Parma 10-12 Sept. 2003

Lectures on Supersymmetric Gauge Theories II:

Non-Abelian Superconductors and Confinement

K. Konishi

- **Confinement in QCD**
- **Seiberg-Witten solutions for $\mathcal{N} = 2$ susy gauge theories**
- **Non-Abelian Superconductor: Monopoles, vortices and Confinement**

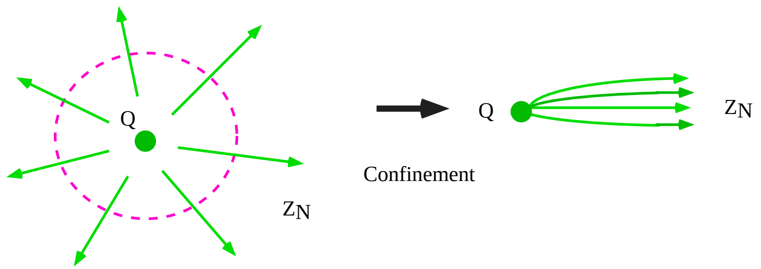
Work with ('98 - '03)

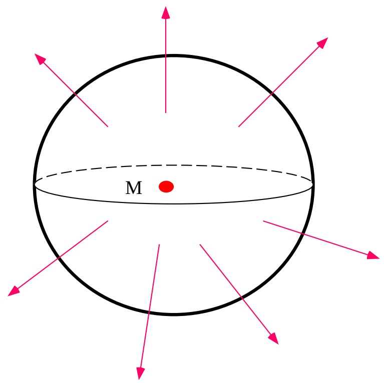
H. Murayama, S. P. Kumar, G. Carlino,

L. Spanu, S. Bolognesi, K. Takenaga,

H. Terao, R. Auzzi, R. Grena, A. Yung

$SU(N)$ YM





$\Pi_1(SU(N)/Z_N) \sim Z_N$

$\Longrightarrow$ **$(Z_N^{(M)}, Z_N^{(E)})$ classification of phases ('t Hooft)**

$SU(N)$ **YM and Solvable Cousins**

- $(Z_N^{(M)}, Z_N^{(E)})$ classification ('t Hooft):

  If field with $x = (a, b)$ condense, particles $X = (A, B)$ with

$$\langle x, X \rangle \equiv a\,B - b\,A \neq 0 \quad (mod\ N)$$

  are confined. (e.g. $\langle \phi_{(0,1)} \rangle \neq 0 \to$ Higgs phase.)

- **Quarks are confined if some field χ exist, s.t.**

$$\langle \chi_{(1,0)} \rangle \neq 0$$

- Softly Broken $N = 4$ (to $N = 1$): all different types of massive vacua, related by $SL(2, Z)$, chiral condensates known

Donagi-Witten, Strassler, Dorey-Kummer

- Softly Broken $N = 2$ Gauge Theories:

  Dynamics particularly transparent

What is χ ? How do they interact ?

XSB**?** θ**;** $\frac{\epsilon'}{\epsilon}$**;** $\Delta I = \frac{1}{2}$ **?**

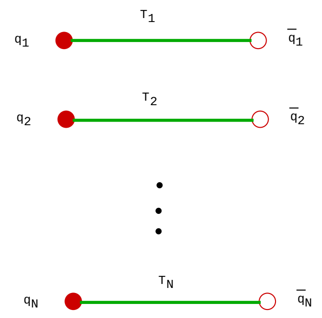
QCD as Dual Superconductor

- $\nexists$ Elementary/soliton monopoles
- Monopoles as topological singularities (lines in $4D$) of Abelian gauge fixing, $SU(3) \to U(1)^2$ ('t Hooft)
- $SU(2) \quad : \quad A_\mu^a = \tilde{\sigma}(x)(\partial_\mu \mathbf{n} \times \mathbf{n})^a + \dots, \qquad \mathbf{n}(\mathbf{r}) = \frac{\mathbf{r}}{r}$

  $\Rightarrow \quad A_i^a = \epsilon_{aij} \frac{r^j}{r^3}$ (Wu-Yang, Cho, Faddeev-Niemi)
- Some evidence in lattice QCD (Di Giacomo, et. al.)
- **Do (Abelian) monopoles carry flavor? ($\mathcal{L}_{eff}$?)**
- **Gauge dependence?**
- **Dynamical $SU(N) \to U(1)^{N-1}$ Breaking? Would imply a richer spectrum of mesons ($T_1 \neq T_2$, etc.)**
- **In Nature and in QCD:**

$$\text{Meson} \sim \sum_{i=1}^{N} |\, q_i \, \bar{q}_i \,\rangle$$

i.e., 1 state vs $\left[\frac{N}{2}\right]$ states ($SU(N) \to U(1)^{N-1} \times Weyl$ not enough).



Dirac's monopoles

- **QED admits pointlike magnetic monopoles if (Dirac)**

$$g\, e = \frac{n}{2}, \qquad n \in \mathbb{Z}, \tag{1}$$

$$\oint_{\partial\Omega} A_i\, dx^i \to \int_{S^2} d\mathbf{S} \cdot \mathbf{H} = 4\pi g, \qquad \mathbf{H} = \nabla \frac{g}{r}.$$

If A **regular then LHS** $\to 0$. **(!?!) Either Dirac string along** $(0,0,0) \to (0,0,-\infty)$ **(invisible if (1) satisfied), or**

- **Cover** S^2 **by two regions** $a : (0 \le \theta < \frac{\pi}{2} + \epsilon)$ **and** $b : (\frac{\pi}{2} - \epsilon < \theta \le \pi)$ **(Wu-Yang)**

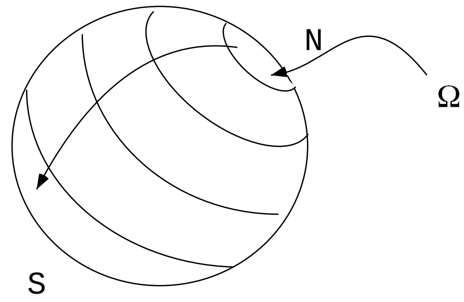
$$(A_\phi)^a = \frac{g}{r \sin\theta}(1 - \cos\theta), \qquad (A_\phi)^b = -\frac{g}{r\sin\theta}(1 + \cos\theta),$$

$$A_i^a = A_i^b - U^\dagger \frac{i}{e} \partial_i U, \qquad U = e^{2ige\phi}, \quad \text{OK if (1))}$$

- **More generally, for dyons** (e_1, g_1)**,** (e_2, g_2)**,**

$$e_1\, g_2 - e_2\, g_1 = \frac{n}{2}, \qquad n \in \mathbb{Z}, \tag{2}$$

- **Topology:** $\Pi_1(U(1)) = \mathbb{Z}$



Dirac monopoles in NA gauge theories

- **Use the $U(1)$ subgroup**
- **But**

$$SU(2) \sim S^3, \qquad \Pi_1(SU(2)) = \mathbf{1}, \qquad SO(3) \sim \frac{S^2}{Z_2}, \qquad \Pi_1(SO(3)) = \mathbb{Z}_2, \quad (3)$$

$\rightarrow$ **NO monopoles in $SU(2)$, $SU(N)$; one type of monopole in $SO(3)$, and so on.**

't Hooft-Polyakov

- $$SU(2) \stackrel{\langle\phi\rangle\neq 0}{\longrightarrow} U(1)$$
$$\mathcal{D}\phi \stackrel{r\to\infty}{\longrightarrow} 0, \quad \Rightarrow \quad \phi \sim U\cdot\langle\phi\rangle\cdot U^{-1};$$
$$A_i^a \sim U\cdot\partial_i U^\dagger \to \epsilon_{aij}\frac{r_j}{r^3}m, \qquad m=1,2,\ldots$$
- (ϕ, A_μ) **represents nontrivial elements of** $\Pi_2(SU(2)/U(1)) = \Pi_1(U(1)) = \mathbb{Z}$
- **Regular, finite energy configurations (*cfr* Dirac)**
- $$H = \int d^3x \left[\frac{1}{4}(F_{ij}^a)^2 + \frac{1}{2}(D_i\phi^a)^2 + \frac{\lambda}{2}(\phi^2 - v^2)^2\right]$$
$$= \int d^3x \left[\frac{1}{4}(F_{ij}^a - \epsilon_{ijk}D_k\phi^a)^2 + \frac{1}{2}F_{ij}^a\epsilon_{ijk}D_k\phi^a + \frac{\lambda}{2}(\phi^2 - v^2)^2\right]$$
- $\frac{1}{2}F_{ij}^a\epsilon_{ijk}D_k\phi^a = \partial_i S_i; \qquad S_i = \frac{1}{2}\epsilon_{ijk}F_{jk}^a\,\phi^a = B_i^a\phi^a$. **(Bogomolny equation)**
- $H \geq \int d^3x \nabla\cdot\mathbf{S} = \frac{4\pi v}{g}m, \quad m = 1,2,\ldots$**.**

