Introduction to Resurgence and Semiclassical Physics

Gerald Dunne

University of Connecticut

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KITP Program: *Resurgent Asymptotics in Physics and Mathematics*, Fall 2017

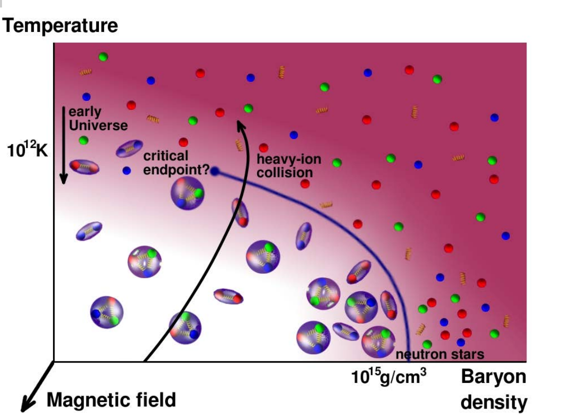
Physical Motivation

- infrared renormalon puzzle in asymptotically free QFT
- non-perturbative physics without instantons: physical meaning of non-BPS saddles
- "sign problem" in finite density QFT
- exponentially improved asymptotics

Bigger Picture

- non-perturbative definition of non-trivial QFT, in the continuum
- analytic continuation of path integrals
- dynamical and non-equilibrium physics from path integrals
- uncover hidden ‘magic’ in perturbation theory

Physical Motivation



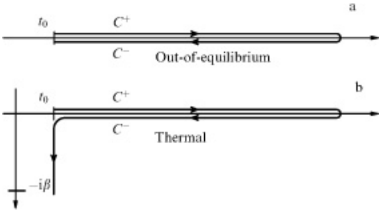
- sign problem: "complex probability"?

$$\int \mathcal{D}A\, e^{-S_{YM}[A]+\ln\det(\not{D}+m+i\,\mu\gamma^0)}$$

- lattice gauge theory problematic at finite baryon density

Physical Motivation

- equilibrium thermodynamics $\leftrightarrow$ Euclidean path integral
- Kubo-Martin-Schwinger: antiperiodic b.c.'s for fermions

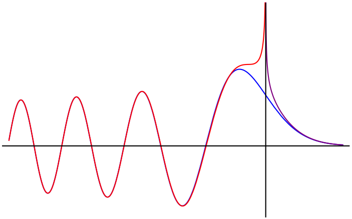


- non-equilibrium physics $\leftrightarrow$ Minkowski path integral
- Schwinger-Keldysh time contours
- quantum transport in strongly-coupled systems

Physical Motivation

- what does a Minkowski path integral mean?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar} S[A]\right)$$



$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}t^3 + x\,t\right)}\,dt \sim \begin{cases} \dfrac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} \quad , \quad x \to +\infty \\[2ex] \dfrac{\sin\left(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} \quad , \quad x \to -\infty \end{cases}$$

Physical Motivation

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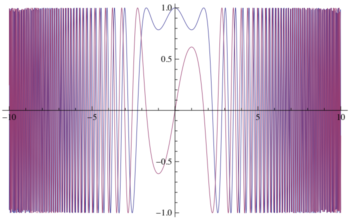
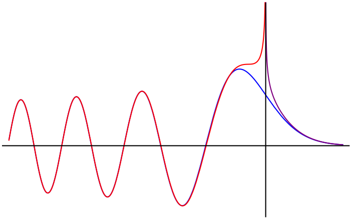
$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar} S[A]\right)$$





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Physical Motivation

- what does a Minkowski path integral mean?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar} S[A]\right)$$

- since we need complex analysis and contour deformation to make sense of oscillatory ordinary integrals, it is natural to expect to require similar tools also for path integrals

Mathematical Motivation

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory generally $\Rightarrow$ divergent series
- series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'
- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, fluids, QM, Matrix Models, QFT, String Theory, ...
- philosophical shift:

  view semiclassical expansions as potentially exact

Trans-series

No function has yet presented itself in analysis, the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms

G. H. Hardy, *Divergent Series*, 1949

- deep result: “this is all we need” (J. Écalle, 1980)
- also as a closed logic system: Dahn and Göring (1980)

Resurgent Trans-Series

- trans-series expansion in QM and QFT applications:

$$f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$

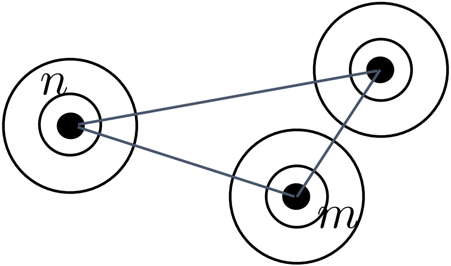
- J. Écalle (1980): set of functions closed under:

(Borel transform) + (analytic continuation) + (Laplace transform)

- *trans-monomial elements*: g^2, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$, are familiar
- "multi-instanton calculus" in QFT
- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ highly correlated
- new: exponentially improved asymptotics

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



Perturbation theory

- hard problem = easy problem + "small" correction
- perturbation theory generally → divergent series

e.g. QM ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

- Zeeman: $c_n \sim (-1)^n \, (2n)!$
- Stark: $c_n \sim (2n)!$
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$
- periodic Sine-Gordon (Mathieu) potential: $c_n \sim n!$
- double-well: $c_n \sim n!$

note generic factorial growth of perturbative coefficients

Perturbation theory

but it works ...

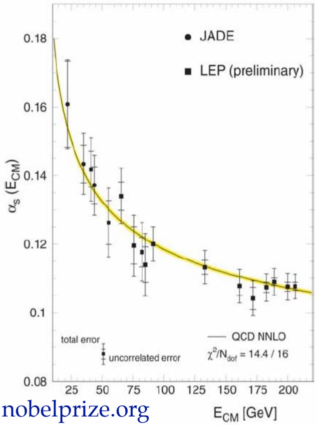
Perturbation theory works

QED perturbation theory:

$$\frac{g-2}{2} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - (0.32848\ldots)\left(\frac{\alpha}{\pi}\right)^2 + (1.18124\ldots)\left(\frac{\alpha}{\pi}\right)^3 - 1.9097(20)\left(\frac{\alpha}{\pi}\right)^4 + 9.16(58)\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{exper}} = 0.001\,159\,652\,180\,73(28)$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{theory}} = 0.001\,159\,652\,181\,78(77)$$



nobelprize.org

QCD: asymptotic freedom

$$\beta(g_s) = -\frac{g_s^3}{16\pi^2}\left(\frac{11}{3}N_C - \frac{4}{3}\frac{N_F}{2}\right)$$

but it is divergent ...

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever … That most of these things [summation of divergent series] are correct, in spite of that, is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question.

N. Abel, 1802 – 1829

The series is divergent; therefore we may be able to do something with it

O. Heaviside, 1850 – 1925

Asymptotic Series vs Convergent Series

$$f(x) = \sum_{n=0}^{N-1} c_n\,(x - x_0)^n + R_N(x)$$

convergent series:

$$|R_N(x)| \to 0 \qquad , \quad N \to \infty \quad , \quad x \quad \text{fixed}$$

asymptotic series:

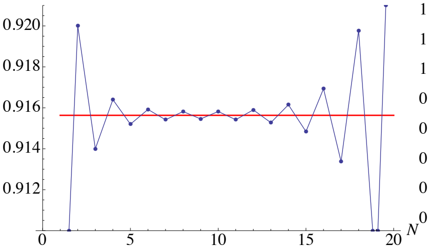
$$|R_N(x)| \ll |x - x_0|^N \qquad , \quad x \to x_0 \quad , \quad N \quad \text{fixed}$$

$\longrightarrow$ "optimal truncation":

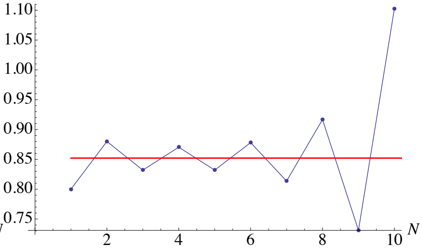
truncate just before least term (x dependent!)

Asymptotic Series vs Convergent Series

$$\sum_{n=1}^{\infty} (-1)^n \, n! \, x^n \sim \frac{1}{x} \, e^{\frac{1}{x}} \, E_1\left(\frac{1}{x}\right)$$



$(x = 0.1)$

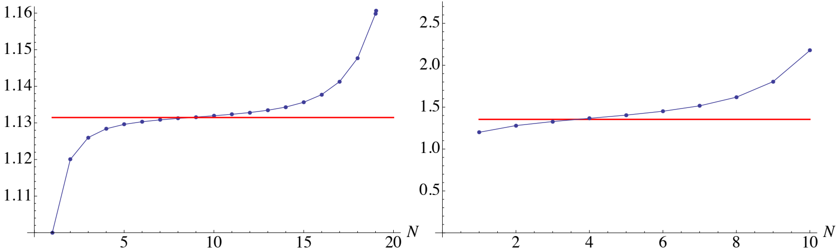


$(x = 0.2)$

optimal truncation order depends on x: $\quad N_{\text{opt}} \approx \frac{1}{x}$

Asymptotic Series vs Convergent Series

$$\sum_{n=0}^{\infty} n!\, x^n \sim \frac{1}{x}\, e^{-\frac{1}{x}}\, Ei\left(\frac{1}{x}\right)$$



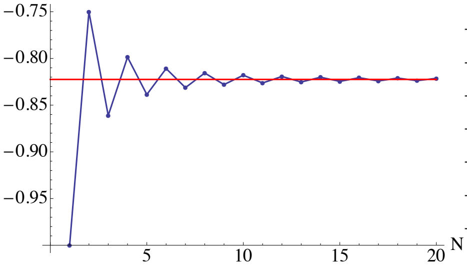
$x = 0.1$ $\qquad$ $x = 0.2$

optimal order depends on x: $\quad N \approx \frac{1}{x}$

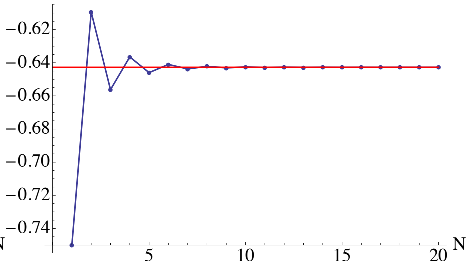
Asymptotic Series vs Convergent Series

contrast with behavior of a convergent series:
more terms always improves the answer, independent of x

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}\, x^n = \mathrm{PolyLog}(2, -x)$$



$(x = 1)$



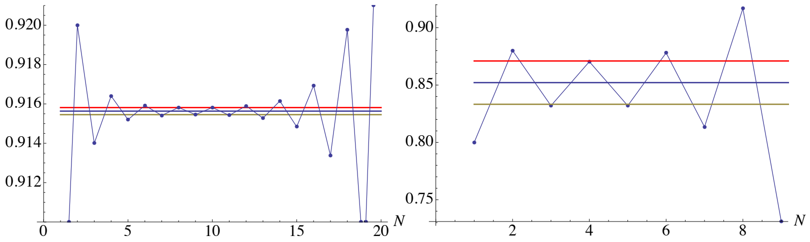
$(x = 0.75)$

Asymptotic Series: exponential precision

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n \sim \frac{1}{x}\, e^{\frac{1}{x}}\, E_1\left(\frac{1}{x}\right)$$

optimal truncation: error term is exponentially small

$$|R_N(x)|_{N\approx 1/x} \approx N!\, x^N\Big|_{N\approx 1/x} \approx N! N^{-N} \approx \sqrt{N} e^{-N} \approx \frac{e^{-1/x}}{\sqrt{x}}$$



$(x = 0.1)$ $(x = 0.2)$

Borel summation: basic idea

write $n! = \int_0^\infty dt\, e^{-t}\, t^n$

alternating factorially divergent series:

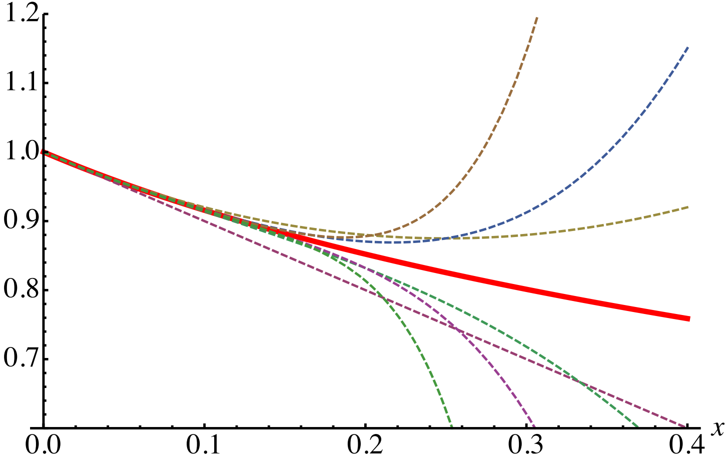
Emile Borel

$$\sum_{n=0}^{\infty} (-1)^n\, n!\, g^n = \int_0^\infty dt\, e^{-t}\, \frac{1}{1+g\,t} \qquad (?)$$

integral convergent for all $g > 0$: “Borel sum” of the series

Borel Summation: basic idea

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n = \int_0^{\infty} dt\, e^{-t}\, \frac{1}{1+x\,t} = \frac{1}{x}\, e^{\frac{1}{x}}\, E_1\left(\frac{1}{x}\right)$$



Borel summation: basic idea

Emile Borel

write $n! = \int_0^\infty dt\, e^{-t}\, t^n$

non-alternating factorially divergent series:

$$\sum_{n=0}^{\infty} n!\, g^n = \int_0^\infty dt\, e^{-t}\, \frac{1}{1 - g\, t} \qquad (??)$$

pole on the Borel axis!

$\Rightarrow$ non-perturbative imaginary part

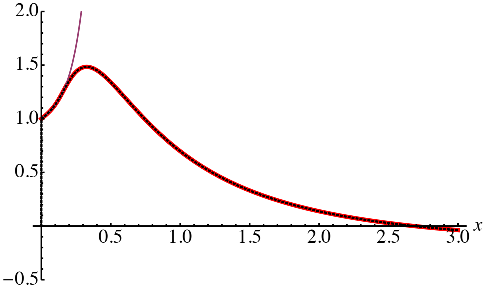
$$\pm \frac{i\,\pi}{g}\, e^{-\frac{1}{g}}$$

but every term in the series is real !?!

