Gravitational waves: theory and sources

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Lectures' content

Lecture 1:

– Einstein equations for weak gravitational fields

– Propagation of GWs: plane wave solution

– Interaction of GWs with free-falling particles

– local Lorentz gauge versus transverse traceless gauge

Lecture 2:

– GW energy-momentum (pseudo) tensor

– Quadrupolar wave generation in linearized Einstein theory

– Indirect detection of GWs: the Hulse-Taylor binary

Lectures' content

Lecture 3:

– **Angular distribution of GWs emitted by binaries**

– **Templates to be used in the search of GWs from compact binaries**

– **Extraction of cosmological parameters using binary black holes as standard candles**

– **How to test alternative theories of gravity using GWs**

On Thursday afternoon: Discussion/comments

Lectures' content

Lecture 4:

– GWs from black-hole's and neutron-star's ring down

– GWs from pulsars, supernovae, low-mass X-ray binaries

– Galactic binaries, supermassive black holes, extreme mass ratio binaries

Lecture 5 & 6:

– GWs from the early Universe: typical frequencies and amplitudes

– Amplification of quantum-vacuum fluctuations

– Stochastic GW background from standard inflationary models

– Examples of stochastic GW background from non-standard inflation

– GWs from first order phase transitions and cosmic strings

References

Landau-Lifshitz: *Teoria dei Campi*, **Chap. 11, 13**

B. Schutz: *A first course in general relativity*, **Chap. 8, 9**

S. Weinberg: *Gravitation and Cosmology*, **Chap. 7,10**

Misner-Thorne-Wheeler: *Gravitation*, **Chap. 8**

Course by Kip Thorne in 2002 at Caltech: Lectures 4, 5 & 6

Brief summary of Einstein equations

$$S = S_g + S_m$$

$$S_g = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R \qquad -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = T_{\mu\nu}$$

$\eta_{\mu\nu} = (-,+,+,+)$ **with** $\mu,\nu = 0,1,2,3$ **and** $i,j = 1,2,3$

by imposing the principle of minimal action

$$\int (G_{\mu\nu} - \frac{8\pi G}{c^4} T_{\mu\nu})\delta g^{\mu\nu}\sqrt{-g} = 0$$

$$\Rightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}\,T_{\mu\nu}$$

Brief summary of Einstein equations [continued]

$$R^{\nu}_{\mu\rho\sigma} = \frac{\partial \Gamma^{\nu}_{\mu\sigma}}{\partial x^{\rho}} - \frac{\partial \Gamma^{\nu}_{\mu\rho}}{\partial x^{\sigma}} + \Gamma^{\nu}_{\lambda\rho}\,\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\nu}_{\lambda\sigma}\,\Gamma^{\lambda}_{\mu\rho}$$

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} \left(\frac{\partial g_{\lambda\nu}}{\partial x^{\rho}} + \frac{\partial g_{\lambda\rho}}{\partial x^{\nu}} - \frac{\partial g_{\rho\nu}}{\partial x^{\lambda}} \right)$$

more explicitly:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \left(\frac{\partial^2 g_{\mu\sigma}}{\partial x^{\nu} \partial x^{\rho}} + \frac{\partial^2 g_{\nu\rho}}{\partial x^{\mu} \partial x^{\sigma}} - \frac{\partial^2 g_{\mu\rho}}{\partial x^{\nu} \partial x^{\sigma}} - \frac{\partial^2 g_{\nu\sigma}}{\partial x^{\mu} \partial x^{\rho}} \right)$$

$$+\frac{1}{2} g_{\lambda\alpha} \left(\Gamma^{\lambda}_{\nu\rho}\,\Gamma^{\alpha}_{\mu\sigma} - \Gamma^{\lambda}_{\nu\sigma}\,\Gamma^{\alpha}_{\mu\rho} \right)$$

Bianchi identity: $R^{\lambda}_{\mu\nu\rho;\sigma} + R^{\lambda}_{\mu\sigma\nu;\rho} + R^{\lambda}_{\mu\rho\sigma;\nu} = 0$

Brief summary of Einstein equations [continued]

Ricci tensor: $R_{\mu\nu} = g^{\rho\sigma}\, R_{\rho\mu\sigma\nu}$

Scalar tensor: $R = g^{\mu\nu} R_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- **Non-linear equations with well-posed initial value structure**

- $4 \times 4 = 16$ **differential equations, but** $G_{\mu\nu}$ **and** $T_{\mu\nu}$ **are symmetric tensors** $\Rightarrow 10$ **differential equations, but because of Bianchi identity** $G_{\mu\nu}{}^{;\nu} = 0 \Rightarrow 6$ **differential equations to be solved when** $T_{\mu\nu}$ **is given**

Einstein equations for weak gravitational fields in flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\mu\rho,\nu\sigma} - h_{\nu\sigma,\mu\rho}\right) + \mathcal{O}(|h|^2)$$

$$G_{\nu\sigma} = R_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}\,R =$$

$$\frac{1}{2}\left[h_{\mu\sigma,\nu}{}^{\mu} + h_{\mu\nu,\sigma}{}^{\mu} - h_{,\nu\sigma} - h_{\nu\sigma,\mu}{}^{\mu} - \eta_{\nu\sigma}h_{\mu\alpha,}{}^{\alpha\mu} + \eta_{\nu\sigma}h_{,\alpha}{}^{\alpha} + \mathcal{O}(|h|^2)\right]$$

