On-Shell Methods in Field Theory

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Lecture I

Tools for Computing Amplitudes

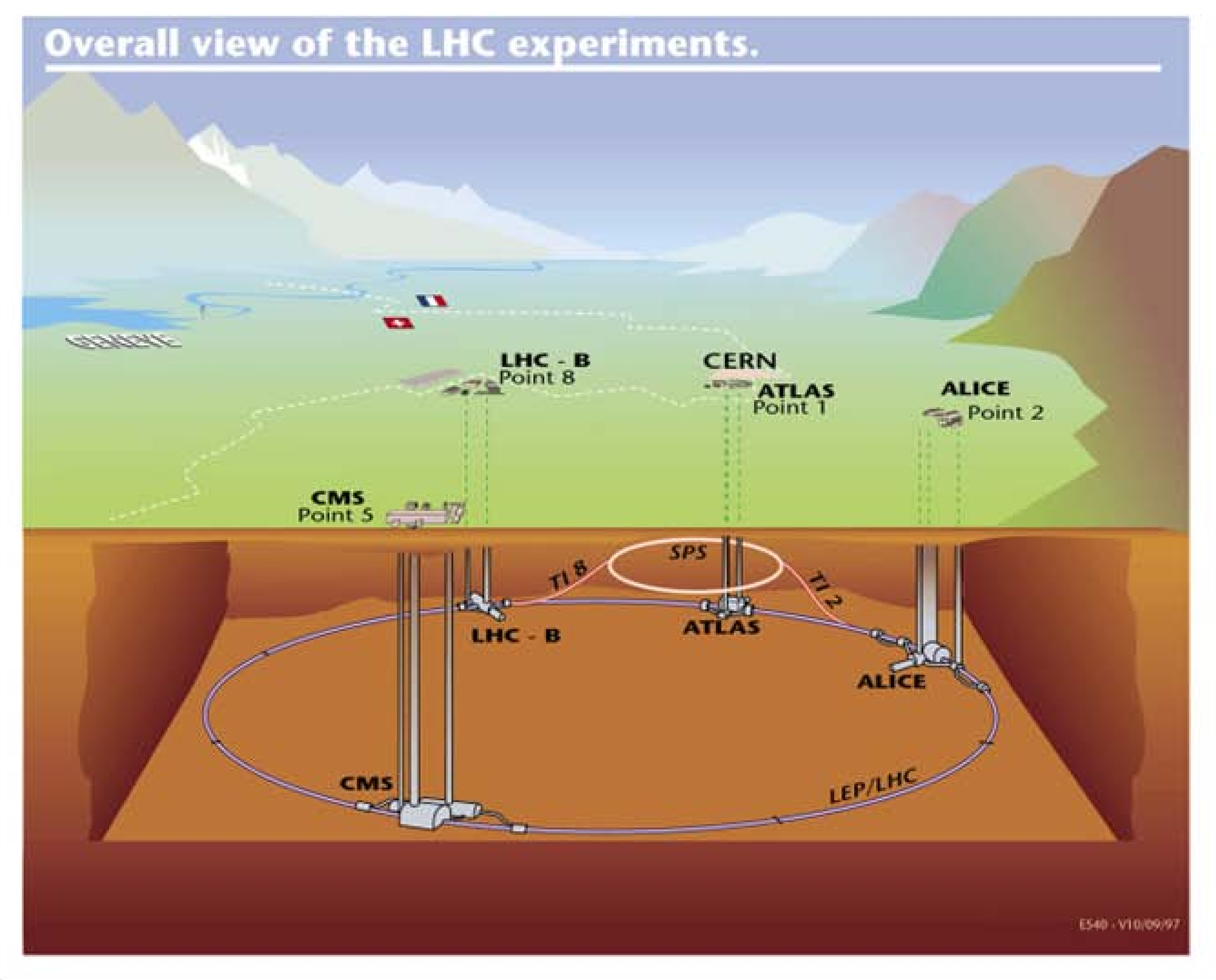
- Focus on gauge theories
  …but they are useful for gravity too

