer: $P = +, \hat{\phi} = 0$!

Simplifies problem! Can neglect P_ϕ entirely!

$$H_{eff} = -\frac{1}{2}\frac{d^2}{d\theta^2} + \frac{\xi^2}{4g^2}[1-\cos(g\theta)].$$

Mathieu equation!

We can get large order asymptotics by looking at

- DLMF.nist.gov
- Stone & Reeve, 1978, PRD
- Ourselves.

Can't resist explaining "Bender–Wu" method, see recent paper, Ünsal – Sulejmanpasic

1608.08256

↳ Mathematica Package "BenderWu".
Use it, love it.

How does it work ?

$$H\psi = \frac{1}{2}\psi''(x) + \frac{1}{g^2}V(gx)\psi(x) = \epsilon\,\psi(x)$$

$\epsilon = \epsilon(g)$.

Idea 1: Expand V around harmonic minimum

$$V \sim \underbrace{V''(0)}_{\omega} x^2 + \ldots$$

Then write $\psi(x) = u(x)\, e^{-\omega x^2/2}$
(Treat later terms as pert.)

$$u(x) = \sum_{\ell=0}^{\infty} u_\ell(x)\, g^\ell$$

$$\epsilon = \sum_{n=0}^{\infty} \epsilon_n g^n.$$

Idea 2: Write $u_\ell(x) = \sum_{k=0}^{k_\ell} A_\ell^k x^k$

depends on level and on ℓ.

Plug, equate powers,

get recursion relation of A_ℓ^k, ϵ_k

$\Rightarrow$ Recursive solution for $\psi(x)$, E,

for <u>any</u> energy level.

$\Rightarrow$ vastly more efficient than e.g. diagrams.

Go back to our $\sin^2(\theta)$ problem.

Use this method, get

$$\epsilon_0^{(0)}(g^2) = \sum \epsilon_k^{(0)} (g^2)^k$$

$$\epsilon_k^{(0)} \sim -\frac{2}{\pi}\left(\frac{1}{4g}\right)^k k!\left(1-\frac{5}{2k}+\dots\right)$$

Renormalon! (At generic N, associated to λ.)

$$\Rightarrow \quad B\mathcal{E}(t) \to -\frac{2}{\pi} \sum \left(\frac{t}{4\varphi}\right)^k \sim -\frac{2}{\pi} \frac{1}{1-t/4\varphi}$$

$$\Rightarrow \quad S_{0\pm}\mathcal{E}^{(0)}(g^2) = \mathrm{Re}(\cdots) \mp \frac{8\varphi}{g^2} \underbrace{e^{-\frac{4\varphi L}{g^2}}}_{e^{-\frac{8\pi}{g^2 N}}}$$

$\boxed{\text{Renormalon!}}$

So who cancels it? Back to NP dynamics.

$$\pi_2(\mathbb{C}P^{N-1}) = \mathbb{Z} \to \exists \text{ instantons (BPS!)}$$

$$S_{inst} = \frac{4\pi}{g^2} = \frac{4\pi N}{\lambda} \to \infty \text{ at } N = \infty$$

So S_{inst} irrelevant at large N!

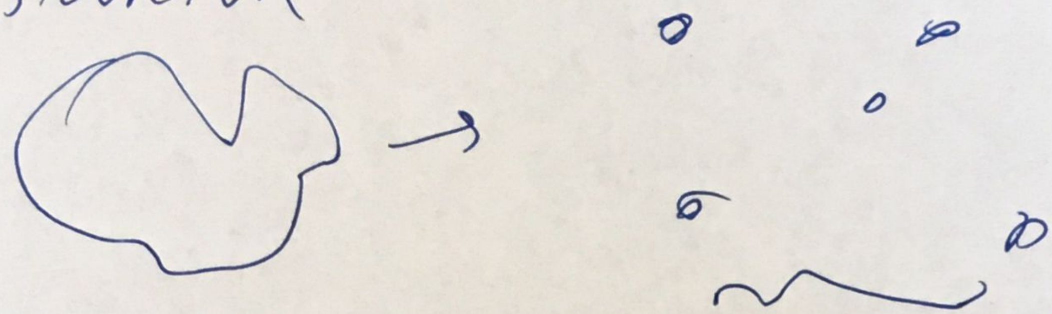
~~Als On~~ Also, instantons have size modulus (zero mode), so no "size"

$\Rightarrow$ can't be separated.