- $\lambda = 0$ **(BPS):**

$$F^a_{ij} - \epsilon_{ijk} D_k \phi^a = 0, \quad \text{or} \quad B^a_k = D_k \phi^a, \qquad H = \frac{4\pi v}{g} m,$$

$$A^a_i = \epsilon_{aij} \frac{r_j}{r^3} A(r), \qquad \phi^a = \frac{r^a}{r} \phi(r),$$

$A(r),\ \phi(r)$ **known explicitly** **(**$A(r) \to -\frac{1}{r}, \quad \phi(r) \to v$**),**

Nonabelian monopoles

$$G \stackrel{\langle\phi\rangle\neq 0}{\longrightarrow} H$$

$$\mathcal{D}\phi \stackrel{r\to\infty}{\longrightarrow} 0, \quad \Rightarrow \quad \phi \sim U\cdot\langle\phi\rangle\cdot U^{-1} \sim \Pi_2(G/H) = \Pi_1(H);$$

$$t^a A_i^a \sim U\cdot\partial_i U^\dagger \Longrightarrow F_{ij} = \epsilon_{ijk}\frac{r_k}{r^3}\beta_\ell T_\ell, \qquad T_i \in \text{Cartan S.A. of} \quad H$$

Topological quantization $\Longrightarrow$ $2\,\alpha\cdot\beta \subset \mathbb{Z}$**:**

$$\beta_i = \text{weight vectors of } \tilde{H} = \text{dual of } H.$$

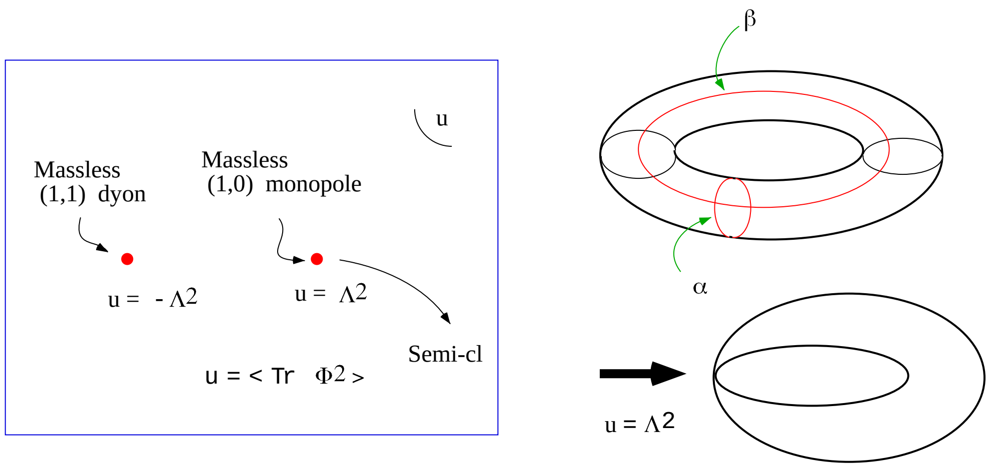
(Goddard-Nuyts-Olive, E. Weinberg)

$$\tilde{H} \Leftrightarrow H$$

$SU(N)/Z_N$	$\Leftrightarrow$	$SU(N)$
$SO(2N)$	$\Leftrightarrow$	$SO(2N)$
$SO(2N+1)$	$\Leftrightarrow$	$USp(2N)$

- **Dirac monopoles (Wu-Yang) for** $|\phi| \to \infty$**;**
- **'t Hooft-Polyakov monopoles for** $G = SU(2)$**,** $H = U(1)$

Seiberg-Witten Solution in $\mathcal{N} = 2$ Gauge Theories



- $SU(2)$: **Lagrangian is Eq.(1) of I with** $\mathcal{W}(\Phi) = 0$

$$\langle \Phi \rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix},$$

- $a \neq 0$ **breaks** $SU(2) \to U(1)$**: at IR,**

$$\mathcal{L}_{eff} = \mathrm{Im}\left[\int d^4\theta\, \bar{A}\, \frac{\partial \mathcal{F}_p(A)}{\partial A} + \int \frac{1}{2} \frac{\partial^2 \mathcal{F}_p(A)}{\partial A^2} W_\alpha W^\alpha \right]$$

where W_α, A **describe** $\mathcal{N} = 2$ $U(1)$ **theory**

- **Define** $A_D \equiv \frac{\partial \mathcal{F}_p(A)}{\partial A}$**: then**

$$\frac{d\,A_D}{d\,u} = \oint_\alpha \frac{dx}{y}, \qquad \frac{d\,A}{d\,u} = \oint_\beta \frac{dx}{y},$$

where ($u \equiv \mathrm{Tr}\,\langle \Phi^2 \rangle$ **describes QMS)**

$$y^2 = (x-u)(x+\Lambda^2)(x-\Lambda^2)$$

- **Exact mass formula (BPS):** $m_{n_m,n_e} = \sqrt{2}\,|\,n_m A_D + n_e A\,|$
- $\mu\,\Phi^2$ **perturbation (Confinement)**

$$\mathcal{W}_{eff} = \sqrt{2}\,A_D\,M\tilde{M} + \mu\,U(A_D) \Rightarrow \langle M \rangle \sim \sqrt{\mu\,\Lambda}$$

- **At the singularities** $u = \pm\Lambda^2$**, instanton sum diverges**

$$\langle \mathrm{Tr}\Phi^2 \rangle = \frac{a^2}{2} + \frac{\Lambda^4}{a^2} + \ldots = \ldots + 1 + 1 + 1 + \ldots$$

- **Dynamical Abelianization,** $SU(N) \to U(1)^{N-1}$ **(***cfr* **QCD)**
- **These "monopoles" are indeed 't Hooft-Polyakov monopoles**

$$\frac{2}{g} Q_e = n_e + \left[-\frac{4}{\pi} \mathrm{Arg}\, a + \frac{1}{2\pi} \sum_{f=1}^{N_f} \mathrm{Arg}\,(m_f^2 - 2a^2) \right] n_m + \ldots$$

- **Jackiw-Rebbi for $n_f = 1, 2, 3$;**

- **Quantum quenching of Quark Numbers** (**Carlino-Terao-Konishi**)

- **Susy breaking (e.g. $m_\lambda \, \lambda \, \lambda$) $\rightarrow$ nontrivial θ dependence**

More general $\mathcal{N} = 2$ models

$SU(n_c)$, $USp(2n_c)$ or $SO(n_c)$ with n_f Quarks

- $$\mathcal{L} = \frac{1}{8\pi} \mathrm{Im}\, \tau_{cl} \left[\int d^4\theta\, \Phi^\dagger e^V \Phi + \int d^2\theta\, \frac{1}{2} W W \right] + \mathcal{L}^{(quarks)} + \Delta\mathcal{L},$$

  where

  $$\Delta\mathcal{L} = \int d^2\theta\ \mu \,\mathrm{Tr}\, \Phi^2, \qquad \tau_{cl} \equiv \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}$$

- **($\mathcal{N}$ =2) gauge multiplet** $\Phi = \phi + \sqrt{2}\,\theta\,\psi + \ldots$; $W_\alpha = -i\lambda + \frac{i}{2}\,(\sigma^\mu\, \bar{\sigma}^\nu)_\alpha^\beta\, F_{\mu\nu}\,\theta_\beta + \ldots$ **both in the adjoint representation;**

- $$\mathcal{L}^{(quarks)} = \sum_i \Big[\int d^4\theta \, \{ Q_i^\dagger e^V Q_i + \tilde{Q}_i^\dagger e^{\tilde{V}} \tilde{Q}_i \} + \int d^2\theta \, \{ \sqrt{2} \tilde{Q}_i \Phi Q^i + m_i \tilde{Q}_i Q^i \} \Big]$$

- **Asymptotic Freedom** $\rightarrow$

  $$n_f \le 2n_c,\ 2n_c + 2,\ n_c - 2, \qquad \text{for} \quad SU(n_c),\ USp(2n_c),\ SO(n_c).$$

- **Global Symmetry ($m_i \to 0$):**

$$G_F = \begin{cases} U(n_f) \times Z_{2n_c - n_f} & SU(n_c); \\ SO(2n_f) \times Z_{2n_c + 2 - n_f} & USp(2n_c); \\ USp(2n_f) \times Z_{2n_c - 2n_f - 4} & SO(n_c) \end{cases}$$