Borel Summation: basic idea

$$\left[\sum_{n=0}^{\infty} n!\, x^n\right] \text{ `` = '' } \int_0^\infty dt\, e^{-t}\, \frac{1}{1-x\,t} \text{ `` = '' } -\frac{1}{x}\, e^{-\frac{1}{x}}\, E_1\left(-\frac{1}{x}\right)$$

$$\text{Borel} \Rightarrow \mathcal{R}e\left[\sum_{n=0}^{\infty} n!\, x^n\right] = \mathcal{P}\int_0^\infty dt\, e^{-t}\, \frac{1}{1-x\,t} = \mathcal{R}e\left[-\frac{1}{x}\, e^{-\frac{1}{x}}\, E_1\left(-\frac{1}{x}\right)\right]$$



- but $E_1\left(-\frac{1}{x}\right)$ also has an imaginary part $= -i\pi$

Borel summation

Borel transform of series $f(g) \sim \sum_{n=0}^{\infty} c_n \, g^n$:

$$\mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series typically has finite radius of convergence.

Borel resummation of original asymptotic series:

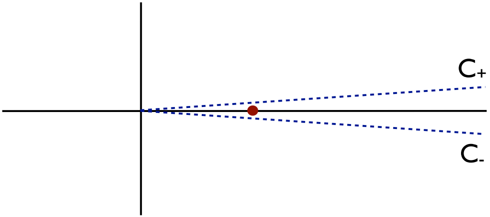
$$\mathcal{S}f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

warning: $\mathcal{B}[f](t)$ may have singularities in (Borel) t plane

Borel singularities

avoid singularities on $\mathbb{R}^+$: directional Borel sums:

$$\mathcal{S}_\theta f(g) = \frac{1}{g} \int_0^{e^{i\theta}\infty} \mathcal{B}[f](t) e^{-t/g} dt$$



go above/below the singularity: $\theta = 0^\pm$

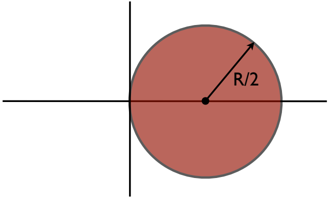
$\longrightarrow$ non-perturbative ambiguity: $\pm \mathrm{Im}[\mathcal{S}_0 f(g)]$

challenge: use physical input to resolve ambiguity

Borel summation: existence theorem (Nevanlinna & Sokal)

$f(z)$ analytic in circle $C_R = \{z : \left|z - \frac{R}{2}\right| < \frac{R}{2}\}$

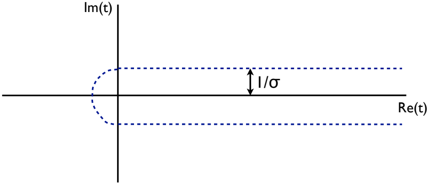
$$f(z) = \sum_{n=0}^{N-1} a_n \, z^n + R_N(z) \qquad , \qquad |R_N(z)| \leq A\,\sigma^N \, N! \, |z|^N$$



Borel transform

$$B(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \, t^n$$

analytic continuation to
$S_\sigma = \{t : |t - \mathbb{R}^+| < 1/\sigma\}$



$$f(z) = \frac{1}{z} \int_0^\infty e^{-t/z} \, B(t) \, dt$$

another view of resurgence:

resurgence can be viewed as a method for making formal asymptotic expansions consistent with global analytic continuation properties

$\Rightarrow$ "the trans-series really IS the function"

(question: to what extent is this true/useful in physics?)

Resurgence: Preserving Analytic Continuation

- zero-dimensional partition functions

$$Z_1(\lambda) = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2\lambda}\sinh^2(\sqrt{\lambda}\,x)} = \frac{1}{\sqrt{\lambda}}\, e^{\frac{1}{4\lambda}}\, K_0\left(\frac{1}{4\lambda}\right)$$

$$\sim \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (-1)^n (2\lambda)^n \frac{\Gamma(n+\frac{1}{2})^2}{n!\,\Gamma\left(\frac{1}{2}\right)^2} \qquad \text{“Borel-summable”}$$

$$Z_2(\lambda) = \int_0^{\pi/\sqrt{\lambda}} dx\, e^{-\frac{1}{2\lambda}\sin^2(\sqrt{\lambda}\,x)} = \frac{\pi}{\sqrt{\lambda}}\, e^{-\frac{1}{4\lambda}}\, I_0\left(\frac{1}{4\lambda}\right)$$

$$\sim \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (2\lambda)^n \frac{\Gamma(n+\frac{1}{2})^2}{n!\,\Gamma\left(\frac{1}{2}\right)^2} \qquad \text{“non-Borel-summable”}$$

- naively: $Z_1(-\lambda) = Z_2(\lambda)$
- connection formula: $K_0(e^{\pm i\pi}\,|z|) = K_0(|z|) \mp i\,\pi\, I_0(|z|)$

$$\Rightarrow \quad Z_1(e^{\pm i\pi}\,\lambda) = Z_2(\lambda) \mp i\, e^{-\frac{1}{2\lambda}}\, Z_1(\lambda)$$

Resurgence: Preserving Analytic Continuation

- Borel summation

$$Z_1(\lambda) = \sqrt{\frac{\pi}{2}}\,\frac{1}{2\lambda}\int_0^\infty dt\, e^{-\frac{t}{2\lambda}}\;{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-t\right)$$

- directional Borel summation

$$\begin{aligned}
&Z_1(e^{i\pi}\,\lambda) - Z_1(e^{-i\pi}\,\lambda)\\
&= \sqrt{\frac{\pi}{2}}\frac{1}{2\lambda}\int_1^\infty dt\, e^{-\frac{t}{2\lambda}}\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;t-i\varepsilon\right) - {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;t+i\varepsilon\right)\right]\\
&\quad = \quad -(2i)\sqrt{\frac{\pi}{2}}\,\frac{1}{2\lambda}\,e^{-\frac{1}{2\lambda}}\int_0^\infty dt\, e^{-\frac{t}{2\lambda}}\;{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-t\right)\\
&\quad = \quad -2\,i\,e^{-\frac{1}{2\lambda}}\,Z_1(\lambda)
\end{aligned}$$

- connection formula: $Z_1(e^{\pm i\pi}\,\lambda) = Z_2(\lambda) \mp i\,e^{-\frac{1}{2\lambda}}\,Z_1(\lambda)$

Resurgence: Preserving Analytic Continuation

Stirling expansion for $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ is divergent

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots + \frac{174611}{6600z^{20}} - \ldots$$

- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$

formal series $\Rightarrow$ $\mathrm{Im}\,\psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2}$

- reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi\cot(\pi z)$

$$\Rightarrow \quad \mathrm{Im}\,\psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi\sum_{k=1}^{\infty} e^{-2\pi k y}$$

"raw" asymptotics inconsistent with analytic continuation

- resurgence: add infinite series of non-perturbative terms this very simple example arises in many QFT and String Theory computations:

Euler-Heisenberg, de Sitter, exact S-matrices, Chern-Simons partition functions, matrix models, ...

$$\sum_{n=1}^{\infty} \frac{1}{\frac{n^2\,\pi^2}{L^2} - \lambda} = -\frac{L^2}{2}\left(\frac{\cot\left(L\sqrt{\lambda}\right)}{L\sqrt{\lambda}} - \frac{1}{L^2\,\lambda}\right)$$

$$= \frac{L}{2\pi\sqrt{\lambda}}\left(\psi\left(1+\frac{L\sqrt{\lambda}}{\pi}\right) - \psi\left(1-\frac{L\sqrt{\lambda}}{\pi}\right)\right)$$

what changes going from linear to nonlinear ODE's ?

• Painlevé functions are the generalization of special functions to nonlinear ODE's: many physical applications: fluids, statistical physics, random matrices, optics, QFT, strings, ...

• resurgent trans-series are the natural language for their asymptotics

see: Mariño, Schiappa, Aniceto, Pasquetti, Vonk, ...
Garoufalidis, Costin, Its, ...

Resurgence in Nonlinear ODEs: e.g. Painlevé II

Painlevé II:

$$w'' = 2w^3(x) + x\, w(x)$$

perturbative solution is non-Borel-summable

$\Rightarrow$ trans-series solution(s)

- Tracy-Widom law for statistics of max. eigenvalue for Gaussian random matrices
- double-scaling limit in 2d Yang-Mills
- double-scaling limit in unitary matrix models
- all-genus solution of 2d supergravity

Transseries Example: Painlevé II (matrix models, fluids ...)

$$w'' = 2w^3(x) + x\,w(x) \qquad , \quad w \to 0 \text{ as } x \to +\infty$$

- $x \to +\infty$ asymptotics: $w \sim \sigma\, Ai(x)$

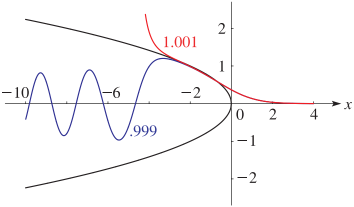
σ = real transseries parameter (flucs Borel summable)

$$w(x) \sim \sum_{n=0}^{\infty} \left(\sigma \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}} \right)^{2n+1} w^{(n)}(x)$$

- $x \to -\infty$ asymptotics: $w \sim \sqrt{-\frac{x}{2}}$

transseries exponentials: $\exp\left(-\frac{2\sqrt{2}}{3}(-x)^{3/2}\right)$

imag. part of transseries parameter fixed by cancellations



- Hastings-McLeod: $\sigma = 1$ unique real solution on $\mathbb{R}$

Transseries Example: Painlevé II (matrix models, fluids ...)

- a puzzle concerning the Hastings-McLeod solution
- as $x \to +\infty$, the perturbative series are Borel summable, and the trans-series exponential factor is

$$\frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}}$$

- but as $x \to -\infty$, the perturbative series are non-Borel summable, and the trans-series exponential factor is

$$\frac{e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}}}{2\sqrt{\pi}\,(-x)^{1/4}}$$

- complicated condensation of pole contributions as pass from one asymptotic region to the other

Resurgence and Hydrodynamics (Heller/Spalinski 2015; Başar/GD, 2015)

- resurgence: generic feature of differential equations
- boost invariant conformal hydrodynamics
- second-order hydrodynamics: $T^{\mu\nu} = \mathcal{E}\, u^\mu\, u^\nu + T^{\mu\nu}_\perp$

$$
\begin{aligned}
\tau \frac{d\mathcal{E}}{d\tau} &= -\frac{4}{3}\mathcal{E} + \Phi \\
\tau_{II}\, \frac{d\Phi}{d\tau} &= \frac{4}{3}\frac{\eta}{\tau} - \Phi - \frac{4}{3}\frac{\tau_{II}}{\tau}\,\Phi - \frac{1}{2}\frac{\lambda_1}{\eta^2}\Phi^2
\end{aligned}
$$

- asymptotic hydro expansion: $\mathcal{E} \sim \frac{1}{\tau^{4/3}}\left(1 - \frac{2\eta_0}{\tau^{2/3}} + \ldots\right)$

- formal series $\to$ trans-series

$$\mathcal{E} \sim \mathcal{E}_{\rm pert} + e^{-S\tau^{2/3}} \times (\text{fluc}) + e^{-2S\tau^{2/3}} \times (\text{fluc}) + \ldots$$

- non-hydro modes clearly visible in the asymptotic hydro series

- trans-series representation:

$$
\begin{aligned}
f(w) &\sim f^{(0)}(w) + \sigma\, w^{\beta}\, e^{-Sw}\, f^{(1)}(w) + \sigma^2\, w^{2\beta}\, e^{-2Sw}\, f^{(2)}(w) + \dots \\
&\sim \sum_{n=0}^{\infty} f^{(n)}(w)\, \sigma^n\, \zeta^n(w) \qquad , \quad \zeta(w) \equiv w^{\beta}\, e^{-Sw}
\end{aligned}
$$

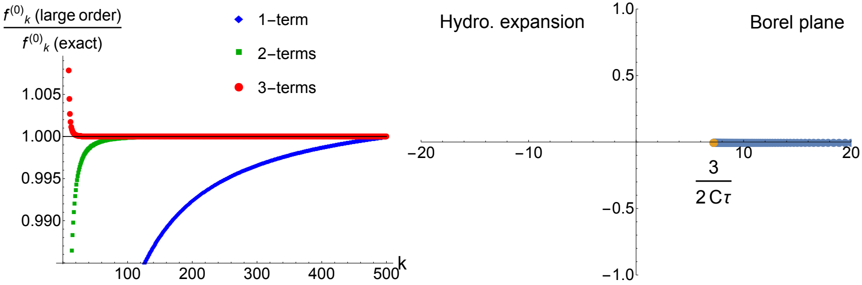
- $\sigma=$ trans-series parameter
- $f^{(n)}(w)=$ fluctuations about n^{th} non-perturbative sector

$$
f^{(n)}(w) \sim f_0^{(n)} + \frac{1}{w} f_1^{(n)} + \frac{1}{w^2} f_2^{(n)} + \frac{1}{w^3} f_3^{(n)} + \dots
$$

- resurgence implies that these expansion coefficients are related (Aniceto/Schiappa)

- study large-order behavior

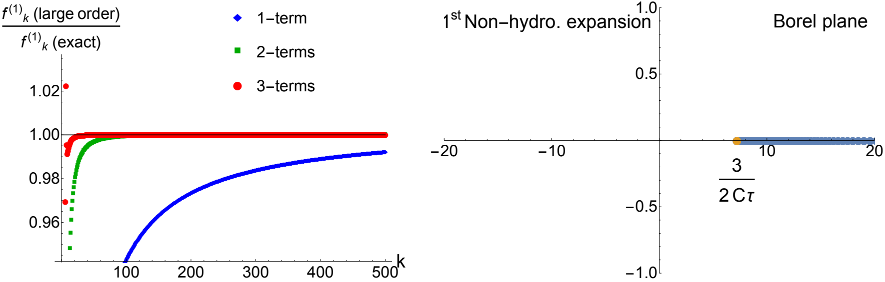
$$f_k^{(0)} \sim S_1 \frac{\Gamma(k+\beta)}{2\pi i\, S^{k+\beta}} \left(f_0^{(1)} + \frac{S}{k+\beta-1} f_1^{(1)} + \frac{S^2}{(k+\beta-1)(k+\beta-2)} f_2^{(1)} + \dots \right)$$