$\Rightarrow$ instanton gas picture is stupid.

What Happens on $\mathbb{R} \times S^1_{twist}$?

instanton



N constituents!

↳ "kink-instantons"

"fractons"

"instanton quarks".

$S = \frac{4\pi}{g^2 N}$; $N-1 \leftrightarrow$ simple roots of $SU(N)$ $\vec{\alpha}_i$

$1 \leftrightarrow$ affine root

↳ only present due to 2D nature.

Size: "M_W" $= \frac{2\pi}{NL}$; fixed!

Dilute gas picture has a chance, given that we also have weak coupling at small L!

Explicit solutions

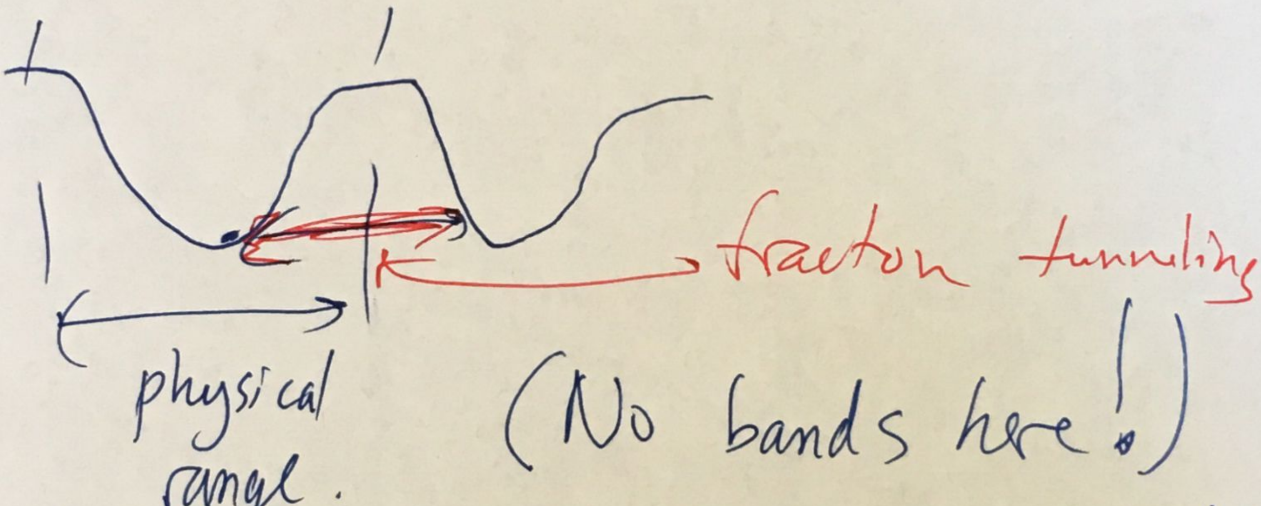
$\mathbb{CP}^1$ $\quad \theta = 2\arctan\left(e^{-\xi(t-t^0)}\right)$

$\phi = \phi_0$. ← zero modes

You won't be surprised by now that these "fractons" generate the gap.

ground : $P=+$, $\phi=0$

1^{st} excited : $P=-$, $\phi=0$.



(No bands here!)

Can be calculated, gap $= E^{(-)} - E^{(+)} \sim e^{-\frac{4\pi}{\lambda}}$!

Where's the renormalon cancellation?

→

Need to consider "neutral bions" again

$$M_i \quad \bar{M}_i$$

$$\bullet \xleftrightarrow[\tau]{} \bullet$$

Plug widely-separated monopole instantons, compute action.

$$\Rightarrow S = 2 \cdot \frac{4\pi}{\lambda} + V_{eff}(\tau)$$

"saddle at infinity"

fluctuation

Now we have to sum τ:

$$[M_i \bar{M}_i] \sim e^{-\frac{8\pi}{\lambda}} \int d\tau \, e^{-V_{eff}}$$

$\xi = \frac{2\pi}{NL}$

$$V_{eff} = -8 \xi L \frac{\vec{\alpha}_i \cdot \vec{\alpha}_j}{g^2} e^{-\xi \tau} + 2 N_f \xi \tau$$

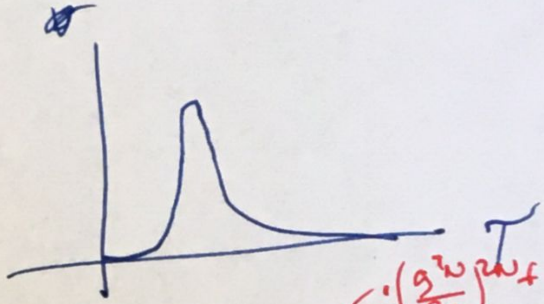
"σ exchange"

fermion zero mode exchange.