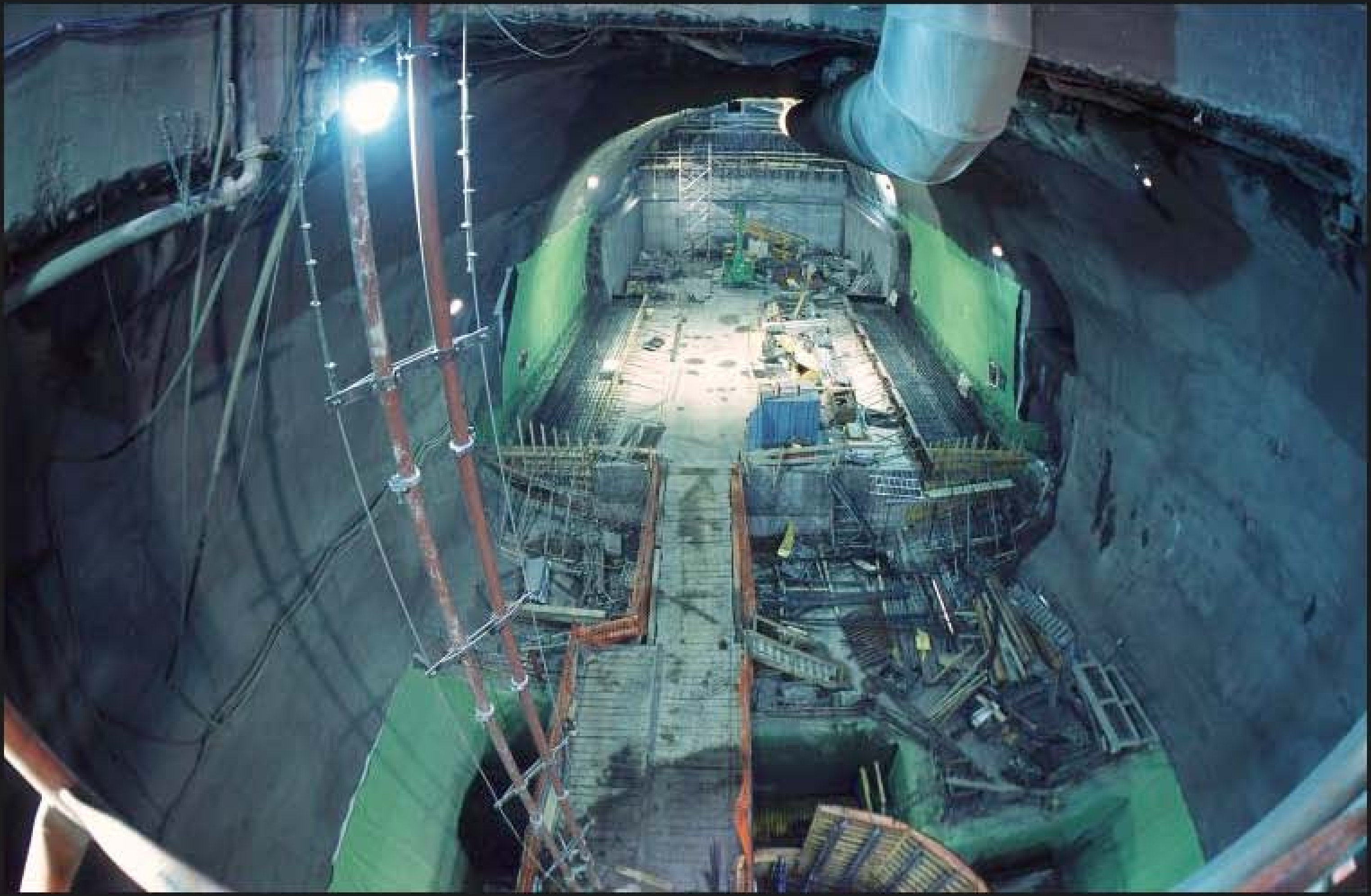
- Motivations and connections
  - Particle physics
  - $\mathcal{N}=4$ supersymmetric gauge theories and AdS/CFT
  - Witten's twistor string