- determines the constant $S_1 = \mathrm{Im}(\sigma)$ numerically

- large-order behavior of fluctuations about first non-hydro sector

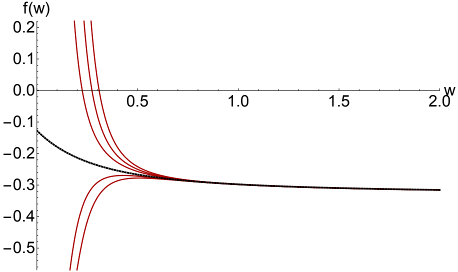
$$f_k^{(1)} \sim 2S_1 \frac{\Gamma(k+\beta)}{2\pi i\, S^{k+\beta}} \left(f_0^{(2)} + \frac{S}{k+\beta-1} f_1^{(2)} + \frac{S^2}{(k+\beta-1)(k+\beta-2)} f_2^{(2)} + \dots \right)$$



- no new free constants at this level!
- resurgent large-order behavior and Borel structure verified to 4-instanton level

• physics: non-hydro modes correspond to quasi-normal-modes in AdS language

• real part of trans-series parameter corresponds to initial condition: note formal late-time hydrodynamical series has no free parameter to associate with initial condition



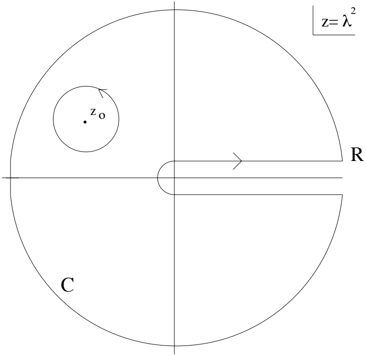
• ⇒ trans-series for metric coefficients in AdS

Borel Summation

back to quantum mechanics

Borel Summation and Dispersion Relations

cubic oscillator: $V = x^2 + \lambda\, x^3$ A. Vainshtein, 1964



$$E(z_0) = \frac{1}{2\pi i}\oint_C dz\,\frac{E(z)}{z - z_0}$$

$$= \frac{1}{\pi}\int_0^R dz\,\frac{Im\,E(z)}{z - z_0}$$

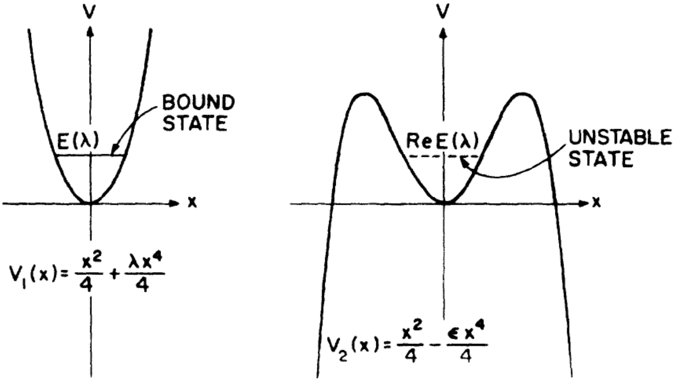
$$= \sum_{n=0}^{\infty} z_0^n \left(\frac{1}{\pi}\int_0^R dz\,\frac{\mathrm{Im}\,E(z)}{z^{n+1}}\right)$$

$$\text{WKB} \Rightarrow \mathrm{Im}\,E(z) \sim \frac{a}{\sqrt{z}}\,e^{-b/z} \quad , \quad z \to 0$$

$$\Rightarrow \qquad c_n \sim \frac{a}{\pi}\int_0^\infty dz\,\frac{e^{-b/z}}{z^{n+3/2}} = \frac{a}{\pi}\,\frac{\Gamma(n+\frac{1}{2})}{b^{n+1/2}} \qquad \checkmark$$

Instability and Divergence of Perturbation Theory

quartic AHO: $V(x) = \frac{x^2}{4} + \lambda \frac{x^4}{4}$ Bender/Wu, 1969



Divergence of perturbation theory

an important part of the story ...

The majority of nontrivial theories are seemingly unstable at some phase of the coupling constant, which leads to the asymptotic nature of the perturbative series

A. Vainshtein (1964)

Borel summation in practice

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \qquad , \qquad c_n \sim \beta^n\, \Gamma(\gamma\, n + \delta)$$

- alternating series: real Borel sum

$$f(g) \sim \frac{1}{\gamma} \int_0^\infty \frac{dt}{t} \left(\frac{1}{1+t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

- nonalternating series: ambiguous imaginary part

$$\mathrm{Re}\, f(-g) \sim \frac{1}{\gamma} \mathcal{P} \int_0^\infty \frac{dt}{t} \left(\frac{1}{1-t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

$$\mathrm{Im}\, f(-g) \sim \pm\frac{\pi}{\gamma} \left(\frac{1}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{1}{\beta g}\right)^{1/\gamma}\right]$$

recall: divergence of perturbation theory in QM

e.g. ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

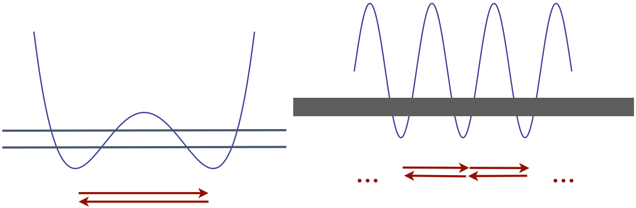
- Zeeman: $c_n \sim (-1)^n \, (2n)!$
- Stark: $c_n \sim (2n)!$
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$
- periodic Sine-Gordon potential: $c_n \sim n!$
- double-well: $c_n \sim n!$

recall: divergence of perturbation theory in QM

e.g. ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

- Zeeman: $c_n \sim (-1)^n \, (2n)!$ stable
- Stark: $c_n \sim (2n)!$ unstable
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$ stable
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$ unstable
- periodic Sine-Gordon potential: $c_n \sim n!$ stable ???
- double-well: $c_n \sim n!$ stable ???

Bogomolny/Zinn-Justin mechanism in QM



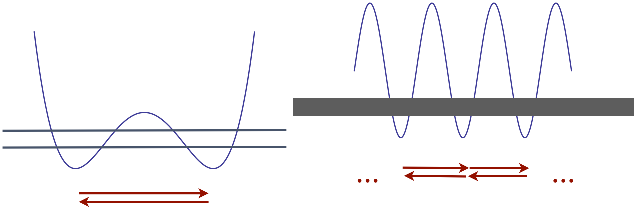
- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a 2-instanton effect

Bogomolny/Zinn-Justin mechanism in QM



- degenerate vacua: double-well, Sine-Gordon, ...

  1. perturbation theory non-Borel summable:
     ill-defined/incomplete

  2. instanton gas picture ill-defined/incomplete:
     $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract

- regularize both by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

Bogomolny/Zinn-Justin mechanism in QM

e.g., double-well: $V(x) = x^2(1 - g\, x)^2$

$$E_0 \sim \sum_n c_n \, g^{2n}$$

- perturbation theory:

$$c_n \sim -3^n \, n! \quad : \quad \text{Borel} \quad \Rightarrow \quad \text{Im}\, E_0 \sim \mp \pi \, e^{-\frac{1}{3g^2}}$$

- non-perturbative analysis: instanton: $g\, x_0(t) = \frac{1}{1+e^{-t}}$
- classical Eucidean action: $S_0 = \frac{1}{6g^2}$
- non-perturbative instanton gas:

$$\text{Im}\, E_0 \sim \pm \pi \, e^{-2\frac{1}{6g^2}}$$

- BZJ cancellation $\Rightarrow$ E_0 is real and unambiguous

"resurgence" $\Rightarrow$ cancellation to all orders

Decoding of Trans-series

$$f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{q=0}^{k-1} c_{n,k,q} \, g^{2n} \left[\exp\left(-\frac{S}{g^2} \right) \right]^k \left[\ln\left(-\frac{1}{g^2} \right) \right]^q$$

- perturbative fluctuations about vacuum: $\sum_{n=0}^{\infty} c_{n,0,0} \, g^{2n}$
- divergent (non-Borel-summable): $c_{n,0,0} \sim \alpha \frac{n!}{(2S)^n}$

$\Rightarrow$ ambiguous imaginary non-pert energy $\sim \pm i \, \pi \, \alpha \, e^{-2S/g^2}$

- but $c_{0,2,1} = -\alpha$: BZJ cancellation !

pert flucs about instanton: $e^{-S/g^2} \, \left(1 + a_1 g^2 + a_2 g^4 + \dots \right)$

divergent:

$a_n \sim \frac{n!}{(2S)^n} \, (a \, \ln n + b) \Rightarrow \pm i \, \pi \, e^{-3S/g^2} \left(a \ln \frac{1}{g^2} + b \right)$

- 3-instanton: $e^{-3S/g^2} \left[\frac{a}{2} \left(\ln\left(-\frac{1}{g^2} \right) \right)^2 + b \, \ln\left(-\frac{1}{g^2} \right) + c \right]$

resurgence: *ad infinitum*, also sub-leading large-order terms

Towards Resurgence in QFT

• resurgence ≡ analytic continuation of trans-series

• effective actions, partition functions, ..., have natural integral representations with resurgent asymptotic expansions

• analytic continuation of external parameters: temperature, chemical potential, external fields, ...

• e.g., magnetic ↔ electric; de Sitter ↔ anti de Sitter, . . .

• matrix models, large N, strings, ... (Mariño, Schiappa, ...)

• soluble QFT: Chern-Simons, ABJM, → matrix integrals

• asymptotically free QFT ? . . . "renormalons"

Divergence from combinatorics

- typical leading growth: $c_n \sim (\pm 1)^n \, \beta^n \, \Gamma(\gamma \, n + \delta)$
- factorial growth of number of Feynman diagrams

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2 - g\, x^4} \, dx = \sum_{n=0}^{\infty} J_n \, g^n \Rightarrow \quad J_n \sim (-1)^n \, (n-1)!$$

- ϕ^4 and ϕ^3: $\quad J_n \sim c^n \, n!$ (Hurst, 1952; Thirring, 1953)

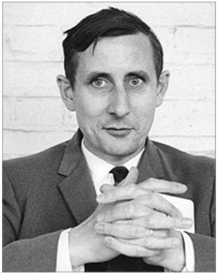
- comment: large N limit in YM/QCD:

number of planar diagrams grows as a power law!

$$J_n^{\text{planar}} \sim c^n$$

(Koplik, Neveu, Nussinov, 1977)

Dyson's argument (QED)



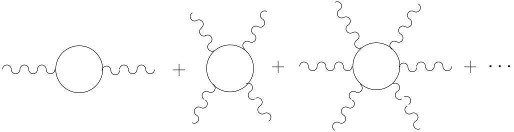
- F. J. Dyson (1952):

physical argument for divergence of QED perturbation theory

$$F(e^2) = c_0 + c_2 e^2 + c_4 e^4 + \ldots$$

Thus [for $e^2 < 0$] every physical state is unstable against the spontaneous creation of large numbers of particles. Further, a system once in a pathological state will not remain steady; there will be a rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization.

- *suggests* perturbative expansion cannot be convergent



- 1-loop QED effective action in uniform emag field
- the birth of *effective field theory*

$$L = \frac{\vec{E}^2 - \vec{B}^2}{2} + \frac{\alpha}{90\pi}\frac{1}{E_c^2}\left[\left(\vec{E}^2 - \vec{B}^2\right)^2 + 7\left(\vec{E}\cdot\vec{B}\right)^2\right] + \dots$$

- encodes nonlinear properties of QED/QCD vacuum

QFT Application: Euler-Heisenberg 1935

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2-\mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B})\cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}.$$

($\mathfrak{E}, \mathfrak{B}$ Kraft auf das Elektron.
$|\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{„137“}\frac{e}{(e^2/mc^2)^2} =$ „Kritische Feldstärke“.)

- Borel transform of a (doubly) asymptotic series
- resurgent trans-series: analytic continuation $B \longleftrightarrow E$
- EH effective action $\sim$ Barnes function $\sim \int \ln\Gamma(x)$

Euler-Heisenberg Effective Action: e.g., constant B field

$$S = -\frac{B^2}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)\exp\left[-\frac{m^2 s}{B}\right]$$

$$S = -\frac{B^2}{2\pi^2}\sum_{n=0}^\infty \frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)}\left(\frac{2B}{m^2}\right)^{2n+2}$$

- characteristic factorial divergence

$$c_n = \frac{(-1)^{n+1}}{8}\sum_{k=1}^\infty \frac{\Gamma(2n+2)}{(k\,\pi)^{2n+4}}$$

- reconstruct correct Borel transform:

$$\sum_{k=1}^\infty \frac{s}{k^2\pi^2(s^2+k^2\pi^2)} = -\frac{1}{2s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)$$

Euler-Heisenberg Effective Action and Schwinger Effect

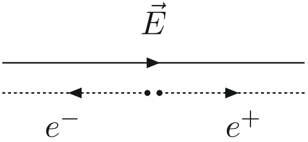
B field: QFT analogue of Zeeman effect

E field: QFT analogue of Stark effect

$B^2 \to -E^2$: series becomes non-alternating

Borel summation $\Rightarrow \text{Im}\, S = \frac{e^2 E^2}{8\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{k\, m^2 \pi}{eE}\right]$

Schwinger effect:



$$2eE\frac{\hbar}{mc} \sim 2mc^2$$

$$\Rightarrow$$

$$E_c \sim \frac{m^2 c^3}{e\hbar} \approx 10^{16} \text{V/cm}$$

WKB tunneling from Dirac sea

$\text{Im}\, S \to$ physical pair production rate

- Euler-Heisenberg series must be divergent

- scalar QED EH in self-dual background ($F = \pm\tilde{F}$):

$$S = \frac{F^2}{16\pi^2}\int_0^\infty \frac{dt}{t}\, e^{-t/F}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

- Gaussian matrix model: $\lambda = g\,N$

$$\mathcal{F} = -\frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

- $c = 1$ String: $\lambda = g\,N$

$$\mathcal{F} = \frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sin^2(t)} - \frac{1}{t^2} - \frac{1}{3}\right)$$