Seiberg-Witten curves in general $\mathcal{N} = 2$ theories

$SU(n_c)$ **(**$USp(2n_c)$**)**

$$y^2 = \prod_{k=1}^{n_c}(x-\phi_k)^2 + 4\Lambda^{2n_c-n_f}\prod_{j=1}^{n_f}(x+m_j), \qquad SU(n_c), \;\; n_f \le 2n_c - 2,$$

and

$$y^2 = \prod_{k=1}^{n_c}(x-\phi_k)^2 + 4\Lambda\prod_{j=1}^{n_f}(x+m_j+\frac{\Lambda}{n_c}), \qquad SU(n_c), \;\; n_f = 2n_c - 1,$$

with $\sum_{k=1}^{n_c}\phi_k = 0$**,**

$USp(2n_c)$**:**

$$xy^2 = \left[x\prod_{a=1}^{n_c}(x-\phi_a^2) + 2\Lambda^{2n_c+2-n_f}m_1\cdots m_{n_f}\right]^2 - 4\Lambda^{2(2n_c+2-n_f)}\prod_{i=1}^{n_f}(x+m_i^2).$$

$SO(n_c)$**:**

$$y^2 = x \prod_{a=1}^{[n_c/2]} (x - \phi_a^2)^2 - 4\Lambda^{2(n_c-2-n_f)} x^{2+\epsilon} \prod_{i=1}^{n_f} (x - m_i^2),$$

$$y^2 = x \prod_{a=1}^{[n_c/2]} (x - \phi_a^2)^2 - 4\Lambda^{2(n_c-2-n_f)} x^{2+\epsilon} \prod_{i=1}^{n_f} (x - m_i^2), \qquad m_i = 0,$$

where $\epsilon = 1$ if n_c is even; $\epsilon = 0$ if n_c is odd.

Genus K ($n_c - 1, n_c, [n_c/2]$ for the above groups) hypertorus

$\mu\,\mathrm{Tr}\,\Phi^2$ perturbation and $\mathcal{N}=1$ vacua

The effective action near the

$$(n_{m1}, n_{m2}, \ldots, n_{mK}; n_{e1}, n_{e2}, \ldots, n_{eK}) = (1,0,\ldots,0;0,\ldots), \;\ldots,\; (0,0,\ldots,1;0,\ldots)$$

singularity is:

$$\mathcal{W} = \sum_{i=1}^{K} \tilde{M}_i \{\sqrt{2}\,a_{Di} + \sum_{k=1}^{n_f} S_k^i\, m_k\} M_i + \mu\, u_2(a_D, a)$$

The vacuum equations:

$$-\frac{\mu}{\sqrt{2}} = \sum_{i=1}^{K} \frac{\partial a_{Di}}{\partial u_2} \tilde{M}_i M_i\,; \qquad 0 = \sum_{i=1}^{K} \frac{\partial a_{Di}}{\partial u_j} \tilde{M}_i M_i, \quad j = 3,4,\ldots,K+1; \tag{4}$$

$$\begin{aligned}
(\sqrt{2}\,a_{D1} + \sum_{k=1}^{n_f} S_k^1\, m_k)\,\tilde{M}_1 &= (\sqrt{2}\,a_{D1} + \sum_{k=1}^{n_f} S_k^1\, m_k)\,M_1 = 0\,; \\
\ldots \qquad \ldots &= \ldots \qquad \ldots \\
(\sqrt{2}\,a_{DK} + \sum_{k=1}^{n_f} S_k^2\, m_k)\,\tilde{M}_K &= (\sqrt{2}\,a_{DK} + \sum_{k=1}^{n_f} S_k^2\, m_k)\,M_K = 0\,.
\end{aligned} \tag{5}$$

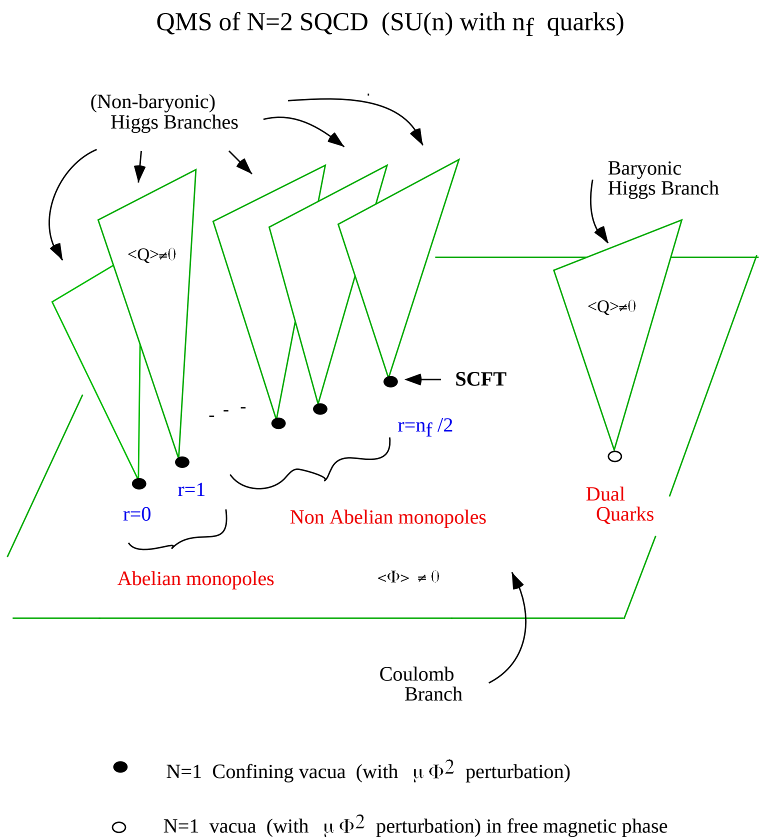
For generic m_i, Eqs.(4) $\to \tilde{M}_i \neq 0$; $M_i \neq 0$, $\forall i$ ($\frac{\partial a_{Di}}{\partial u_j}$ and $\frac{\partial a_{Di}}{\partial u_2}$ satisfy no special relations.) This means that

$$\sqrt{2}\, a_{Di} + \sum_{k=1}^{n_f} S_k^i\, m_k = 0 \qquad \forall i \tag{6}$$

i.e., **all K monopoles are simultaneously massless, and condense**. Confinement à la 't Hooft-Mandelstam, corresponding to **dual superconductor** in the maximal Abelian subgroup.

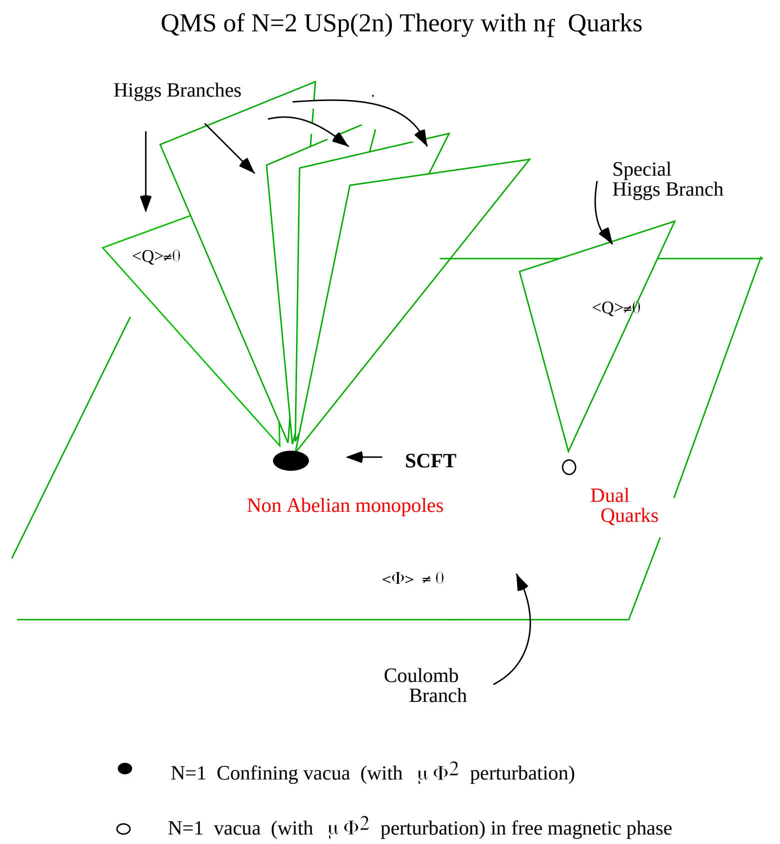
Actually, non-abelian dual superconductors in the $m_i \to 0$ limit

Vacua of general $\mathcal{N} = 2$ models



- $G_{eff} \sim SU(r) \times U(1)^{n_c - r - 1}$; n_f dual quarks* in $\underline{r}$
- Confining for $r \leq \frac{n_f}{2}$; SCFT at $r = \frac{n_f}{2}$ **

*) **What are they?** **) **What DoF?**



- All r vacua (at finite m) collapse into a single SCFT at $m \to 0$;
- All confining vacua (with $\mu\,\Phi^2$) are of this type;
- Global $SO(2\,n_f) \to U(n_f)$ Symmetry Breaking
  $(cfr\ \langle\bar{\psi}\,\psi\rangle^{(QCD)} \neq 0)$

label (r)	Deg.Freed.	Eff. Gauge Group	Phase	Global Symmetry
0 (NB)	monopoles	$U(1)^{n_c-1}$	Confinement	$U(n_f)$
1 (NB)	monopoles	$U(1)^{n_c-1}$	Confinement	$U(n_f-1)\times U(1)$
$2,..,[\frac{n_f-1}{2}]$ (NB)	dual quarks	$SU(r)\times U(1)^{n_c-r}$	Confinement	$U(n_f-r)\times U(r)$
$n_f/2$ (NB)	rel. nonloc.	-	Almost SCFT	$U(n_f/2)\times U(n_f/2)$
BR	dual quarks	$SU(\tilde{n}_c)\times U(1)^{n_c-\tilde{n}_c}$	Free Magnetic	$U(n_f)$

Table 1: Phases of $SU(n_c)$ gauge theory with n_f flavors. $\tilde{n}_c \equiv n_f - n_c$.