First imagine we arbitrarily look at

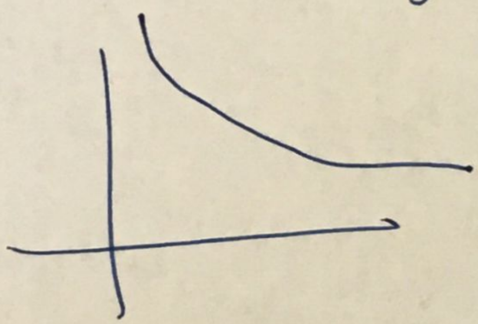
$$V_{eff} \sim + \frac{4}{g^2} \xi L e^{-\xi\tau} + 2N_f \xi \tau$$

$$I = \int_{-\infty}^{\infty} d\tau \, \underbrace{e^{-V_{eff}(\tau)}}$$



$$I = \left(\frac{g^2 N}{8\pi}\right)^{2N_f} \Gamma(2N_f), \text{ if } \int_0^\infty \ldots \quad \left(\frac{g^2 N}{8\pi}\right)^{2N_f} \Big(\Gamma(2N_f) - \underbrace{\Gamma(2N_f, \tfrac{8\pi}{\lambda})}_{\text{wrong!}}\Big)$$

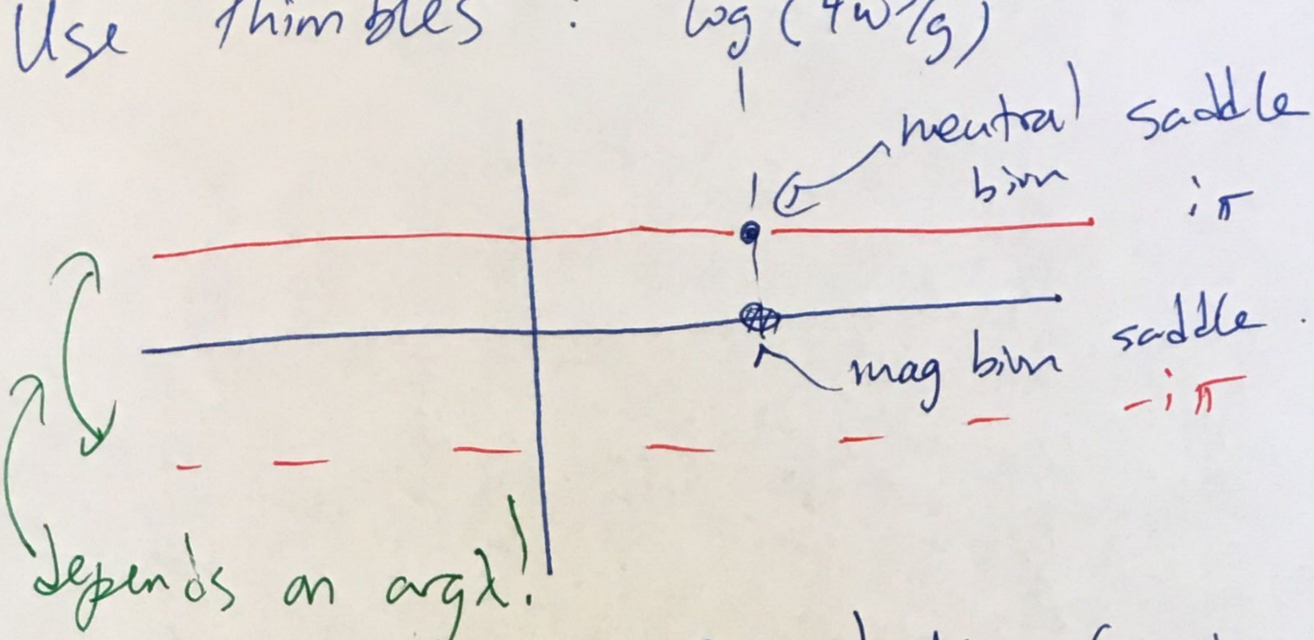
But for us, $V_{eff} = -\frac{4}{g^2} \xi L e^{-\xi\tau} + 2N_f \xi \tau$



Disaster again.