- Chern-Simons matrix model:

$$\mathcal{F} = -\frac{1}{4}\sum_{m\in\mathbb{Z}}\int_0^\infty \frac{dt}{t}\, e^{-2(\lambda+2\pi\, i\, m)\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

de Sitter/ anti de Sitter effective actions (Das & GD, hep-th/0607168)

- explicit expressions (multiple gamma functions)

$$\mathcal{L}_{AdS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(AdS_d)} \left(\frac{K}{m^2}\right)^n$$

$$\mathcal{L}_{dS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(dS_d)} \left(\frac{K}{m^2}\right)^n$$

- changing sign of curvature: $a_n^{(AdS_d)} = (-1)^n a_n^{(dS_d)}$
- odd dimensions: convergent
- even dimensions: divergent

$$a_n^{(AdS_d)} \sim \frac{\mathcal{B}_{2n+d}}{n(2n+d)} \sim 2(-1)^n \frac{\Gamma(2n+d-1)}{(2\pi)^{2n+d}}$$

- pair production in dS_d with d even

QED/QCD effective action and the "Schwinger effect"

- formal definition:

$$\Gamma[A] = \ln \det\, (i \not{D} + m) \qquad\qquad D_\mu = \partial_\mu - i \frac{e}{\hbar c} A_\mu$$

- vacuum persistence amplitude

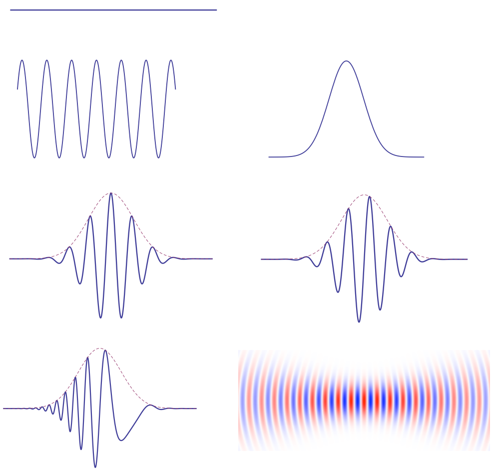
$$\langle O_{\rm out} \,|\, O_{\rm in} \rangle \equiv \exp\left(\frac{i}{\hbar}\, \Gamma[A]\right) = \exp\left(\frac{i}{\hbar} \left\{ {\rm Re}(\Gamma) + i\, {\rm Im}(\Gamma) \right\}\right)$$

- encodes nonlinear properties of QED/QCD vacuum
- vacuum persistence probability

$$\left| \langle O_{\rm out} \,|\, O_{\rm in} \rangle \right|^2 = \exp\left(-\frac{2}{\hbar}\, {\rm Im}(\Gamma)\right) \approx 1 - \frac{2}{\hbar}\, {\rm Im}(\Gamma)$$

- probability of vacuum pair production $\approx \frac{2}{\hbar}\, {\rm Im}(\Gamma)$
- cf. Borel summation of perturbative series, & instantons

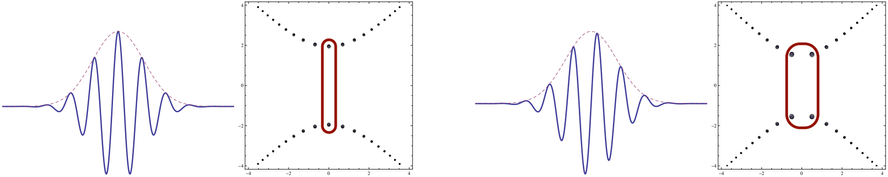
Schwinger Effect: Beyond Constant Background Fields



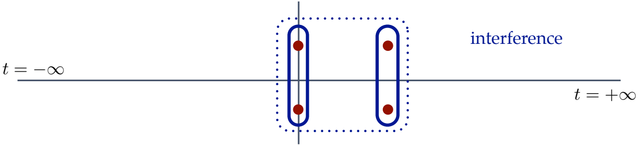
- constant field
- sinusoidal or single-pulse
- envelope pulse with sub-cycle structure; carrier-phase effect
- chirped pulse; Gaussian beam , ...

- structured fields require complex instantons (saddles)
- physics: optimization and quantum control

Carrier Phase Effect from Stokes Phenomenon Dumlu, GD, 2010



- interference produces momentum spectrum structure



$$P \approx 4 \sin^2(\theta)\, e^{-2\,\mathrm{Im} W} \qquad \theta:\ \text{interference phase}$$

- double-slit interference, in time domain, from vacuum
- Ramsey effect: N alternating sign pulses $\Rightarrow$ N-slit system $\Rightarrow$ coherent N^2 enhancement Akkermans, GD, 2012

Worldline Instantons

GD, Schubert, 2005

To maintain the relativistic invariance we describe a trajectory in space-time by giving the four variables $x_\mu(u)$ as functions of some fifth parameter (somewhat analogous to the proper-time) *Feynman, 1950*

- worldline representation of effective action

$$\Gamma = -\int d^4x \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint_x \mathcal{D}x \, \exp\left[-\int_0^T d\tau \, \left(\dot{x}_\mu^2 + A_\mu \, \dot{x}_\mu \right) \right]$$

- double-steepest descents approximation:
- worldline instantons (saddles): $\ddot{x}_\mu = F_{\mu\nu}(x)\,\dot{x}_\nu$
- proper-time integral: $\frac{\partial S(T)}{\partial T} = -m^2$

$$\operatorname{Im}\Gamma \approx \sum_{\text{saddles}} e^{-S_{\text{saddle}}(m^2)}$$

- multiple turning point pairs $\Rightarrow$ complex instantons (saddles)

Dumlu, GD, 2011

QFT: Renormalons

reviews: Beneke, 1998; Shifman, 2014

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

$$c_n \sim (\pm 1)^n \frac{n!}{(2S)^n}$$

QFT: new physical effects occur, due to running of couplings with momentum

- faster source of divergence: “renormalons”

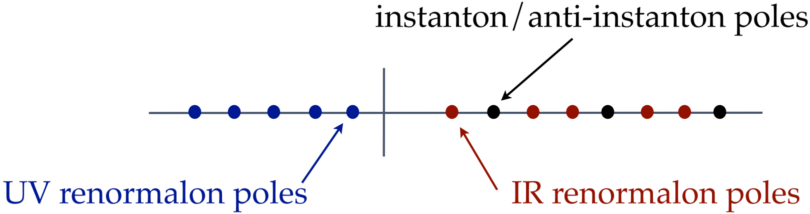
$$c_n \sim (\pm 1)^n \frac{\beta_0^n \, n!}{(2S)^n}$$

- both positive and negative Borel poles

IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory: $\longrightarrow \quad \pm i\, e^{-\frac{2S}{\beta_0\, g^2}}$

instantons on $\mathbb{R}^2$ or $\mathbb{R}^4$: $\longrightarrow \quad \pm i\, e^{-\frac{2S}{g^2}}$



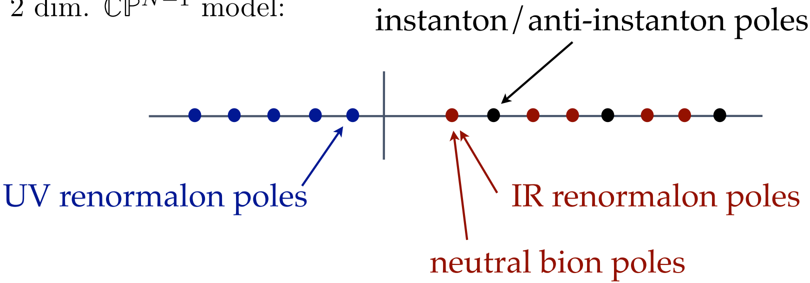
appears that BZJ cancellation cannot occur

asymptotically free theories remain inconsistent

't Hooft, 1980; David, 1981

resolution: there is another problem with the non-perturbative instanton gas analysis (Argyres, Ünsal 1206.1890; GD, Ünsal, 1210.2423)

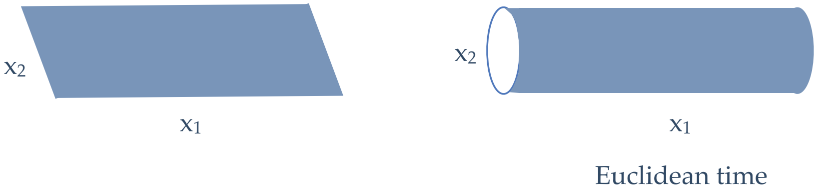
- scale modulus of instantons
- spatial compactification and principle of continuity
- 2 dim. $\mathbb{CP}^{N-1}$ model:



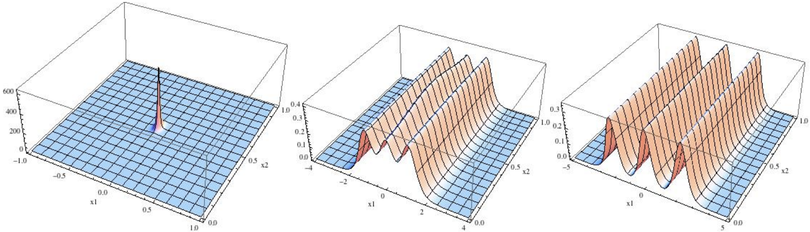
cancellation occurs ! (GD, Ünsal, 1210.2423, 1210.3646)

Topological Molecules in Spatially Compactified Theories

$\mathbb{CP}^{N-1}$: regulate scale modulus problem with (spatial) compactification: $\mathbb{R}^2 \to \mathbb{S}^1_L \times \mathbb{R}^1$



$\mathbb{Z}_N$ twist: instantons fractionalize: $S_{\text{inst}} \longrightarrow \frac{S_{\text{inst}}}{N} = \frac{S_{\text{inst}}}{\beta_0}$



Perturbative Analysis

- weak-coupling semi-classical analysis
- perturbative $\to$ effective QM problem
- perturbation theory diverges & non-Borel summable
- perturbative sector: directional Borel summation

$$B_{\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\, B\mathcal{E}(t)\, e^{-t/g^2} = \mathrm{Re}\, B\mathcal{E}(g^2) \mp i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

- compare:

$$\left[\mathcal{I}_i\bar{\mathcal{I}}_i\right]_\pm = \left(\ln\left(\frac{g^2\, N}{8\pi}\right) - \gamma\right)\frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}} \pm i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

exact ("BZJ") cancellation !

explicit application of resurgence to nontrivial QFT

Non-perturbative Physics Without Instantons

Dabrowski, GD, 1306.0921, Cherman, Dorigoni, GD, Ünsal, 1308.0127, 1403.1277, GD, Ünsal, 1505.07803

- $O(N)$ & principal chiral model have no instantons !
- but they have finite action non-BPS saddles
- Yang-Mills, $\mathbb{CP}^{N-1}$, $O(N)$, principal chiral model, ... all have non-BPS solutions with finite action

(Din & Zakrzewski, 1980; Uhlenbeck 1985; Sibner, Sibner, Uhlenbeck, 1989)

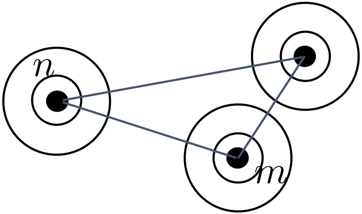
- "unstable": negative modes of fluctuation operator
- what do these mean physically ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from directional Borel sums of perturbation theory

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\rm saddle}]} \times (\text{fluctuations}) \times (\text{qzm})$$

The Bigger Picture: Decoding the Path Integral

what is the origin of resurgent behavior in QM and QFT ?



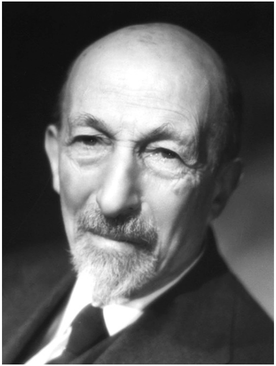
to what extent are (all?) multi-instanton effects encoded in perturbation theory? And if so, why?

- QM & QFT: basic property of all-orders steepest descents integrals

- Lefschetz thimbles: analytic continuation of path integrals

The shortest path between two truths in the real domain passes through the complex domain

Jacques Hadamard, 1865 - 1963



All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals:

<u>*hyperasymptotics*</u> (Berry/Howls 1991, Howls 1992)

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}}\, e^{-\frac{1}{g^2} f_n}\, T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

$$T^{(n)}(g^2) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2 v/(F_{nm})} T^{(m)} \left(\frac{F_{nm}}{v} \right)$$

- exact resurgent relation between fluctuations about n^{th} saddle and about neighboring saddles m

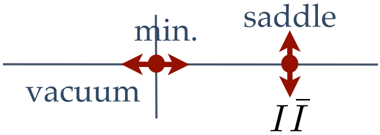
$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \dots \right.$$

- universal factorial divergence of fluctuations (Darboux)
- fluctuations about different saddles explicitly related !

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(g^2) = \int_0^{\pi} dz\, e^{-\frac{1}{g^2}\sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$T_r^{(0)} = \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)}$$
$$\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1 - \frac{\frac{1}{4}}{(r-1)} + \frac{\frac{9}{32}}{(r-1)(r-2)} - \frac{\frac{75}{128}}{(r-1)(r-2)(r-3)} + \right.$$

- low order coefficients about saddle z_1:

$$T^{(1)}(g^2) \sim i\sqrt{\pi}\left(1 - \frac{1}{4}g^2 + \frac{9}{32}g^4 - \frac{75}{128}g^6 + \dots\right)$$

- fluctuations about the two saddles are explicitly related could something like this work for path integrals?

“functional Darboux theorem” ?

- multi-dimensional case is already non-trivial and interesting

Pham (1965); Delabaere/Howls (2002)

- Picard-Lefschetz theory

- do a computation to see what happens ...