	Deg.Freed.	Eff. Gauge Group	Phase	Global Symmetry
1st Group	rel. nonloc.	-	Almost SCFT	$U(n_f)$
2nd Group	dual quarks	$USp(2\tilde{n}_c)\times U(1)^{n_c-\tilde{n}_c}$	Free Magnetic	$SO(2n_f)$

Table 2: Phases of $USp(2n_c)$ gauge theory with n_f flavors with $m_i \to 0$. $\tilde{n}_c \equiv n_f - n_c - 2$.

Q.N. of the NA Monopoles

$$SU(3) \xrightarrow{\langle\phi\rangle} \frac{SU(2)\times U(1)}{\mathbb{Z}_2}, \qquad \langle\phi\rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}$$

't Hooft-Polyakov solutions in $SU_U(2), SU_V(2) \subset SU(3)$

$\Rightarrow$ **two $SU(3)$ solutions* with** $\Pi_1(\frac{SU(2)\times U(1)}{\mathbb{Z}_2}) = \mathbb{Z}$

monopoles	$\tilde{SU}(2)$	$\tilde{U}(1)$
$\tilde{q}$	$\underline{2}$	1

$$SU(n) \xrightarrow{\langle\phi\rangle} SU(r)\times U^{n-r}(1), \qquad \langle\phi\rangle = \begin{pmatrix} v_1 \mathbf{1}_{r\times r} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & v_2 & 0 & \dots \\ \mathbf{0} & 0 & \ddots & \dots \\ \mathbf{0} & 0 & \dots & v_{n-r+1} \end{pmatrix}$$

- **Degenerate r-plet of monopole solutions** (q);**
- **The same charge structure in the r-vacua of $\mathcal{N}=2$ SQCD**

monopoles	$\tilde{SU}(r)$	$\tilde{U}_0(1)$	$\tilde{U}_1(1)$	$\tilde{U}_2(1)$	$\dots$	$\tilde{U}_{n-r-1}(1)$
q	$\underline{r}$	1	0	0	$\dots$	0
e_1	$\underline{1}$	0	1	0	$\dots$	0
e_2	$\underline{1}$	0	0	1	0	0
$\vdots$	$\underline{1}$	0	$\dots$			0
e_{n-r-1}	$\underline{1}$	0	0	$\dots$	$\dots$	1

- **Flavor q.n. of N.A. monopoles? $\Leftarrow$ Jackiw-Rebbi**

BPS monopoles

* $SU(3)$

$$SU(3)\xrightarrow{\langle\phi\rangle}SU(2)\times U(1), \qquad \langle\phi\rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}.$$

A broken $SU_U(2)$ subgroup $\rightarrow$

$$t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad t^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \quad \frac{t^3 + \sqrt{3}t^8}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

a solution

$$\phi(\mathbf{r}) \;=\; \begin{pmatrix} -\frac{1}{2}v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2}v \end{pmatrix} + 3\,v(t_4, t_5, \frac{t_3}{2} + \frac{\sqrt{3}t_8}{2}) \cdot \hat{r}\phi(r),$$

$$\vec{A}(\mathbf{r}) \;=\; (t_4, t_5, \frac{t_3}{2} + \frac{\sqrt{3}t_8}{2}) \wedge \hat{r}A(r),$$

where $\phi(r)$ and $A(r)$ are BPS- 't Hooft's functions with $\phi(\infty) = 1,\ \phi(0) = 0,\ A(\infty) = -1/r$.

A second solution with the same energy by using another $SU_V(2)$ group.

Nonabelian Monopoles Are Subtle

- $\nexists$ "Colored dyons" (?!) [1] (Abouelsaood, Coleman, E. Weinberg, Balachandran, ...)

  i.e. In the background of a non-Abelian monopole not possible to construct globally defined $T^1 - T^3$, isomorphic to unbroken $SU(2)$

- Monopoles are multiplets of the dual $\tilde{H}$ group, not of H. The no-go theorem $\rightarrow$

$$G_{gauge} \neq H \otimes \tilde{H}$$

- Not justified to study $G \stackrel{\langle\phi\rangle \neq 0}{\longrightarrow} H$ as a limit of max.ly broken cases;

- NA monopoles never really semi-classical, even if $\langle\phi\rangle \gg \Lambda_H$:

  - If H broken $\Rightarrow$ approximately degenerate set of monopoles *e.g.*, Pure $\mathcal{N} = 2$, $SU(3)$

  - If H unbroken $\Rightarrow$ N.A. monopoles in irreps of $\tilde{H}$. $\heartsuit$

- $\heartsuit$ **realized in the r vacua of $\mathcal{N} = 2$ SQCD with $SU(r) \times U(1)^{n_c - r + 1}$ gauge group.**

[1] No charge fractionalization (Goldstone-Wilczek, Niemi-Paranjape-Semenoff, Witten effect) for non-Abelian charges.

It occurs only for $r < \frac{n_f}{2}$ $\Leftarrow$ **Sign-flip of the beta function:**

$$b_0^{(dual)} \propto -2\,r + n_f > 0, \qquad b_0 \propto -2\,n_c + n_f < 0.$$

- When sign flip not possible (pure $\mathcal{N}=2$ YM, generic vacua of $\mathcal{N}=2$ theories) $\Rightarrow$ Dynamical Abelianization!

- **QM'ly, NA monopoles requires massless fermions**

Vortices

$\mathbb{Z}_N$ Vortices

Spanu, Konishi

- **Condensation of NA monopoles** $\Leftrightarrow$ **Confinement**
- $SU(N) \Rightarrow \mathbb{Z}_N$ **(in general,** $G \Rightarrow \mathcal{C}$**, a discrete center)**
- $\exists$ **Vortex if** $\Pi_1(G/\mathcal{C})$ **nontrivial, e.g.** $\Pi_1(SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N$

$$A_i \sim \frac{i}{g} U(\phi)\partial_i U^\dagger(\phi); \quad \phi_A \sim U \phi_A^{(0)} U^\dagger, \qquad U(\phi) = \exp i \sum_j^r \beta_j T_j \phi$$

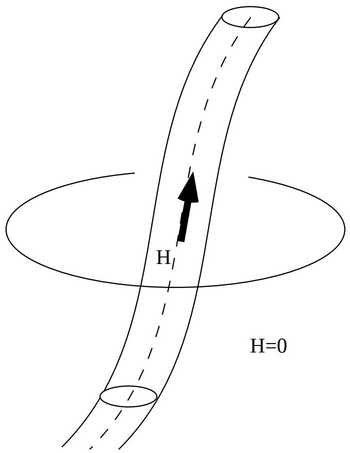
- **Quantization**

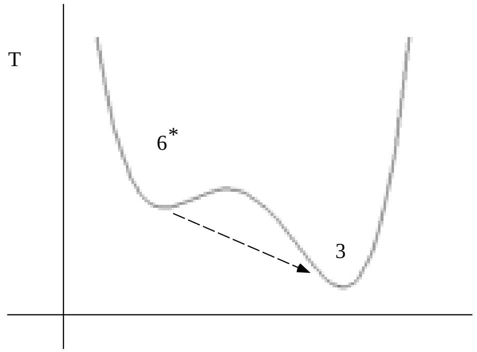
$$U(2\pi) \in \mathbb{Z}_N, \qquad \alpha \cdot \beta \in \mathbb{Z},$$