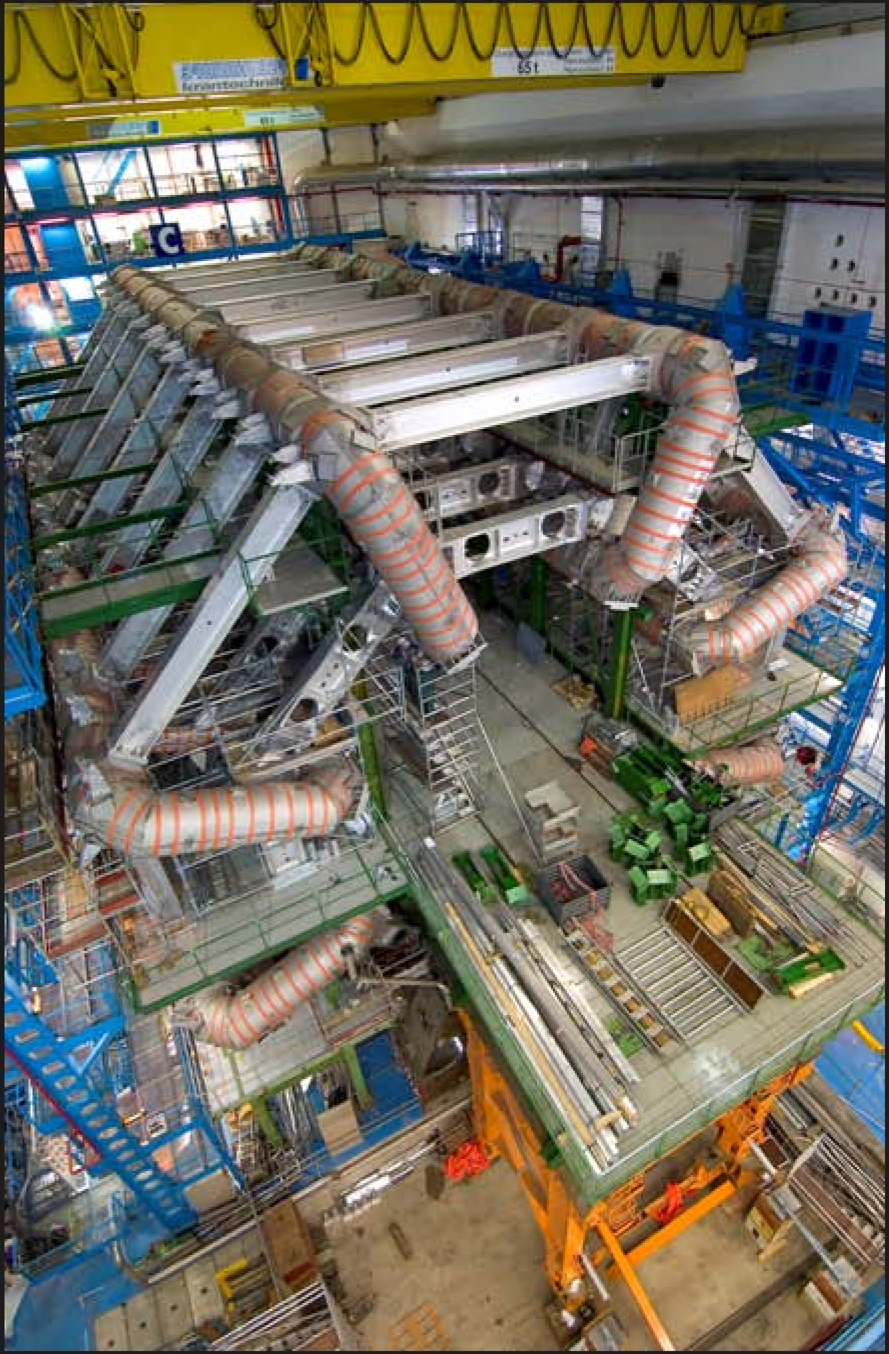
Particle Physics

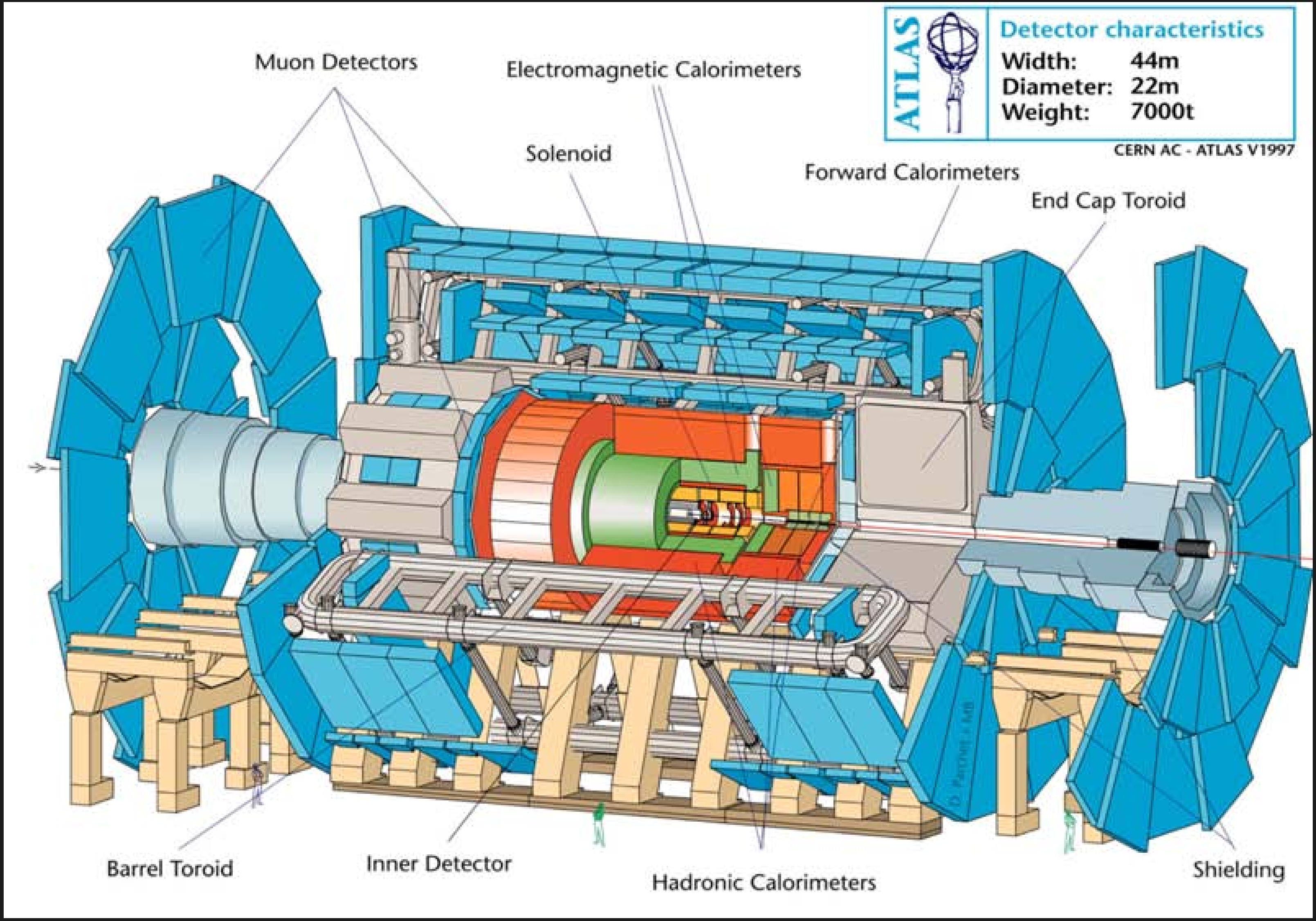
- Why do we
- Why do we
- What quan

Now 450 to 600 days away…