Resurgence in (Infinite Dim.) Path Integrals (GD, Ünsal, 1401.5202)

- periodic potential: $V(x) = \frac{1}{g^2} \sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n! \left(1 - \frac{5}{2} \cdot \frac{1}{n} - \frac{13}{8} \cdot \frac{1}{n(n-1)} - \dots \right)$$

- instanton/anti-instanton saddle point:

$$\text{Im}\, E \sim \pi\, e^{-2\frac{1}{2g^2}} \left(1 - \frac{5}{2} \cdot g^2 - \frac{13}{8} \cdot g^4 - \dots \right)$$

- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6} \cdot \frac{1}{3} \cdot \frac{1}{n} - \frac{1277}{72} \cdot \frac{1}{3^2} \cdot \frac{1}{n(n-1)} - \dots \right)$$

- instanton/anti-instanton saddle point:

$$\text{Im}\, E \sim \pi\, e^{-2\frac{1}{6g^2}} \left(1 - \frac{53}{6} \cdot g^2 - \frac{1277}{72} \cdot g^4 - \dots \right)$$

Uniform WKB

- usual WKB: $\psi = \exp\left[\pm\frac{i}{\hbar}\sum_{n=0}^{\infty}\hbar^n S_n\right]$ (Liouville/Green)

$$S_0 = \int^x \sqrt{2m(E-V)} \quad , \quad S_1 = \pm i \ln S_0' \quad , \quad \dots$$

$$\psi \approx \frac{1}{\sqrt{S_0'}} \exp\left[\pm\frac{i}{\hbar}S_0\right]\Big(1+\dots\Big)$$

- more efficient expansion:

$$\psi = \frac{1}{\sqrt{\varphi'}} \exp\left[\pm\frac{i}{\hbar}\varphi\right] \quad , \quad \varphi = \sum_{n=0}^{\infty}\hbar^{2n} S_{2n}$$

- but: still singular at turning points
- better approach: uniform WKB (e.g. one turning point)

$$\psi = \frac{1}{\sqrt{\varphi'}} Ai\left[-\frac{1}{\hbar^{2/3}}\varphi\right] \quad , \quad \varphi_0 = \left(\frac{3}{2}\int^x \sqrt{2m(E-V)}\right)^{2/3}$$

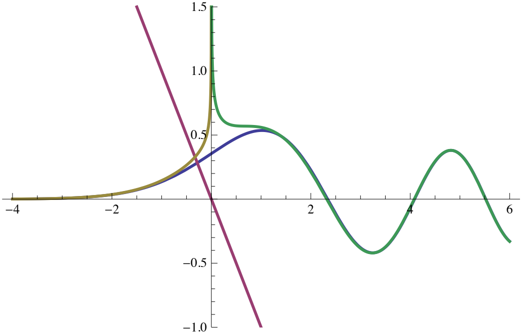
- smooth through turning point; Airy connection formulae

Uniform WKB

- uniform WKB: e.g. one turning point

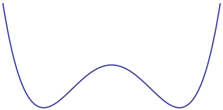
$$\psi = \frac{1}{\sqrt{\varphi'}} Ai\left[-\frac{1}{\hbar^{2/3}}\varphi\right] \quad , \quad \varphi_0 = \left(\frac{3}{2}\int^x \sqrt{2m(E-V)}\right)^{2/3}$$

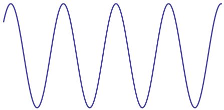
- smooth through turning point



Uniform WKB & Resurgent Trans-Series (1306.4405, 1401.5202)

$$-\frac{d^2}{dx^2}\psi+\frac{V(g\,x)}{g^2}\psi=E\,\psi\rightarrow -g^4\,\frac{d^2}{dy^2}\psi(y)+V(y)\psi(y)=g^2\,E\,\psi(y)$$





- weak coupling: degenerate harmonic classical vacua
- non-perturbative effects: $g^2 \leftrightarrow \hbar \quad \Rightarrow \quad \exp\left(-\frac{c}{g^2}\right)$
- approximately harmonic

$\Rightarrow$ uniform WKB with parabolic cylinder functions

- ansatz (with parameter ν): $\psi(y)=\frac{D_\nu\left(\frac{1}{g}u(y)\right)}{\sqrt{u'(y)}}$

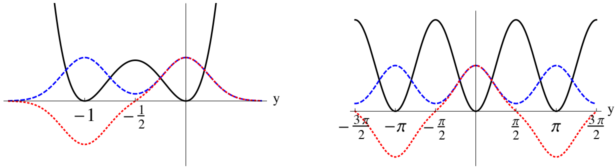
"similar looking equations have similar looking solutions"

Uniform WKB & Resurgent Trans-Series

- perturbative expansion for E and $u(y)$:

$$E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$$

- $\nu = N$: usual perturbation theory (not Borel summable)
- global analysis $\Rightarrow$ boundary conditions:



- midpoint $\sim \frac{1}{g}$; non-Borel summability $\Rightarrow$ $\quad g^2 \to e^{\pm i\,\epsilon}\, g^2$
- trans-series encodes analytic properties of D_ν

  $\Rightarrow$ generic and universal

Uniform WKB & Resurgent Trans-Series

$$D_\nu(z) \sim z^\nu\, e^{-z^2/4}\,(1+\dots) + e^{\pm i\pi\nu}\,\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\,z^{-1-\nu}\,e^{z^2/4}\,(1+\dots)$$

$\longrightarrow$ exact quantization condition

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}\,2}{g^2}\right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi\,g^2}}\,\mathcal{P}(\nu, g^2)$$

$\Rightarrow$ ν is only exponentially close to N (here $\xi \equiv \frac{e^{-S/g^2}}{\sqrt{\pi\,g^2}}$):

$$\nu = N + \frac{\left(\frac{2}{g^2}\right)^N \mathcal{P}(N, g^2)}{N!}\,\xi$$

$$-\frac{\left(\frac{2}{g^2}\right)^{2N}}{(N!)^2}\left[\mathcal{P}\frac{\partial\mathcal{P}}{\partial N} + \left(\ln\left(\frac{e^{\pm i\pi}\,2}{g^2}\right) - \psi(N+1)\right)\mathcal{P}^2\right]\xi^2 + O(\xi^3)$$

- insert: $E = E(\nu, g^2) = \sum_{k=0}^\infty g^{2k} E_k(\nu)$ $\Rightarrow$ trans-series

Connecting Perturbative and Non-Perturbative Sector

Zinn-Justin/Jentschura conjecture:

generate *entire trans-series* from just two functions:

(i) perturbative expansion $E = E_{\rm pert}(\hbar, N)$
(ii) single-instanton fluctuation function $\mathcal{P}_{\rm inst}(\hbar, N)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

$$E(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} e^{-S/\hbar}\, \mathcal{P}_{\rm inst}(\hbar, N) + \ldots$$

in fact ... (GD, Ünsal, 1306.4405, 1401.5202) fluctuation factor:

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3} \left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\hbar^2}{S}\right)\right]$$

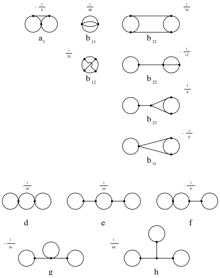
$\Rightarrow$ perturbation theory $E_{\rm pert}(\hbar, N)$ encodes everything !

Resurgence at work

- fluctuations about $\mathcal{I}$ (or $\bar{\mathcal{I}}$) saddle are determined by those about the vacuum saddle, to all fluctuation orders

- "QFT computation": 3-loop fluctuation about $\mathcal{I}$ for double-well and Sine-Gordon:

Escobar-Ruiz/Shuryak/Turbiner 1501.03993, 1505.05115



$$\text{DW}: \quad e^{-\frac{S_0}{\hbar}}\left[1-\frac{71}{72}\,\hbar-0.607535\,\hbar^2-\ldots\right]$$

$$\text{resurgence}: e^{-\frac{S_0}{\hbar}}\left[1+\frac{1}{72}\,\hbar\left(-102N^2-174N-71\right)\right.$$

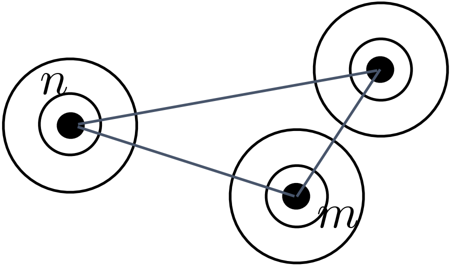
$$\left.+\frac{1}{10368}\hbar^2\left(10404N^4+17496N^3-2112N^2-14172N-6299\right)+\ldots\right]$$

- known for all N and to essentially any loop order, directly from perturbation theory !

- diagramatically mysterious ...

Connecting Perturbative and Non-Perturbative Sector

all orders of multi-instanton trans-series are encoded in perturbation theory of fluctuations about perturbative vacuum



why ? turn to path integrals again
... look for a semiclassical explanation

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}\, S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2}S_{\text{real}}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

Stokes phenomenon: intersection numbers $\mathcal{N}_k$ can change with phase of parameters

Thimbles from Gradient Flow

gradient flow to generate steepest descent thimble:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

- keeps $Im[S]$ constant, and $Re[S]$ is monotonic

$$\frac{\partial}{\partial \tau}\left(\frac{S-\bar{S}}{2i}\right) = -\frac{1}{2i}\int\left(\frac{\delta S}{\delta A}\frac{\partial A}{\partial \tau} - \overline{\frac{\delta S}{\delta A}}\,\overline{\frac{\partial A}{\partial \tau}}\right) = 0$$

$$\frac{\partial}{\partial \tau}\left(\frac{S+\bar{S}}{2}\right) = -\int\left|\frac{\delta S}{\delta A}\right|^2$$

- Chern-Simons theory (Witten 2010)
- comparison with complex Langevin (Aarts 2013, ...)
- lattice (Aurora, 2013; Tokyo/RIKEN): Bose-gas ✓

Thimbles and Gradient Flow: an example

PHYSICAL REVIEW D **88,** 051501(R) (2013)

Monte Carlo simulations on the Lefschetz thimble: Taming the sign problem

Marco Cristoforetti,[1,2] Francesco Di Renzo,[3] Abhishek Mukherjee,[1,2] and Luigi Scorzato[1,2]
[1]*ECT*/FBK, strada delle tabarelle 286, 38123 Trento, Italy*
[2]*LISC/FBK, via sommarive 18, 38123 Trento, Italy*
[3]*Università di Parma and INFN gruppo collegato di Parma, Viale G.P. Usberti n.7/A, 43124 Parma, Italy*
(Received 31 March 2013; published 16 September 2013)

CRISTOFORETTI *et al.*

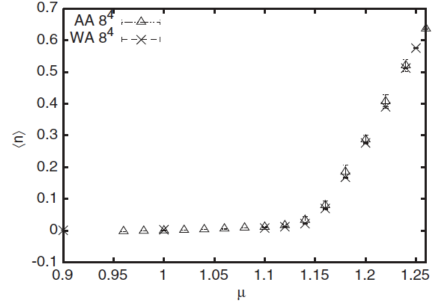
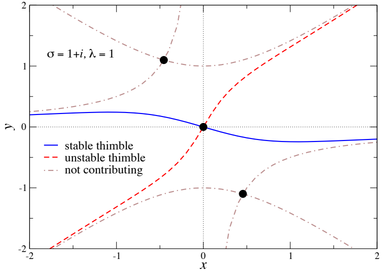


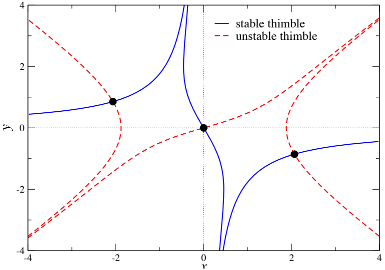
FIG. 3. Comparison of the average density $\langle n \rangle$ obtained with the worm algorithm (WA) [22] with the Aurora algorithm (AA) presented here, for the lattice $V = 8^4$. We thank C. Gattringer and T. Kloiber for providing us their results.

Thimbles, Gradient Flow and Resurgence

$$Z = \int_{-\infty}^{\infty} dx \, \exp\left[- \left(\frac{\sigma}{2} x^2 + \frac{x^4}{4} \right) \right]$$

(Aarts, 2013; GD, Unsal, ...)





- contributing thimbles change with phase of σ
- need all three thimbles when $|arg[\sigma]| > \frac{\pi}{2}$
- integrals along thimbles are related (resurgence)
- resurgence: preferred unique "field" choice

Path integrals with complex saddles: zero dim. prototype

d=0 partition function:

$$\mathcal{Z}(g^2|m) = \frac{1}{g\sqrt{\pi}} \int_{-\mathbb{K}}^{\mathbb{K}} dz\, e^{-\frac{1}{g^2}\, \mathrm{sd}^2(z|m)}$$

$$V(z|m) = \frac{1}{g^2}\, \mathrm{sd}^2(g\, z|m)$$

- duality property:

$$V(z|m)\big|_{g^2} = V(z|1-m)\big|_{-g^2}$$

- perturbative series $\sum_n a_n(m) g^{2n}$ satisfies duality:

$$a_n(m) = (-1)^n a_n(1-m)$$

Path integrals with complex saddles: zero dim. prototype

$$\mathcal{Z}(g^2|0)\Big|_{\rm pert} = 1+\frac{g^2}{4}+\frac{9g^4}{32}+\frac{75g^6}{128}+\frac{3675g^8}{2048}+\frac{59535g^{10}}{8192}+\dots$$

$$\mathcal{Z}\left(g^2|1\right)\Big|_{\rm pert} = 1-\frac{g^2}{4}+\frac{9g^4}{32}-\frac{75g^6}{128}+\frac{3675g^8}{2048}-\frac{59535g^{10}}{8192}+\dots$$

$$\mathcal{Z}\left(g^2\middle|\frac{1}{4}\right)\Big|_{\rm pert} = 1+\frac{g^2}{8}+\frac{9g^4}{64}+\frac{105g^6}{512}+\frac{1995g^8}{4096}+\frac{48195g^{10}}{32768}+\dots$$

$$\mathcal{Z}\left(g^2\middle|\frac{3}{4}\right)\Big|_{\rm pert} = 1-\frac{g^2}{8}+\frac{9g^4}{64}-\frac{105g^6}{512}+\frac{1995g^8}{4096}-\frac{48195g^{10}}{32768}+\dots$$

$$\mathcal{Z}\left(g^2\middle|\frac{1}{2}\right)\Big|_{\rm pert} = 1+0g^2+\frac{3g^4}{32}+0g^6+\frac{315g^8}{2048}+0g^{10}+\dots$$

- duality relation: $\mathcal{Z}(g^2|m)=\mathcal{Z}(-g^2|1-m)$

  non-alternating for $m<\frac{1}{2}$ alternating for $m>\frac{1}{2}$

puzzles: Borel summable? "instantons" ?