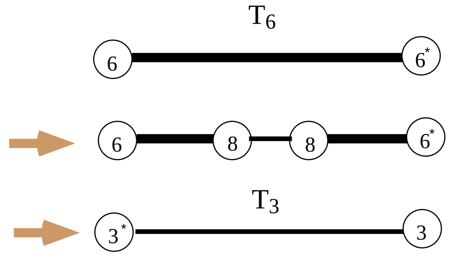
- **Solutions are irreps of** $\tilde{G} = SU(N)$: **carry** $\mathbb{Z}_N$ **(**N**-ality)**
- $\mathbb{Z}_N$ **vortices are non-BPS (***cfr.* **ANO)**
- **Tension ratios**[2]

$$T_k \propto \sin \frac{\pi k}{N} \qquad ? \qquad T_\ell + T_m < T_{\ell+m} \quad ?$$

[2] Douglas-Shenker, Hanany-Strassler-Zaffaroni, Herzog-Klebanov, Del Debbio-Panagopoulos-Rossi-Vicari, Lucini-Teper, Auzzi-Konishi


H
H=0


T
6*
3


T6
6
6*
6
8
8
6*
T3
3*
3

Non-Abelian Vortices Hanany-Tong, Auzzi-Boplognesi-Evslin-Konishi-Yung '03

$SU(3) \to SU(2) \times U(1)/\mathbb{Z}_2$:

$\mathcal{N} = 2$ **theory with** $4 \leq n_f \leq 5$ **with large bare mass** m **(with adj mass** $\mu\,\Phi^2$**):**

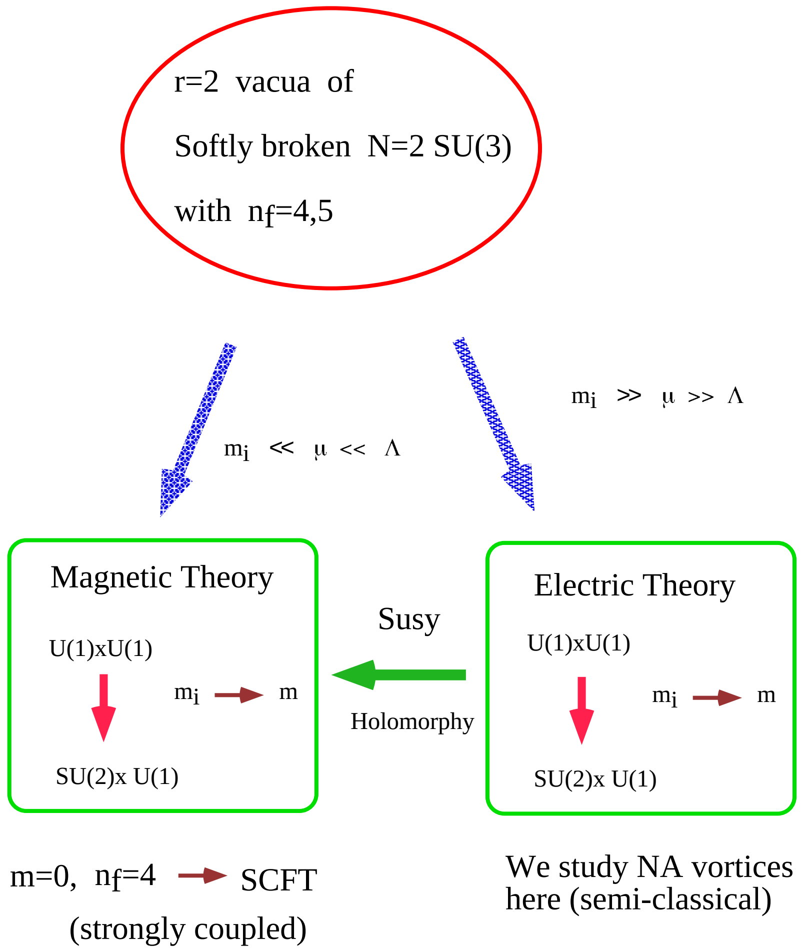
$$\Phi = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & -2m \end{pmatrix}, \qquad < q^{kA} >=< \bar{\tilde{q}}^{kA} >= \sqrt{\frac{\xi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $\xi = \sqrt{\mu\, m} \ll m$. **For vortex solution, set** $\Phi = \langle\Phi\rangle$; $q = \tilde{q}^\dagger$; **and** $q \to \frac{1}{2}q$**:**

$$S = \int d^4x \left[\frac{1}{4g_2^2} \left(F^a_{\mu\nu}\right)^2 + \frac{1}{4g_1^2} \left(F^8_{\mu\nu}\right)^2 + \left|\nabla_\mu q^A\right|^2 + \frac{g_2^2}{8} \left(\bar{q}_A \tau^a q^A\right)^2 + \frac{g_1^2}{24} \left(\bar{q}_A q^A - 2\xi\right)^2 \right],$$

$$\begin{aligned} T \;&=\; \int d^2x \,(\sum_{a=1}^{3} \left[\frac{1}{2g_2} F^{(a)}_{ij} \pm \frac{g_2}{4} (\bar{q}_A \tau^a q^A) \epsilon_{ij} \right]^2 + \left[\frac{1}{2g_1} F^{(8)}_{ij} \pm \frac{g_1}{4\sqrt{3}} \left(|q^A|^2 - 2\xi \right) \epsilon_{ij} \right]^2 \\ &+\; \frac{1}{2} \left| \nabla_i\, q^A \pm i\epsilon_{ij} \nabla_j\, q^A \right|^2 \pm \frac{\xi}{2\sqrt{3}} \tilde{F}^{(8)} \,) \end{aligned} \qquad (7)$$

Example of Non-Abelian Vortices in $\mathcal{N} = 2$ SQCD



Non-Abelian Bogomolny equations (Auzzi, Bolognesi, Evslin, Konishi, Yung, hep-th/0307287)

$$\frac{1}{2g_2} F_{ij}^{(a)} \pm \frac{g_2}{4} (\bar{q}_A \tau^a q^A) \epsilon_{ij} = 0, \qquad (a = 1,2,3); \qquad \frac{1}{2g_1} F_{ij}^{(8)} \pm \frac{g_1}{4\sqrt{3}} \left(|q^A|^2 - 2\xi \right) \epsilon_{ij} = 0,$$

$$\nabla_i \, q^A + i \varepsilon \epsilon_{ij} \nabla_j \, q^A = 0, \qquad A = 1, 2.$$

Abelian (particular) solutions of $SU(3) \to U(1) \times U(1)$ by *e.g.* setting $A_\mu^1 = A_\mu^2 = 0$, and with squark fields of the 2×2 color-flavor diag. form:

$$q^{kA}(x) = \bar{\tilde{q}}^{kA}(x) \neq 0, \qquad \text{only for} \quad k = A = 1, 2.$$

$$q^{kA}(x) = \begin{pmatrix} e^{i\,n\,\varphi} \phi_1(r) & 0 \\ 0 & e^{i\,k\,\varphi} \phi_2(r) \end{pmatrix},$$

$$A_i^3(x) = -\varepsilon \epsilon_{ij} \frac{x_j}{r^2} \; \left((n-k) - f_3(r) \right), \qquad A_i^8(x) = -\sqrt{3} \; \varepsilon \epsilon_{ij} \frac{x_j}{r^2} \; \left((n+k) - f_8(r) \right)$$

where

$$r \frac{\mathrm{d}}{\mathrm{d}r} \phi_1(r) - \frac{1}{2} \left(f_8(r) + f_3(r) \right) \phi_1(r) = 0, \qquad r \frac{\mathrm{d}}{\mathrm{d}r} \phi_2(r) - \frac{1}{2} \left(f_8(r) - f_3(r) \right) \phi_2(r) = 0,$$

$$-\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} f_8(r) + \frac{g_1^2}{6} \left(\phi_1(r)^2 + \phi_2(r)^2 - 2\xi \right) = 0, \qquad -\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} f_3(r) + \frac{g_2^2}{2} \left(\phi_1(r)^2 - \phi_2(r)^2 \right) = 0.$$

with boundary conds for the gauge fields:

$$f_3(0) = \varepsilon_{n,k}\,(n-k)\,, \quad f_8(0) = \varepsilon_{n,k}\,(n+k)\,, \qquad f_3(\infty) = 0, \quad f_8(\infty) = 0$$

and the requirement that the squark fields be everywhere regular. Also

$$\phi_1(\infty) = \sqrt{\xi}, \quad \phi_2(\infty) = \sqrt{\xi}$$

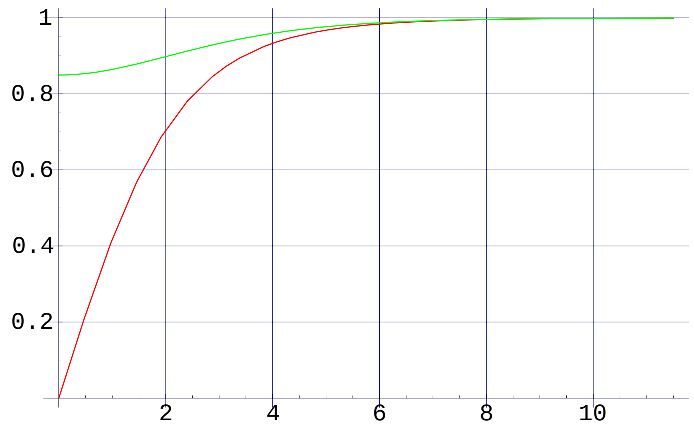


Figure 1: Vortex profile functions $\phi_1(r)$ and $\phi_2(r)$ of the $(1,0)$-string. Note $\phi_1(0) = 0$.