Same solution!

Use thimbles :



<u>NB</u> : Thimble infinite! No log in action!

You <u>must</u> integrate all the way.

↳ To get match to eg. uniform WKB.

Do integral, result is

$$\sim \left(-\frac{g^2 N}{8\pi}\right)^{2N_f} \Gamma(2N_f) \cdot e^{-\frac{8\pi}{\lambda}}$$

$$\sim \left[\frac{1}{2\epsilon} + \left(-\gamma_E \pm i\pi + \log\left(\frac{8\pi}{\lambda}\right)\right)\right] e^{-8\pi/\lambda}$$

renormalon !

If we now do everything carefully, can verify that

$$S_{\pm} P + e^{-8\pi/\lambda}(\text{thimble } \pm)$$

$$= \underbrace{\text{Real}}_{\text{from P + NP}} + \underbrace{(\mp(\cdots) \pm (\cdots))}_{0}$$

as expected

renorm = gap · 2

Cancellation of renormalon ambiguity, in a QFT, using semi-classics.

Challenges: · Do this to all orders. Not been done.

· Extrapolate to large L

· Solve QCD

Switch gears.

- CP^{N-1} has instantons
- So does QM.
- They fractionalize, drive renormalons, and mass gaps.

But some QFTs don't have instantons, but do have mass gaps, renormalons.
Then what!? Who fractionalizes?

Stick to 2D, look at PCM.

$$S = \frac{1}{\lambda}\int d^2x \; \mathrm{tr}\, \partial_\mu U \, \partial^\mu U^\dagger$$

- Asymp free.
- Has gap.
- Integrable.
- Matrix-like large N limit: $U \in SU(N)$!
- has $SU(N)_L \times SU(N)_R$ symmetry,
- D=2 reduction of χ-PT, with $f_\pi^2 = \frac{1}{\lambda}$. λ dimensionless.

But $\pi_2(SU(N)) = 0$.

No instantons. What to do?

Turns out, $\exists$ unitons!

$$S = \frac{8\pi}{\lambda} k \,, \quad k \in \mathbb{Z}.$$

NOT BPS, solve 2^{nd} order EoM.

associated to $\pi_3(SU(N)) = \mathbb{Z}$.

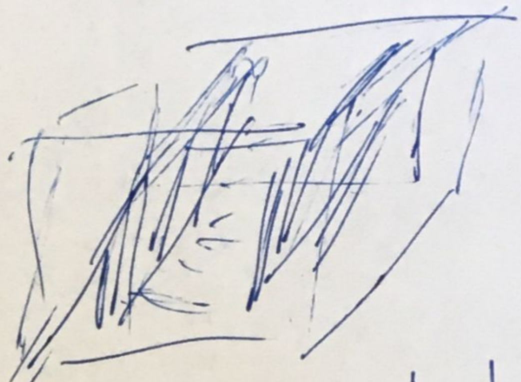
Discovered by Uhlenbeck, 1984.

$$CP^{N-1} \xrightarrow[\text{geodesic}]{} SU(N)$$

uniton

$\longrightarrow$

$\mathbb{R}\times S^1$



twist vital.

If we do $\mathbb{Z}_N$ twist

$$U(x+L) = \Omega\, U(x)\, \Omega^{+}$$

~~take~~ center sym.

get fractionalization.

why this twist? Same argument as before: ~ extremize $\log Z$, minimize finite vol effects.

~ get smooth L dep

Can analyze just like CP^{N-1} (61)

- Fractons drive mass gap on $\mathbb{R} \times S^1_L$.
- $[F_i \bar{F}_i] \sim$ renormalons!
- ambiguity cancellation,

$\mathcal{J}^2 \sim$ renomalon, again as expected.

"$\mathcal{J} \sim \Lambda$" renormalon $\langle \partial_\mu u \, \partial^\mu u \rangle \sim$ "Λ^2".

- Can't be an accident!

Can also do it for $O(N)$, Grassmanians, etc

Conclusions:

- $\exists$ resurgence in non-super-symmetric QFTs!
- related to mass gap.
- Thimbles <u>vital</u> to get it right.

To do:

- Deeper understanding?
- Higher orders?
- Other theories?
- Larger L?
- Solve YM!