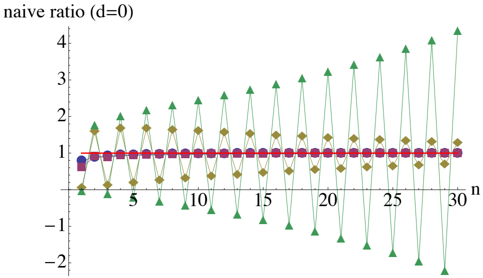
Path integrals with complex saddles: zero dim. prototype

$$\mathcal{Z}(g^2|m) = \frac{2}{g\sqrt{\pi}} \int_0^{\mathbb{K}} dz\, e^{-\frac{1}{g^2}\,\mathrm{sd}^2(z|m)}$$

- large-order behavior about 0 from saddle point $B = \mathbb{K}$:

$$S_B = \frac{1}{1-m} \qquad \Rightarrow \qquad a_n \sim \frac{(n-1)!}{\pi S_B^{n+1/2}}$$

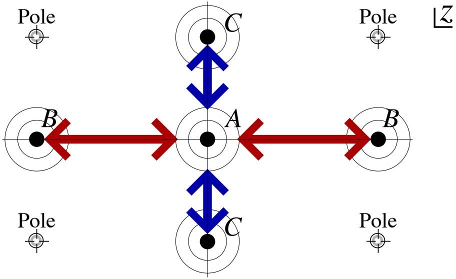
- compare with actual series:



disaster !

Path integrals with complex saddles: zero dim. prototype

- resolution: there is another saddle (complex !)



- note: $\mathcal{Z}(g^2|m)$ satisfies a third-order ODE
- $\Rightarrow$ 3 actions: three-term trans-series

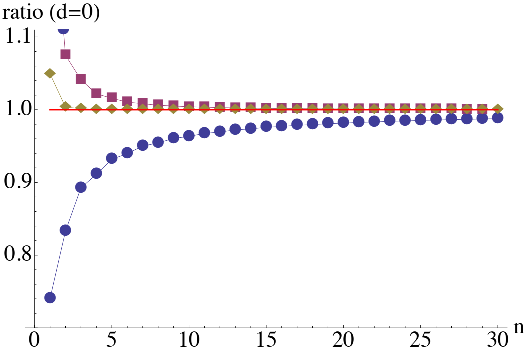
$$\mathcal{Z}(g^2|m) \quad \equiv \quad \sigma_A\,\Phi_A(g^2) + \sigma_B\,e^{-S_B/g^2}\,\Phi_B(g^2) + \sigma_C\,e^{-S_C/g^2}\,\Phi_C(g^2)$$

- coefficients of perturbative expansions are connected

Path integrals with complex saddles: zero dim. prototype

$$a_n \sim \frac{(n-1)!}{\pi}\left(\frac{1}{S_B^{n+1/2}} + (-1)^n \frac{1}{|S_C|^{n+1/2}}\right)$$

$\Rightarrow$ improved asymptotics:



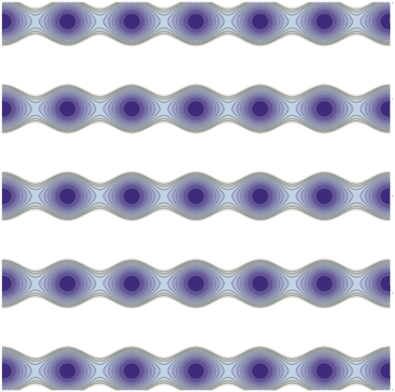
conclusion: perturbation series feels *all* saddles, both real and complex

Ghost Instantons: Analytic Continuation of Path Integrals

(Başar, GD, Ünsal, arXiv:1308.1108)

$$\mathcal{Z}(g^2|m) = \int \mathcal{D}x\, e^{-S[x]} = \int \mathcal{D}x\, e^{-\int d\tau \left(\frac{1}{4}\dot{x}^2 + \frac{1}{g^2}\,\mathrm{sd}^2(g\,x|m)\right)}$$

- doubly periodic potential: *real* & *complex* instantons



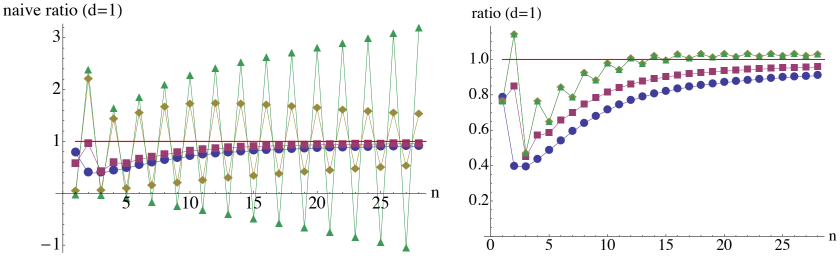
instanton actions:

$$S_{\mathcal{I}}(m) = \frac{2\arcsin(\sqrt{m})}{\sqrt{m(1-m)}}$$

$$S_{\mathcal{G}}(m) = \frac{-2\arcsin(\sqrt{1-m})}{\sqrt{m(1-m)}}$$

- large order growth of perturbation theory:

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$



without ghost instantons *with ghost instantons*

- complex instantons directly affect perturbation theory, even though they are not in the original path integral measure

Resurgence and Matrix Models, Topological Strings

Mariño, Schiappa, Weiss: *Nonperturbative Effects and the Large-Order Behavior of Matrix Models and Topological Strings* 0711.1954; Mariño, *Nonperturbative effects and nonperturbative definitions in matrix models and topological strings* 0805.3033

- resurgent Borel-Écalle analysis of partition functions, Wilson loops, etc ... in matrix models

$$Z(g_s, N) = \int dU \, \exp\left[\frac{1}{g_s} \mathrm{tr}\, V(U)\right]$$

- two variables: g_s and N ('t Hooft coupling: $\lambda = g_s N$)
- e.g. Gross-Witten-Wadia: $V = U + U^{-1}$
- double-scaling limit: Painlevé II
- 3rd order phase transition at $\lambda = 2$: condensation of instantons
- similar in 2d Yang-Mills on Riemann surface

Resurgence in the Gross-Witten-Wadia Model

Buividovich, GD, Valgushev 1512.09021

- unitary matrix model $\equiv$ 2d $U(N)$ lattice gauge theory
- third order phase transition at $\lambda = 2$

$$Z = \int \mathcal{D}U \, \exp\left[\frac{N}{\lambda}\mathrm{Tr}(U + U^\dagger)\right]$$

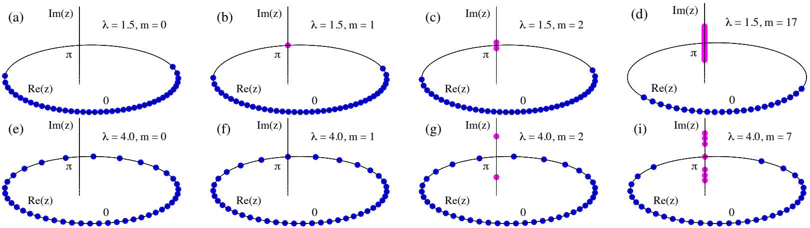
- in terms of eigenvalues e^{iz_i} of U

$$Z = \prod_{i=1}^{N} \int_{-\pi}^{\pi} dz_i \, e^{-S(z_i)}$$

$$S(z_i) \equiv -\frac{2N}{\lambda}\sum_i \cos(z_i) - \sum_{i<j} \ln \sin^2\left(\frac{z_i - z_j}{2}\right)$$

- at large N search numerically for saddles: $\frac{\partial S}{\partial z_i} = 0$

- phase transition driven by complex saddles



- eigenvalue tunneling into the complex plane
- weak-coupling: "instanton" is $m = 1$ configuration
  - has negative mode $\Rightarrow$ resurgent trans-series
- strong-coupling: dominant saddle is $m = 2$, complex !

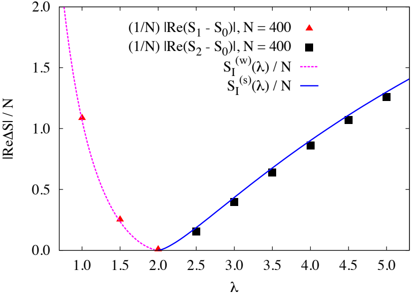
Resurgence in the Gross-Witten-Wadia Model 1512.09021

- weak-coupling "instanton" action from string eqn

$$S_I^{(weak)} = 4/\lambda\sqrt{1-\lambda/2} - \operatorname{arccosh}\left((4-\lambda)/\lambda\right), \quad \lambda < 2$$

- strong-coupling "instanton" action from string eqn

$$S_I^{(strong)} = 2\operatorname{arccosh}\left(\lambda/2\right) - 2\sqrt{1-4/\lambda^2}, \quad \lambda \geq 2$$



- interpolated by Painlevé II (double-scaling limit)

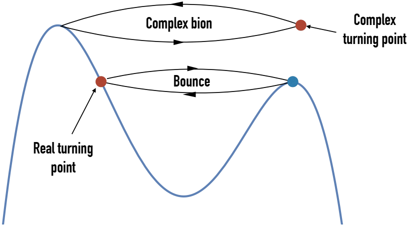
Complex Saddles in Path Integrals

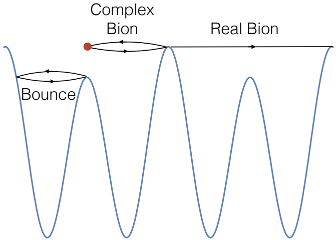
(Behtash, GD, Schäfer, Sulejmanpasic, Ünsal 1510.00978, 1510.03435)

- puzzle 1: how do approximate bion solutions yield exact SUSY answers?

- puzzle 2: how to explain SUSY breaking for DW semiclassically?

- puzzle 3: how to explain SUSY non-breaking for SG semiclassically?





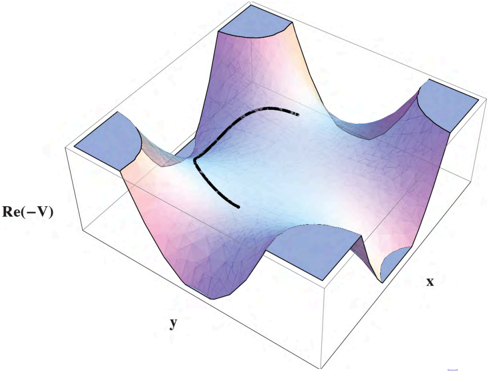
Complex Saddles in Path Integrals

- complex classical equations of motion

$$\frac{d^2 z}{dt^2} = \frac{\partial V}{\partial z} \qquad \text{or equivalently} \qquad \frac{d^2 x}{dt^2} = +\frac{\partial V_r}{\partial x}$$

$$\frac{d^2 y}{dt^2} = -\frac{\partial V_r}{\partial y}$$

- very different from 2d motion !



- gradient flow:

$$\frac{\partial z(t,u)}{\partial u} = +\frac{\delta \bar{\mathcal{S}}}{\delta \bar{z}} = +\left(\frac{d^2 \bar{z}}{dt^2} - \frac{\partial \bar{V}}{\partial \bar{z}}\right) ,$$
$$\frac{\partial \bar{z}(t,u)}{\partial u} = +\frac{\delta \mathcal{S}}{\delta z} = +\left(\frac{d^2 z}{dt^2} - \frac{\partial V}{\partial z}\right)$$

- behavior of action

$$\frac{\partial \mathrm{Im}[\mathcal{S}]}{\partial u} = 0$$

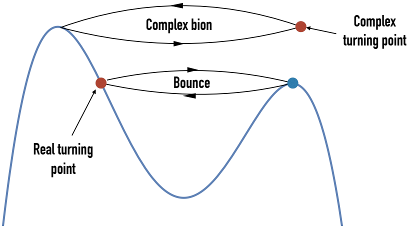
$$\frac{\partial \mathrm{Re}[-\mathcal{S}]}{\partial u} \leq 0$$

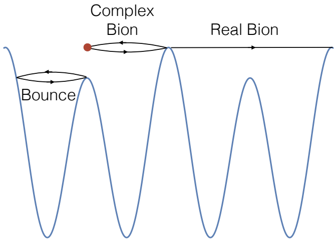
Complex Saddles in Path Integrals

- complex classical saddle solutions come from the effective potential

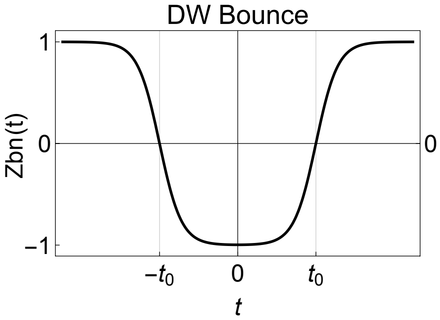
$$V_{\text{eff}} = (W')^2 \pm W''$$

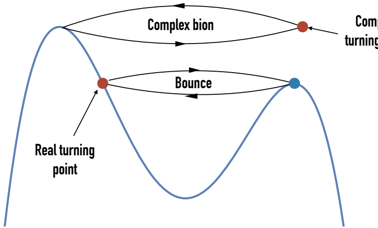
- arises from integrating out the fermions





Complex Saddles in Path Integrals: SUSY Double Well



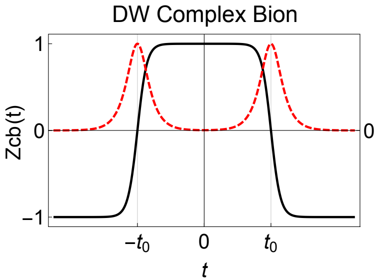


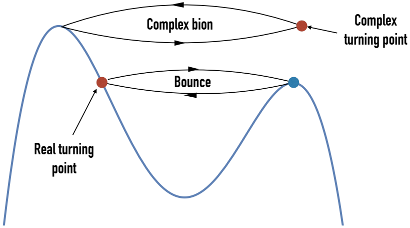
$$z_{bounce}(t) = z_3^{\rm cr} - \frac{(z_3^{\rm cr} - z_T)}{2} \coth\left(\omega_{\rm bn}\frac{t_0}{2}\right)\left(\tanh\left[\omega_{\rm bn}\frac{(t+t_0)}{2}\right] - \tanh\left[\omega_{\rm bn}\frac{(t-t_0)}{2}\right]\right)$$

- real bounce goes to a real t.p. and back
- but has sub-leading relevance for ground state properties