Figure 2: The profile functions $f_3(r)$ and $f_8(r)$ for the $(1,0)$-string.

Exact Symmetry

The $SU(3) \to SU(2) \times U(1) \to \emptyset$ theory has unbroken (both by inter. and by VEVS) global symmetry, $SU(2)_{C+F}$.

$SU(2)_{C+F}$ broken (to a $U(1)$) by a vortex configuration $\Rightarrow$ continuous vortex zero modes (moduli) of

$$SU(2)/U(1) = S^2 = \mathbf{CP}^1$$

Minimum vortex of generic orientation:

$$q^{kA} = U \begin{pmatrix} e^{i\varphi}\phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1} = e^{\frac{i}{2}\varphi(1+n^a\tau^a)}\, U \begin{pmatrix} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1},$$

$$\mathbf{A}_i(x) = U[-\frac{\tau^3}{2}\,\epsilon_{ij}\,\frac{x_j}{r^2}\,[1 - f_3(r)]]U^{-1} = -\frac{1}{2}\,n^a\tau^a\epsilon_{ij}\,\frac{x_j}{r^2}\,[1 - f_3(r)],$$

$$A_i^8(x) = -\sqrt{3}\,\epsilon_{ij}\,\frac{x_j}{r^2}\,[1 - f_8(r)]$$

where[3]

$$U \in SU(2)_{C+F}$$

The tension

$$T = 2\pi\xi$$

independent of U.

[3]Explicitly, if $n^a = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$, the rotation matrix is given by $U = \exp -i\beta\,\tau_3/2\,\exp -i\alpha\,\tau_2/2$.

Remarks

- **Reduction of the vortex spectrum (meson spectrum): (Fig)**

$$\Pi_1(\frac{U(1)\times U(1)}{\mathbb{Z}_2}) = \mathbb{Z}^2$$

  to

$$\Pi_1(\frac{SU(2)\times U(1)}{\mathbb{Z}_2}) = \mathbb{Z}$$

- Transition from abelian ($m_i \neq m_j$) to nonabelian ($m_i = m$) superconductivity **reliably** and **quantum mechanically** analysed

- (Indirect) solution for the "existence problem" of nonabelin **monopoles**

- Dynamics of vortex zero modes

$$\mathbf{n} \to \mathbf{n}(z,t)$$

$$S_\sigma^{(1+1)} = \beta \int d^2x \, \frac{1}{2} \, (\partial \, n^a)^2 + \text{fermions}.$$

  $O(3) = \mathbf{CP}^1$ sigma model! Dual (Shifman et.al.; Vafa-Hori) to a chiral theory with two vacua $\longrightarrow$ No spontaneous breaking of $SU(2)_{C+F} \Leftrightarrow$ confining, dual $SU(2)$ (Witten index = 2).

Reduction of the vortex spectrum (meson spectrum)

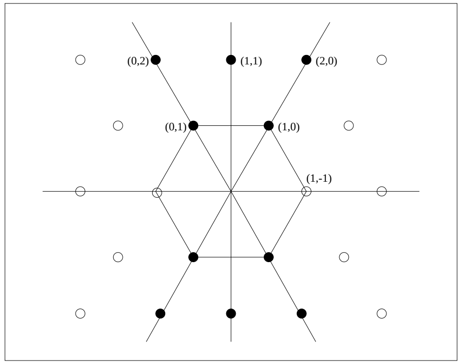


Figure 3: Lattice of (n, k) vortices in the theory $SU(3) \to U(1)^2$.

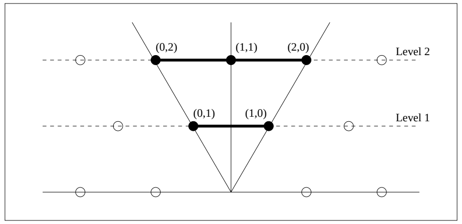


Figure 4: Reduced lattice of $\mathbb{Z}$ vortices $SU(3) \to SU(2) \times U(1)$.

Subtle are Non-Abelian Vortices (too)

Auzzi, Bolognesi, Evslin, Konishi, Yung; Hanany, Tong

- General setting: **Gauge** group broken as

$$G \stackrel{\langle\phi\rangle\neq 0}{\longrightarrow} H \stackrel{\langle\phi'\rangle\neq 0}{\longrightarrow} \emptyset, \qquad \langle\phi\rangle \gg \langle\phi'\rangle,$$

- An exact **global** symmetry $H_{C+F} \subset H \otimes G_F$ (not spontaneously broken), but broken by the vortex to G_0

  $\Rightarrow$ **Vortex zero modes (moduli)** $\sim H_{C+F}/G_0$

- $SU(N) \to \frac{SU(N-1)\times U(1)}{\mathbb{Z}_{N-1}} \to \emptyset$ system with $2N > N_f \geq 2(N-1)$

  $\Rightarrow$ Vortex with $2(N-2)$-parameter family of zeromodes

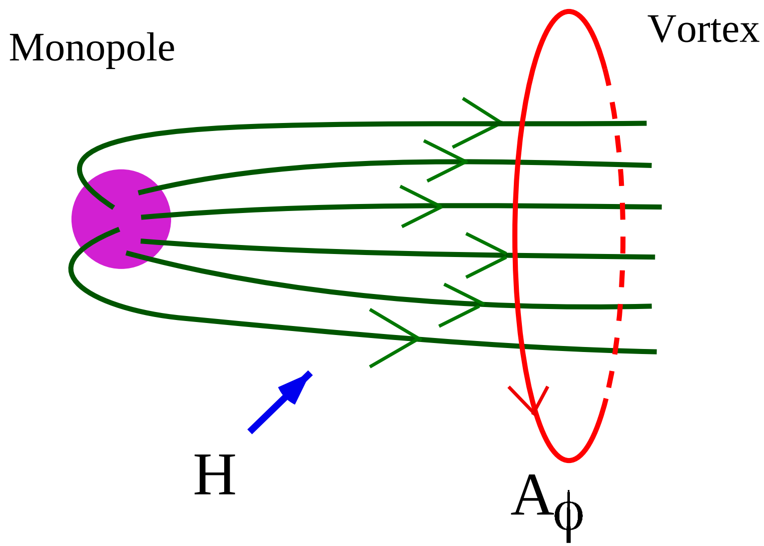
$$\frac{SU(N-1)}{SU(N-2)\times U(1)} \sim \mathbf{CP}^{N-2}.$$

- Vortices with non-Abelian quantum numbers

- Monopoles of G/H are confined by magnetic vortices of $H \to \emptyset$;

- **Both** described by $\Pi_1(H)$

- **Q-M'ly, non-Abelian vortices also requires massless quark flavors!**
- Monopoles can be attached at the ends of the vortex (Figure) ($\mathbb{Z}_{N-1}$ factor crucial)



$\mu \neq 0$

Almost Superconf. Confining Vacua

Auzzi, Grena, Konishi

<u>Sextet Vacua of $SU(3)$, $n_f = 4$ Model</u>

$$y^2 = \prod_{i=1}^{3}(x-\phi_i)^2 - \prod_{a=1}^{4}(x+m_a) \equiv (x^3 - Ux - V)^2 - \prod_{a=1}^{4}(x+m_a).$$

For *equal bare quark masses* ($m_a = m$), it simplifies:

$$y^2 = \prod_{i=1}^{3}(x-\phi_i)^2 - (x+m)^4 \equiv (x^3 - Ux - V)^2 - (x+m)^4.$$

The sextet vacua: $diag\,\phi = (-m, -m, 2m)$, i.e.,

$$U = \langle \mathrm{Tr}\,\Phi^2 \rangle = 3\,m^2; \qquad V = \langle \mathrm{Tr}\,\Phi^3 \rangle = 2\,m^3,$$

where the curve exhibits a singular behavior, $y^2 \propto (x+m)^4$ corresponding to the unbroken $SU(2)$.