Lorentz gauge can always be imposed ...

$$x'^{\mu} = x^{\mu} + \xi^{\mu}$$

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\, h$$

$$\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu} - \eta^{\mu\rho}\, \xi^{\nu}{}_{,\rho} - \eta^{\lambda\nu}\, \xi^{\mu}{}_{,\lambda} + \eta^{\mu\nu} \xi^{\rho}{}_{,\rho}$$

$$\bar{h}'^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \eta^{\lambda\nu}\, \xi^{\mu}{}_{,\lambda\nu}$$

$$\eta^{\lambda\nu}\, \xi^{\mu}{}_{,\lambda\nu} = \bar{h}^{\mu\nu}{}_{,\nu}$$

Imposing transverse-traceless gauge

We choose $\xi_\mu = B_\mu \, e^{ik_\alpha x^\alpha}$ with $k_\alpha k^\alpha = 0$ $(\eta^{\rho\nu}\partial_\rho\partial_\nu\xi_\mu = 0)$

$$\mathcal{A}'_{\mu\nu} = \mathcal{A}_{\mu\nu} - iB_\mu k_\nu - iB_\nu k_\mu + i\eta_{\mu\nu}B^\rho k_\rho$$

We impose:

1. $\mathcal{A}'_{\mu\nu}k^\nu = 0$
2. $\mathcal{A}'_{\mu\nu}\eta^{\mu\nu} = 0$
3. If U^ν is a constant timelike unit vector $(U_\nu U^\nu = -1)$ we impose $\mathcal{A}'_{\mu\nu}U^\mu = 0$

This set of equations determine B_μ

Linearly and circularly polarized gravitational waves

- **Linearly polarized GW:**

$$\mathbf{e}_+ = \mathbf{e}_x \wedge \mathbf{e}_x - \mathbf{e}_y \wedge \mathbf{e}_y \qquad \text{and} \qquad \mathbf{e}_\times = \mathbf{e}_x \wedge \mathbf{e}_y + \mathbf{e}_y \wedge \mathbf{e}_x$$

$$(\mathbf{u} \wedge \mathbf{v})(\lambda, \mathbf{q}) = (\lambda \cdot \mathbf{u})(\mathbf{q} \cdot \mathbf{v})$$

- **Circularly polarized GW:**

$$\mathbf{e}_R = \mathbf{e}_+ + i\mathbf{e}_\times \qquad \text{and} \qquad \mathbf{e}_L = \mathbf{e}_+ - i\mathbf{e}_\times$$

Equation of geodesic deviation

Pair of nearby freely-falling particles traveling on trajectories $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu$

$$0 = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau}$$

$$0 = \frac{d^2(x^\mu + \xi^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x + \xi) \frac{d(x^\mu + \xi^\mu)}{d\tau} \frac{d(x^\lambda + \xi^\lambda)}{d\tau}$$

Taking the difference and limiting to first order in ξ

$$\nabla_U \nabla_U \xi^\lambda = R^\lambda_{\nu\mu\rho} \xi^\mu U^\nu U^\rho \qquad U^\alpha = \frac{dx^\alpha}{d\tau}$$

From TT gauge to local Lorentz gauge: non-static tidal potential

If $\bar{x}^\mu$ and $\bar{g}^{\mu\nu}$ refer to TT gauge $(h^{\rm TT}_{xy} = 0, h^{\rm TT}_{xx} \equiv h^{\rm TT} \neq 0)$:

$$\bar{g}^{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{\rm TT} & 0 & 0 \\ 0 & 0 & -h^{\rm TT} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The local Lorentz metric $g_{\mu\nu}$ should reduce to Minkowski metric at the origin and all its first derivatives must vanish

$$\bar{t} = t - \dot{h}^{\rm TT}\,(x^2 - y^2)/4$$
$$\bar{x} = x - h^{\rm TT}\,x/2$$
$$\bar{y} = y + h^{\rm TT}\,y/2$$
$$\bar{z} = z + \dot{h}^{\rm TT}\,(x^2 - y^2)/4$$

$$g^{\mu\nu} = \eta_{\mu\nu} - 2\begin{pmatrix} \Phi(t) & 0 & 0 & \Phi(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Phi(t) & 0 & 0 & \Phi(t) \end{pmatrix}$$

$$\Phi(t) = -\frac{1}{4}\ddot{h}^{\rm TT}\,(x^2 - y^2) \Rightarrow \frac{d^2\xi^j}{dt^2} = -\frac{\partial^2\Phi}{\partial x^j \partial x^k}\,\xi^k$$

GWs interacting with free-falling particles

- If the two particles are originally separated along x

$$\frac{\partial^2 \xi^x}{\partial^2 t} = \frac{L}{2} \frac{\partial^2 h_{xx}^{\rm TT}}{\partial^2 t}$$

$$\frac{\partial^2 \xi^y}{\partial^2 t} = \frac{L}{2} \frac{\partial^2 h_{xy}^{\rm TT}}{\partial^2 t}$$

- If the two particles are originally separated along y

$$\frac{\partial^2 \xi^x}{\partial^2 t} = \frac{L}{2} \frac{\partial^2 h_{xy}^{\rm TT}}{\partial^2 t}$$

$$\frac{\partial^2 \xi^y}{\partial^2 t} = -\frac{L}{2} \frac{\partial^2 h_{xx}^{\rm TT}}{\partial^2 t}$$