Complex Saddles in Path Integrals: SUSY Double Well

- no REAL solution exists from



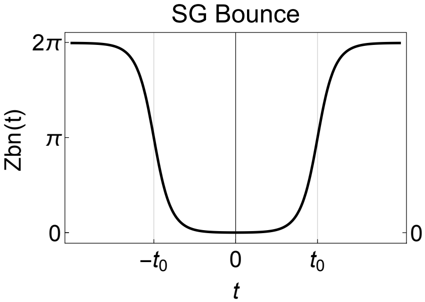


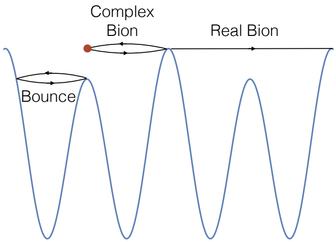
$$z_{\rm cb}(t) = z_1^{\rm cr} - \frac{(z_1^{\rm cr} - z_T)}{2} \coth\left(\frac{\omega_{\rm cb} t_0}{2}\right) \left(\tanh\left[\omega_{\rm cb}\frac{(t+t_0)}{2}\right] - \tanh\left[\omega_{\rm cb}\frac{(t-t_0)}{2}\right]\right)$$

- complex bion goes to a complex t.p. and back
- characteristic size of the approximate bion
- action has imaginary part

$$S_{\rm cb} = {\rm Re}[S_{\rm cb}] \pm i N_f \pi, \qquad {\rm Im}\,[S_{\rm cb}] = \pm N_f \pi$$

Complex Saddles in Path Integrals: SUSY Sine-Gordon



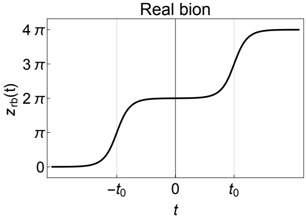


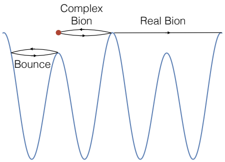
$$z_{\rm bn}(t) = 4\pi a - 4a\left[\arctan\left(\exp\left[-\omega_{\rm bn}(t-t_0)\right]\right) + \arctan\left(\exp\left[\omega_{\rm bn}(t+t_0)\right]\right)\right]$$

- real bounce goes to a real t.p. and back
- but has sub-leading relevance for ground state properties

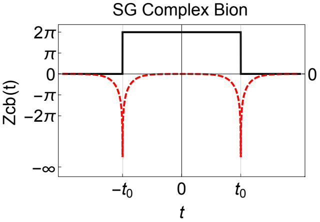
Complex Saddles in Path Integrals: SUSY Sine-Gordon

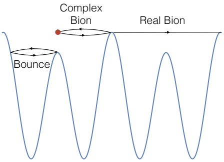
- SUSY Sine-Gordon has two different bion solutions





$$z_{\text{real bion}}(t) = 2\pi + 4\left(\arctan\left(\exp\left[\omega_{\text{rb}}(t+t_0)\right]\right) - \arctan\left(\exp\left[-\omega_{\text{rb}}(t-t_0)\right]\right)\right)$$





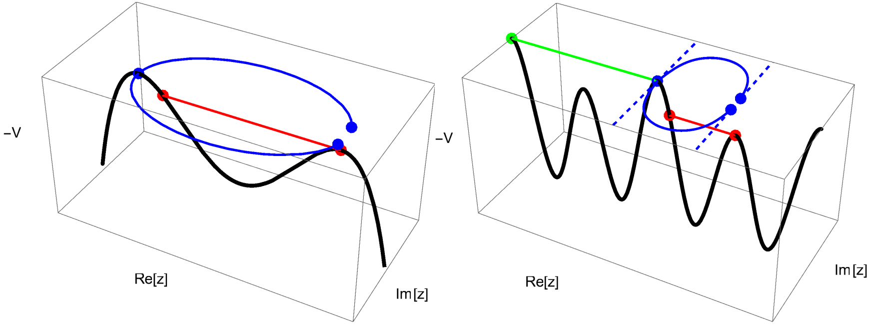
$$z_{\text{complex bion}}(t) = 2\pi \pm 4\left[\arctan\left(\exp\left(\omega_{\text{cb}}(t+t_0)\right)\right) + \arctan\left(\exp\left(-\omega_{\text{cb}}(t-t_0)\right)\right)\right.$$

Necessity of Complex Saddles

(Behtash, GD, Schäfer, Sulejmanpasic, Ünsal (1510.00978), (1510.03435)

SUSY QM: $g\,\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\left(W'\right)^2 \pm \frac{g}{2} W''$

- $W = \frac{1}{3}x^3 - x \rightarrow$ tilted double-well
- $W = \cos\frac{x}{2} \rightarrow$ double Sine-Gordon
- new (exact) complex saddles



Necessity of Complex Saddles

(Behtash, GD, Schäfer, Sulejmanpasic, Ünsal (1510.00978), (1510.03435)

SUSY QM: $g\,\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\left(W'\right)^2 \pm \frac{g}{2}W''$

- complex saddles have complex action:

$$S_{\text{complex bion}} \sim 2S_I + i\pi$$

- $W = \cos\frac{x}{2} \rightarrow$ double Sine-Gordon

$$E_{\text{ground state}} \sim 0 - 2\,e^{-2S_I} - 2\,e^{-i\pi}e^{-2S_I} = 0 \qquad \checkmark$$

- $W = \frac{1}{3}x^3 - x \rightarrow$ tilted double-well

$$E_{\text{ground state}} \sim 0 - 2\,e^{-i\pi}e^{-2S_I} > 0 \qquad \checkmark$$

semiclassics: complex saddles required for SUSY algebra

- similar effects in QFT, also non-SUSY

Connecting weak and strong coupling

physics question:

does weak coupling analysis contain enough information to extrapolate to strong coupling ?

. . . even if the degrees of freedom re-organize themselves in a very non-trivial way?

classical asymptotics is clearly not enough: could resurgent asymptotics be enough?

Connecting weak and strong coupling

- often, weak coupling expansions are divergent, but strong-coupling expansions are convergent (generic behavior for special functions)

- e.g. Euler-Heisenberg

$$\Gamma(B) \sim -\frac{m^4}{8\pi^2} \sum_{n=0}^{\infty} \frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)} \left(\frac{2eB}{m^2}\right)^{2n+4}$$

$$\begin{aligned}\Gamma(B) &= \frac{(eB)^2}{2\pi^2}\left\{-\frac{1}{12}+\zeta'(-1)-\frac{m^2}{4eB}+\frac{3}{4}\left(\frac{m^2}{2eB}\right)^2-\frac{m^2}{4eB}\ln(2\pi)\right.\\ &+\left[-\frac{1}{12}+\frac{m^2}{4eB}-\frac{1}{2}\left(\frac{m^2}{2eB}\right)^2\right]\ln\left(\frac{m^2}{2eB}\right)-\frac{\gamma}{2}\left(\frac{m^2}{2eB}\right)^2\\ &\left.+\frac{m^2}{2eB}\left(1-\ln\left(\frac{m^2}{2eB}\right)\right)+\sum_{n=2}^{\infty}\frac{(-1)^n\zeta(n)}{n(n+1)}\left(\frac{m^2}{2eB}\right)^{n+1}\right\}\end{aligned}$$

Physics Motivation

- trans-series expansion is a double-expansion: can be organized in different ways

$$\begin{aligned} F(N, g^2) &= \sum_n g^{2n} p_n^{(0)}(N) + e^{-\frac{1}{g^2}} \sum_n g^{2n} p_n^{(1)}(N) + \dots \\ &= \sum_{h=0}^{\infty} \frac{1}{N^{2h-2}} f_h(N\, g^2) \\ &= \sum_k \frac{1}{g^{2k}} c_k(N) \end{aligned}$$

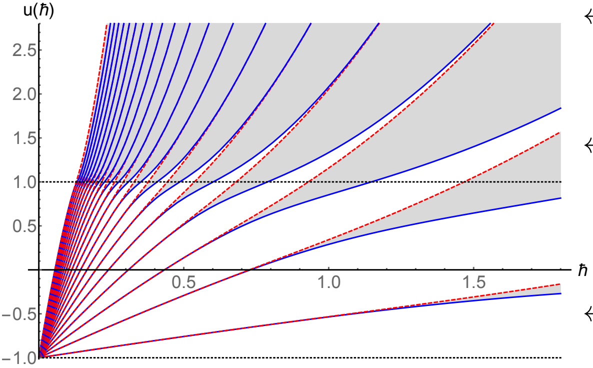
- how does a divergent trans-series at weak coupling turn into a convergent series at strong-coupling?

- what happens to the resurgent structure?

- separated by a phase transition: "instantons condense"

Resurgence in $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ Theories (Başar, GD, 1501.05671)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$$



⟵ electric sector (convergent)

⟵ magnetic sector

⟵ dyonic sector (divergent)

- energy: $u = u(N, \hbar)$; 't Hooft coupling: $\lambda \equiv N\,\hbar$
- very different physics for $\lambda \gg 1$, $\lambda \sim 1$, $\lambda \ll 1$

Resurgence of $\mathcal{N}=2$ SUSY SU(2)

- moduli parameter: $u = \langle \mathrm{tr}\,\Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial \mathcal{W}}{\partial a}$
- Nekrasov twisted superpotential $\mathcal{W}(a, \hbar, \Lambda)$:
- Mathieu equation: (Mironov/Morozov)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \Lambda^2 \cos(x)\,\psi = u\,\psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

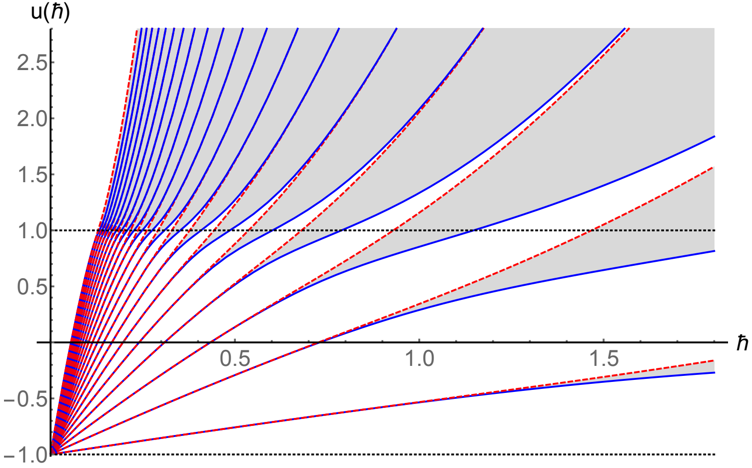
- Matone relation:

$$u(a,\hbar) = \frac{i\pi}{2}\Lambda\frac{\partial \mathcal{W}(a,\hbar,\Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

- $\mathcal{N} = 2^*$ $\leftrightarrow$ Lamé equation

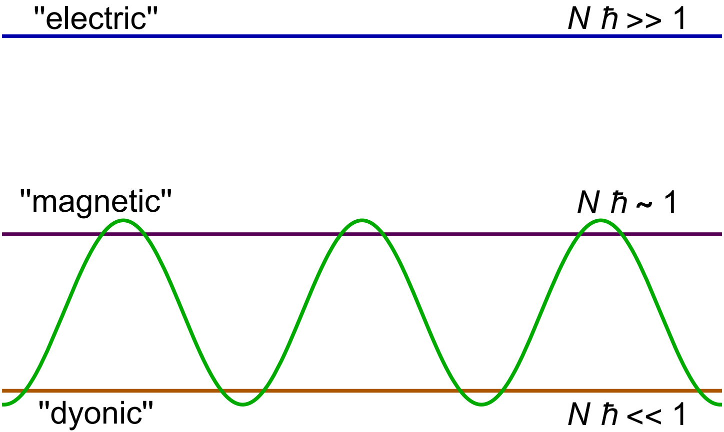
Mathieu Equation Spectrum: ($\hbar$ plays role of g^2)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$



Mathieu Equation Spectrum: ($\hbar$ plays role of g)

- 3 different spectral regions: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$



Mathieu Equation Spectrum

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$$

- small N: divergent, non-Borel-summable $\to$ trans-series

$$\begin{aligned} u(N,\hbar) \quad &\sim \quad -1 + \hbar\left[N+\frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2 + \frac{1}{4}\right] \\ &\quad -\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3 + \frac{3}{4}\left(N+\frac{1}{2}\right)\right] - \dots \end{aligned}$$

- large N: convergent expansion: $\longrightarrow$?? trans-series ??

$$\begin{aligned} u(N,\hbar) \sim \frac{\hbar^2}{8}\Bigg(N^2 &+ \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 \\ &+ \frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12} + \dots\Bigg) \end{aligned}$$

- note: poles in coefficients

Mathieu Equation Spectrum: far above the barrier

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$$

- narrow gaps high in the spectrum: complex instantons
- Dyhkne: same formula for band/gap splittings

$$\Delta u \sim \frac{2}{\pi}\frac{\partial u}{\partial N}\, e^{-\frac{2\pi}{\hbar}\mathrm{Im}\, a_0^D}$$

$$\begin{aligned}\Delta u_N^{\rm gap} &\sim \frac{\hbar^2}{4}\frac{1}{\left(2^{N-1}(N-1)!\right)^2}\left(\frac{2}{\hbar}\right)^{2N}\left[1+O\left(\left(\frac{2}{\hbar}\right)^4\right)\right] \\ &\sim \frac{N\,\hbar^2}{2\pi}\left(\frac{e}{N\,\hbar}\right)^{2N} \qquad , \quad N\gg 1\end{aligned}$$

Keldysh Approach in QED

Brézin/Itzykson, 1970; Popov, 1971

- Schwinger effect in $E(t) = \mathcal{E}\cos(\omega t)$
- adiabaticity parameter: $\gamma \equiv \frac{m\,\omega}{\mathcal{E}}$
- WKB $\Rightarrow$ $P_{\rm QED} \sim \exp\left[-\pi\,\frac{m^2}{\hbar\,\mathcal{E}}\,g(\gamma)\right]$

$$P_{\rm QED} \sim \begin{cases} \exp\left[-\pi\,\frac{m^2}{\hbar\,\mathcal{E}}\right] \quad , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \\ \left(\frac{\mathcal{E}}{\omega\, m}\right)^{4m/\hbar\omega} \quad , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}$$