Mass Formula

$$M_{(g_1,g_2;q_1,q_2)} = \sqrt{2}\,|g_1\,a_{D1} + g_2\,a_{D2} + q_1\,a_1 + q_2\,a_2|.$$

$$a_{D1} = \oint_{\alpha_1} \lambda, \qquad a_{D2} = \oint_{\alpha_2} \lambda, \qquad a_1 = \oint_{\beta_1} \lambda, \qquad a_2 = \oint_{\beta_2} \lambda,$$

where the (meromorphic) one-form λ is given by

$$\lambda = \frac{x}{2\pi} d \, \log \frac{\prod (x - \phi_i) - y}{\prod (x - \phi_i) + y}.$$

Expansion near the SCFT Point

$$U = 3m^2 + u, \qquad V = 2m^3 + v,$$

The discriminant of the curve factorizes as

$$\Delta = \Delta_s \, \Delta_+ \, \Delta_-, \qquad \Delta_s = (m \, u - v)^4$$

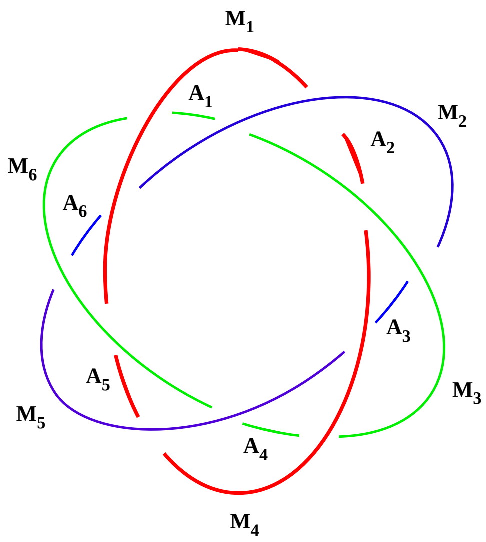
and the loci of $\Delta = 0$ are

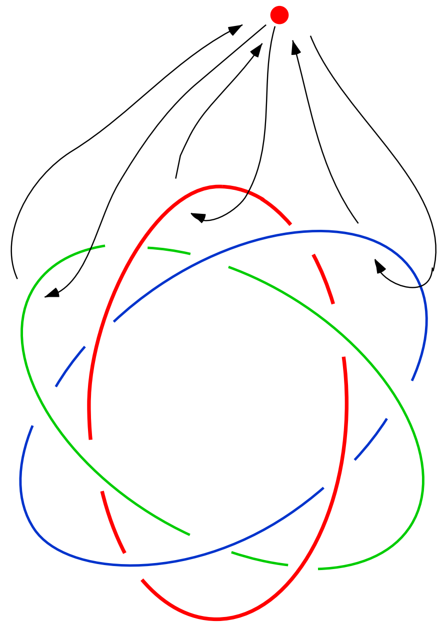
$$v = m \, u, \qquad v = m \, u + \frac{u^2}{4}, \qquad v = m \, u - \frac{u^2}{4}.$$

By rescaling $\quad u = m \, \tilde{u}, \; v = m^2 \, \tilde{v}$, intersecting them with a S^3

$$|\tilde{u}|^2 + |\tilde{v}|^2 = 1.$$

and making a stereographic projection from $S^3 \to R^3$





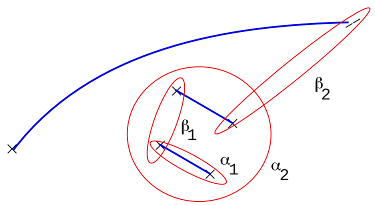
Monodromy and Charges

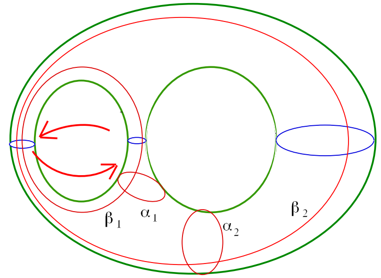
Monodromy around $M_1 \Rightarrow$

$$\alpha_1 \to \alpha_1, \qquad \beta_1 \to \beta_1 - 4\alpha_1, \qquad \alpha_2 \to \alpha_2, \qquad \beta_2 \to \beta_2.$$

The monodromy transformation:

$$\begin{pmatrix} a_{D1} \\ a_{D2} \\ a_1 \\ a_2 \end{pmatrix} \to M_1 \begin{pmatrix} a_{D1} \\ a_{D2} \\ a_1 \\ a_2 \end{pmatrix}, \qquad M_1 = \tilde{M}_1^4, \quad \tilde{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$





From

$$M = \begin{pmatrix} \mathbf{1} + \vec{q} \otimes \vec{g} & \vec{q} \otimes \vec{q} \\ -\vec{g} \otimes \vec{g} & \mathbf{1} - \vec{g} \otimes \vec{q} \end{pmatrix} \tag{9}$$

the (four) massless particles at the singularity $\tilde{v} = \tilde{u}$ have charges

$$(g_1, g_2; q_1, q_2) = (1, 0; 0, 0).$$

Analogously:

$$M_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_6 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}, \quad etc.$$

Conjugations:

$$M_1 = M_6^{-1} A_5 M_6, \quad A_2 = M_2^{-1} M_1 M_2, \quad M_4 = M_3^{-1} A_2 M_3, \quad A_5 = M_5^{-1} M_4 M_5,$$

$$M_2 = M_1^{-1} A_6 M_1, \quad A_3 = M_3^{-1} M_2 M_3, \quad M_5 = M_4^{-1} A_3 M_4, \quad A_6 = M_6^{-1} M_5 M_6,$$

$$M_3 = M_2^{-1} A_1 M_2, \quad A_4 = M_4^{-1} M_3 M_4, \quad M_6 = M_5^{-1} A_4 M_5, \quad A_1 = M_1^{-1} M_6 M_1$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}, \quad M_4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & 1 & 0 \\ 4 & -4 & 0 & 1 \end{pmatrix},$$

$$M_5 = \begin{pmatrix} -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & 3 & 0 \\ 4 & -4 & -2 & 1 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -16 & 4 & 5 & 0 \\ 4 & -1 & -1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 3 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & -1 & 0 \\ 4 & -4 & 2 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -3 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -16 & 12 & 5 & 0 \\ 12 & -9 & -3 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} -3 & 4 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & 5 & 0 \\ 4 & -4 & -4 & 1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

N.B.

$$M_1 = (\tilde{M}_1)^4, \quad M_4 = (\tilde{M}_4)^4, \quad A_2 = (\tilde{A}_2)^4, \quad A_5 = (\tilde{A}_5)^4,$$

with

$$\tilde{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{M}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix},$$

$$\tilde{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_5 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

$\Rightarrow$ **Charges**

$$M_1:\ (1,0;0,0)^4,\quad M_4:\ (-1,1;0,0)^4,\quad M_2:\ (-2,0;1,0),\quad M_5:\ (2,-2;-1,0),$$
$$A_2:\ (-1,0;1,0)^4,\quad A_5:\ (1,-1;-1,0)^4,\quad A_3:\ (-2,2;-1,0),\quad A_6:\ (2,0;1,0),$$
$$M_3:\ (0,1;-1,0),\quad M_6:\ (0,1;1,0),\quad A_4:\ (4,-3;-1,0),\quad A_1:\ (-4,1;1,0),$$

(A) **Which q.n. with respect to** $SU(2)\times U(1)$ **?**

(B) **Which of them are actually there at the SCFT Point as LEEDF ?**

(C) **How do they give** $\beta = 0$ **?**

(D) **How do they interact ?**

Ans. to (A):

$$\tilde{m}_1 = m_1;\quad \tilde{q}_1 = q_1 - \frac{1}{2}\,q_2;\qquad U_1(1)\subset SU(2);$$

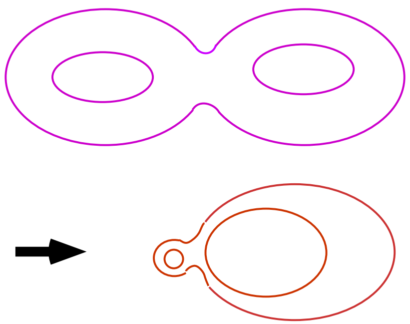
and

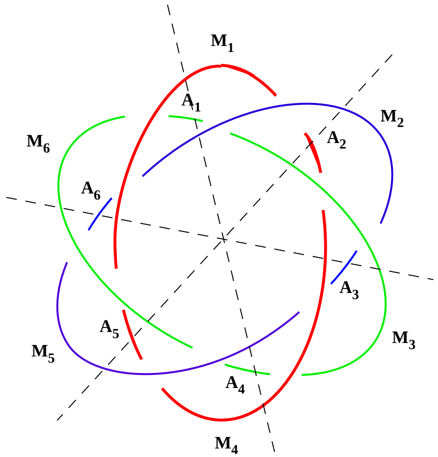
$$\tilde{m}_2 = m_1 + 2\,m_2;\quad \tilde{q}_2 = \frac{1}{2}\,q_2,\qquad U_2(1);$$