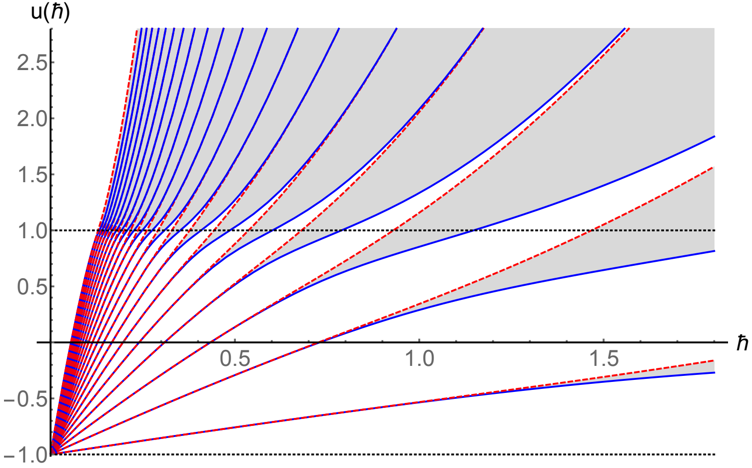
- semi-classical instanton (saddle) interpolates between non-perturbative ‘tunneling pair-production” and perturbative “multi-photon pair production”

- exact mapping $\Rightarrow$ physical interpretation of different non-pert expressions

$$\hbar \leftrightarrow \frac{4\omega^2}{\mathcal{E}} \quad ; \quad N \leftrightarrow \frac{m}{\omega} \quad ; \quad u = 1 + 2\gamma^2$$

Mathieu Equation Spectrum

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$$



Mathieu Equation Spectrum: far above the barrier

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi = u\,\psi \qquad , \quad u\gg 1 \quad , \quad N\hbar \gg 1$$

- gap edges $u_N^{(\pm)}(\hbar)$

$$u_0 = \frac{\hbar^2}{8}\left(0-\frac{1}{\hbar^2}+\frac{7}{4\hbar^6}-\frac{58}{9\hbar^{10}}+\frac{68687}{2304\hbar^{14}}+\dots\right)$$

$$u_1^{(-)} = \frac{\hbar^2}{8}\left(1-\frac{4}{\hbar^2}-\frac{2}{\hbar^4}+\frac{1}{\hbar^6}-\frac{1}{6\hbar^8}-\frac{11}{36\hbar^{10}}+\frac{49}{144\hbar^{12}}+\dots\right)$$

$$u_1^{(+)} = \frac{\hbar^2}{8}\left(1+\frac{4}{\hbar^2}-\frac{2}{\hbar^4}-\frac{1}{\hbar^6}-\frac{1}{6\hbar^8}+\frac{11}{36\hbar^{10}}+\frac{49}{144\hbar^{12}}+\dots\right)$$

$$u_2^{(-)} = \frac{\hbar^2}{8}\left(4-\frac{4}{3\hbar^4}+\frac{5}{54\hbar^8}-\frac{289}{19440\hbar^{12}}+\frac{21391}{6998400\hbar^{16}}+\dots\right)$$

$$u_2^{(+)} = \frac{\hbar^2}{8}\left(4+\frac{20}{3\hbar^4}-\frac{763}{54\hbar^8}+\frac{1002401}{19440\hbar^{12}}-\frac{1669068401}{6998400\hbar^{16}}+\dots\right)$$

- convergent (“strong-coupling”) expansions!

Beyond Large N: Multi-instantons at strong coupling

- convergent expansion, but coefficients have poles:

$$u(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \dots\right.$$

$$u_2^{(+)} = \frac{\hbar^2}{8}\left(4 + \frac{20}{3\hbar^4} - \frac{763}{54\hbar^8} + \frac{1002401}{19440\hbar^{12}} - \frac{1669068401}{6998400\hbar^{16}} + \dots\right)$$

$$u_2^{(-)} = \frac{\hbar^2}{8}\left(4 - \frac{4}{3\hbar^4} + \frac{5}{54\hbar^8} - \frac{289}{19440\hbar^{12}} + \frac{21391}{6998400\hbar^{16}} + \dots\right)$$

- average: $\frac{20-4}{2\cdot 3} = \frac{8}{3} = \frac{2^4}{2\cdot(2^2-1)}$

$$\frac{8}{\hbar^2}u_2^{(\pm)} = \left(4 + \frac{8}{3\hbar^4}\right) \pm \frac{4}{\hbar^4}\left(1 - \frac{16}{9\hbar^4}\right) - \frac{379}{54\hbar^8}\left(1 - \frac{62632}{17055\hbar^4}\right) \pm \frac{11141}{432\hbar^{12}}\left(1 - \right.$$

- this is an instanton expansion
- pole develops at 2-instanton order in all fluctuations

Beyond Large N: Multi-instantons at strong coupling

- convergent expansion, but coefficients have poles:

$$u(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12} + \dots\right)$$

$$u_4^{(+)} = \frac{\hbar^2}{8}\left(16 + \frac{8}{15\hbar^4} + \frac{433}{3375\hbar^8} - \frac{45608}{5315625\hbar^{12}} + \dots\right)$$

$$u_4^{(-)} = \frac{\hbar^2}{8}\left(16 + \frac{8}{15\hbar^4} - \frac{317}{3375\hbar^8} + \frac{80392}{5315625\hbar^{12}} + \dots\right)$$

- average: $\frac{433-317}{2\cdot 3375} = \frac{58}{3375} = \frac{2^8(5\cdot 4^2+7)}{32\cdot(4^2-1)^3\cdot(4^2-4)}$

$$\frac{8}{\hbar^2}u_4^{(\pm)} = \left(16 + \frac{8}{15\hbar^4} + \frac{58}{3375\hbar^8} + \frac{17932}{5315625\hbar^{12}}\right) \pm \frac{1}{9\hbar^8}\left(1 + \frac{8}{75\hbar^4} + \dots\right)$$

- pole develops at 2-instanton order: instanton expansion

Beyond Large N: Multi-instantons at strong coupling

$$u(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \dots\right.$$

- re-organize as a multi-instanton expansion

$$u_N^{(\pm)}(\hbar) = \frac{\hbar^2 N^2}{8}\sum_{n=0}^{N-1}\frac{\alpha_n(N)}{\hbar^{4n}} \pm \frac{\hbar^2}{8}\frac{1}{(2^{N-1}(N-1)!)^2}\left(\frac{2}{\hbar}\right)^{2N}\sum_{n=0}^{N-1}\frac{\beta_n(N)}{\hbar^{4n}} + \dots$$

- fluctuation series are very similar
- 1-instanton gap splitting: (Basar, GD, Unsal, 2014)

$$\Delta u_N \equiv \frac{1}{(2^{N-1}(N-1)!)^2}\frac{\partial u}{\partial N}e^{A(N,\hbar)} \quad \Rightarrow \quad \frac{\partial A}{\partial \hbar^2} = -\frac{4}{\hbar^4}\frac{\partial u}{\partial N}$$

- 1-inst. flucts. determined by pert. exp. (polynomial !)

- resurgent multi-instanton structure in convergent region

Small g and Large N

- often we study theories with both g and N
- 't Hooft limit: $\lambda \equiv N\, g$ fixed
- planar limit of QCD/YM: $J_n \sim n!$ but $J_n^{\rm planar} \sim c^n$
- e.g. Bessel functions:

$$Z_N\left(\frac{1}{g}\right) \equiv I_N\left(N\frac{1}{Ng}\right) \sim \begin{cases} \sqrt{\frac{g}{2\pi}}\, e^{1/g} & , \quad g \to 0,\ N \text{ fixed} \\ \frac{1}{\sqrt{2\pi N}}\left(\frac{e}{2Ng}\right)^N & , \quad N \to \infty,\ g \text{ fixed} \end{cases}$$

- uniform asymptotics: $N \to \infty \quad , \quad N\, g$ fixed

$$Z_N\left(\frac{1}{g}\right) = I_N\left(N\frac{1}{Ng}\right) \sim \frac{\exp\left[\sqrt{N^2+\frac{1}{g^2}}\right]}{\sqrt{2\pi}\left(N^2+\frac{1}{g^2}\right)^{\frac{1}{4}}}\left(\frac{\frac{1}{N\,g}}{1+\sqrt{1+\frac{1}{(N\,g)^2}}}\right)^N$$

- analogue of Keldysh tunneling/multi-photon transition

Perturbative/Non-perturbative connection (Başar, GD, 1501.05671)

- Zinn-Justin: $B(u,\hbar)$, $A(u,\hbar)$ determine full trans-series
- GD, Ünsal: $u(B,\hbar)$ encodes $A(B,\hbar)$:

$$\frac{\partial u}{\partial B} = -\frac{\hbar}{16}\left(2B + \hbar\frac{\partial A}{\partial \hbar}\right)$$

- identifications:

$$a \leftrightarrow \frac{\hbar}{2}B \quad , \quad a_D \leftrightarrow \frac{\hbar}{4\pi}A + \text{shift} \quad , \quad \Lambda \sim \frac{1}{\hbar}$$

$$\frac{\partial u(a,\hbar)}{\partial a} = \frac{\pi i}{2}\left(a^D(a,\hbar) - a\frac{\partial a^D(a,\hbar)}{\partial a} - \hbar\frac{\partial a^D(a,\hbar)}{\partial \hbar}\right)$$

- simple proof from Nekrasov $\mathcal{F}$ and Matone relation

$$u \sim \Lambda\frac{\partial \mathcal{F}}{\partial \Lambda} \quad \Rightarrow \quad \frac{\partial u}{\partial a} \sim \Lambda\frac{\partial}{\partial \Lambda}\frac{\partial \mathcal{F}}{\partial a} = \Lambda\frac{\partial a_D}{\partial \Lambda}$$

- quantum geometry: $a(u,\hbar)$ and $a_D(u,\hbar)$ related

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis, via trans-series
- trans-series 'encode' analytic continuation information
- expansions about different saddles are intimately related
- there is extra un-tapped 'magic' in perturbation theory

- matrix models, large N, strings, SUSY QFT
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ SUSY gauge theory

- fundamental & generic property of steepest descents
- moral: go complex and consider all saddles, not just minima

A Few References: books

- J.C. Le Guillou and J. Zinn-Justin (Eds.), *Large-Order Behaviour of Perturbation Theory*
- C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*
- R. B. Dingle, *Asymptotic expansions: their derivation and interpretation*
- O. Costin, *Asymptotics and Borel Summability*
- R. B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*
- E. Delabaere, "Introduction to the Ecalle theory", In *Computer Algebra and Differential Equations* **193**, 59 (1994), London Math. Soc. Lecture Note Series
- M. Mariño, *Instantons and Large N : An Introduction to Non-Perturbative Methods in Quantum Field Theory*, (Cambridge University Press, 2015).

A Few References: papers

- C.M. Bender and T.T. Wu, "Anharmonic oscillator", Phys. Rev. **184**, 1231 (1969); "Large-order behavior of perturbation theory", Phys. Rev. Lett. **27**, 461 (1971).
- E. B. Bogomolnyi, "Calculation of instanton–anti-instanton contributions in quantum mechanics", Phys. Lett. B **91**, 431 (1980).
- M. V. Berry and C. J. Howls, "Hyperasymptotics for integrals with saddles", Proc. R. Soc. A **434**, 657 (1991)
- J. Zinn-Justin & U. D. Jentschura, "Multi-instantons and exact results I: Conjectures, WKB expansions, and instanton interactions," Annals Phys. **313**, 197 (2004), quant-ph/0501136, "Multi-instantons and exact results II" Annals Phys. **313**, 269 (2004), quant-ph/0501137
- E. Delabaere and F. Pham, "Resurgent methods in semi-classical asymptotics", Ann. Inst. H. Poincaré **71**, 1 (1999)
- E. Witten, "Analytic Continuation Of Chern-Simons Theory," arXiv:1001.2933

A Few References: papers

- M. Mariño, R. Schiappa and M. Weiss, "Nonperturbative Effects and the Large-Order Behavior of Matrix Models and Topological Strings," Commun. Num. Theor. Phys. **2**, 349 (2008) arXiv:0711.1954
- S. Pasquetti and R. Schiappa, "Borel and Stokes Nonperturbative Phenomena in Topological String Theory and c=1 Matrix Models," Annales Henri Poincaré **11**, 351 (2010) arXiv:0907.4082
- I. Aniceto, R. Schiappa and M. Vonk, "The Resurgence of Instantons in String Theory," Commun. Num. Theor. Phys. **6**, 339 (2012), arXiv:1106.5922
- M. Mariño, "Lectures on non-perturbative effects in large N gauge theories, matrix models and strings," arXiv:1206.6272
- I. Aniceto and R. Schiappa, "Nonperturbative Ambiguities and the Reality of Resurgent Transseries", Commun. Math. Phys. **335**, 183 (2015), arXiv:1308.1115
- I. Aniceto, J. G. Russo and R. Schiappa, "Resurgent Analysis of Localizable Observables in Supersymmetric Gauge Theories", arXiv:1410.5834
- D. Dorigoni, "An Introduction to Resurgence, Trans-Series and Alien Calculus," arXiv:1411.3585.

A Few References: papers

- G. V. Dunne & M. Unsal, "New Methods in QFT and QCD: From Large-N Orbifold Equivalence to Bions and Resurgence," Annual Rev. Nucl. Part. Science 2016, arXiv:1601.03414
- A. Behtash, G. V. Dunne, T. Schaefer, T. Sulejmanpasic & M. Unsal, "Toward Picard-Lefschetz Theory of Path Integrals, Complex Saddles and Resurgence," Annals of Mathematical Sciences and Applications 2016, arXiv:1510.03435
- G. V. Dunne & M. Ünsal, "Resurgence and Trans-series in Quantum Field Theory: The CP(N-1) Model," JHEP **1211**, 170 (2012), and arXiv:1210.2423
- G. V. Dunne & M. Ünsal, "Generating Non-perturbative Physics from Perturbation Theory," arXiv:1306.4405; "Uniform WKB, Multi-instantons, and Resurgent Trans-Series," arXiv:1401.5202.
- G. Basar, G. V. Dunne and M. Unsal, "Resurgence theory, ghost-instantons, and analytic continuation of path integrals," JHEP **1310**, 041 (2013), arXiv:1308.1108
- G. Basar & G. V. Dunne, "Resurgence and the Nekrasov-Shatashvili Limit: Connecting Weak and Strong Coupling in the Mathieu and Lamé Systems", JHEP **1502**, 160 (2015), arXiv:1501.05671.