$\Rightarrow$

Matrix	Charge
M_1, M_4	$(\pm 1, 1, 0, 0)^4$
A_2, A_5	$(\pm 1, -1, \mp 1, 0)^4$
M_2, M_5	$(\pm 2, 2, \mp 1, 0)$
A_3, A_6	$(\pm 2, -2, \pm 1, 0)$
M_3, M_6	$(0, 2, \pm 1, 0)$
A_1, A_4	$(\pm 4, -2, \mp 1, 0)$

Superconformal Limit ($u = 0, v = 0$) :





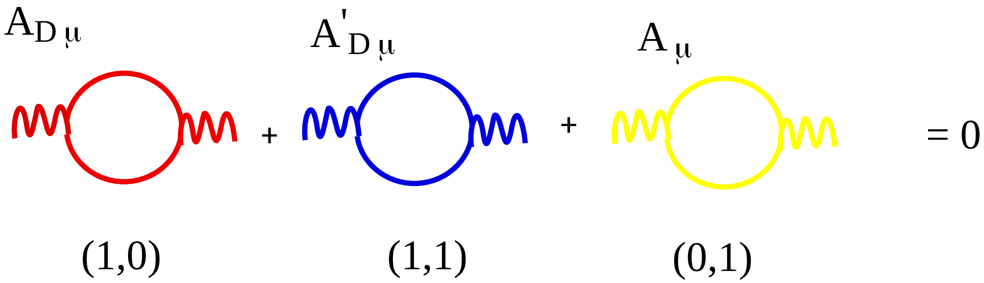
- Large torus: $\tau_{22} \to 1$ (Weakly interacting $U(1)$ theory);

- Small torus ($SU(2)$) : τ_{11} depends on the way u, $v \to 0$!
- τ_{11} depends only on ρ where $v = \epsilon^2$; $u = \epsilon\rho$
- At different phase of $\epsilon \Rightarrow$ different sections ($SU(2, Z)$-related descriptions of the same physics!) (**Ans. to (B)**)

$+2i$	$(0,1)$	$(4,-1)$	$(0,1)$	$(0,-1)$	$(-4,1)$	$(0,-1)$	$\ldots$
2	$(2,1)$	$(2,-1)$	$(2,-1)$	$(-2,-1)$	$(-2,1)$	$(-2,1)$	$\ldots$
$-2i$	$(0,-1)$	$(-4,1)$	$(0,-1)$	$(0,1)$	$(4,-1)$	$(0,1)$	$\ldots$
-2	$(-2,-1)$	$(-2,1)$	$(-2,1)$	$(2,1)$	$(2,-1)$	$(2,-1)$	$\ldots$
∞	$(\pm 1,0)^4$	$(\mp 1,0)^4$	$(\pm 1,\mp 1)^4$	$(\mp 1,0)^4$	$(\pm 1,0)^4$	$(\mp 1,\pm 1)^4$	$\ldots$

♡ **Define SCFT in the limit,** $\quad \epsilon \to 0, \quad \rho \to 0.$

Renormalization-Group Fixed Point



- Inversion formula

$$\frac{1}{2} = \frac{\theta_{00}^4(0,\tau_{11})}{\theta_{10}^4(0,\tau_{11})} \qquad \to \qquad \tau_{11} = \frac{\pm 1 + i}{2}, \quad \frac{\pm 3 + i}{10}, \quad \ldots$$

Other solutions by $SL(2,Z)$ transformations $\tau \to \tau + 2;\ \tau \to \frac{\tau}{1-2\tau}$

- **Cancellation of b_0 (Consider $U_1(1) \subset SU(2)$)**

i) $(\mp 1, 0)^4$ cancel the contr. of the gauge multiplets ;

ii) $(\pm 2, \pm 1)$ and $(0, \pm 1)$ cancel (*cfr*: Argyres-Douglas)

$$\sum_i (q_i + m_i \tau)^2 = 1 + (2\,\tau + 1)^2 = 0, \qquad \text{for} \quad \tau^* = \frac{-1+i}{2}$$

iii) In the second section $(\pm 4, \mp 1)$ and $(\pm 2, \mp 1)$ cancel for $\tau^* = \frac{3+i}{10}$!

iv) Different sections $\Rightarrow$ Different description of the Same physics

- **Low Energy Theory is an interacting SCFT with**

$SU(2) \times U(1)$ **Gauge Group and 4 magnetic monopole doublets, one dyon doublet and one electric doublet.** (**Ans. to (C)**)

Unequal masses: Six Colliding $\mathcal{N} = 1$ Local Vacua:

- **Each $\mathcal{N} = 1$ theory is a local $U(1)^2$ theory with $M_i, \tilde{M}_i$, ($i = 1, 2$) $\Rightarrow$ 12 hypermultiplets (as in the SCFT);**
- **Effect of $\mathcal{N} = 1$ perturbation $\mu \, \mathrm{Tr}\, \Phi^2$ in terms of an effective Lagrangian:**

$$\mathcal{P} = \sum_{i=1}^{2} \sqrt{2}\, A_{D_i}\, M_i \tilde{M}_i + \mu\, U(A_{D1}, A_{D2}) + \text{mass terms}$$

$\Rightarrow \langle M_i \rangle \neq 0$, $\langle \tilde{M}_i \rangle \neq 0$ **(Confinement);**

- **But in the $m_i \to m$ (SCFT Limit)**

$$\langle M_i \rangle \to 0, \quad \langle \tilde{M}_i \rangle \to 0,$$

!!?? Deconfinement? (cfr. Gorski, Yung, Vainshtein)

- **We do know (the large μ analysis, vacuum counting, and holomorphic dependence of physic on μ) that**

$$G_F = SU(4) \times U(1) \Rightarrow U(2) \times U(2)$$

- *Order parameter of the symmetry breaking?*

ANS: Condensation of $SU(2)$ doublets $\mathcal{M}^i_\alpha$, ($\alpha = 1, 2$, $i = 1, \ldots, 4$)

$$\langle \mathcal{M}^i_\alpha \, \mathcal{M}^j_\beta \rangle = \epsilon_{\alpha\beta} \, C^{ij} \neq 0, \quad \text{or} \quad \langle \mathcal{M}^i_\alpha \rangle = \delta^i_\alpha \, v \neq 0$$

due to the $SU(2)$ interactions. (Probable Ans. to (D))

Summarizing:

Softly broken $\mathcal{N}=2$, $SU(n_c)$ gauge theories with n_f quarks $\Rightarrow$ confining vacua with:

- Physics quite different for

  (i) $r=0,1 \Rightarrow$ Weakly coupled Abelian monopoles;

  (ii) $r<\frac{n_f}{2} \Rightarrow$ Weakly coupled non-Abelian monopoles;

  (iii) $r=\frac{n_f}{2} \Rightarrow$ Strongly coupled non-Abelian monopoles,

- Both at generic r - vacua and at the SCFT ($r=\frac{n_f}{2}$) vacua,

$$\langle \mathcal{M}_{\alpha}^{i} \rangle = \delta_{\alpha}^{i}\, v \neq 0, \qquad (\alpha=1,2,\ldots,r; \quad i=1,2,\ldots,n_f)$$

  ("Color-Flavor-Locking")

- Gauge invariant condensates are

$$\epsilon^{\alpha_1\alpha_2\ldots\alpha_r}\mathcal{M}_{\alpha_1}^{i_1}\mathcal{M}_{\alpha_2}^{i_2}\ldots\mathcal{M}_{\alpha_r}^{i_r} \sim U(1) \text{ monopole?}$$

QCD

- **No dynamical Abelianization**
- **QCD with n_f flavor ($\tilde{n}_c = 2, 3$, $n_f = 2, 3$)**

$$b_0 = 11\, n_c - 2\, n_f \quad \Rightarrow \quad \tilde{b}_0 = 11\, \tilde{n}_c - n_f$$

No sign flip (no weakly-coupled nonabelian monopoles)

- **Strongly-interacting nonabelian superconductor**
- **Hint from r-vacua and from the almost SCF vacua**

$$\langle \mathcal{M}^i_{L,\alpha} \rangle = \delta^i_\alpha\, v_R \neq 0, \quad \langle \mathcal{M}^\alpha_{R,i} \rangle = \delta^\alpha_i\, v_L \neq 0, \quad (\alpha = 1, 2, \ldots \tilde{n}_c; i = 1, 2, \ldots n_f)$$

- **A better picture might be**

$$\langle \mathcal{M}^i_{L,\alpha}\, \mathcal{M}^\alpha_{R,j} \rangle = \text{const.}\, \delta^i_j \neq 0;$$

for $\tilde{n}_c = 2,\ n_f = 2$

$$G_F = SU_L(2) \times SU_R(2) \Rightarrow SU_V(2)$$