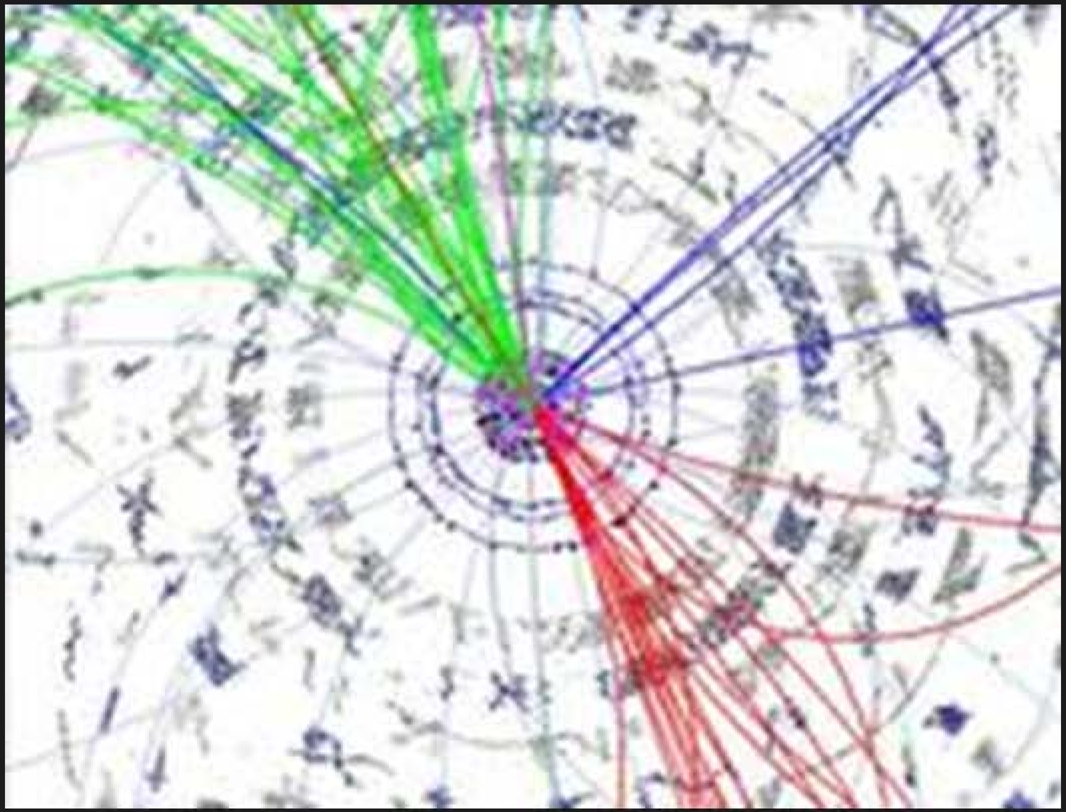

Overall view of the LHC experiments.
LHC - B
Point 8
CERN
ATLAS
Point 1
ALICE
Point 2
CMS
Point 5
SPS
TI 8
TI 2
LHC - B
ATLAS
ALICE
CMS
LEP/LHC
E540 - V10/09/97

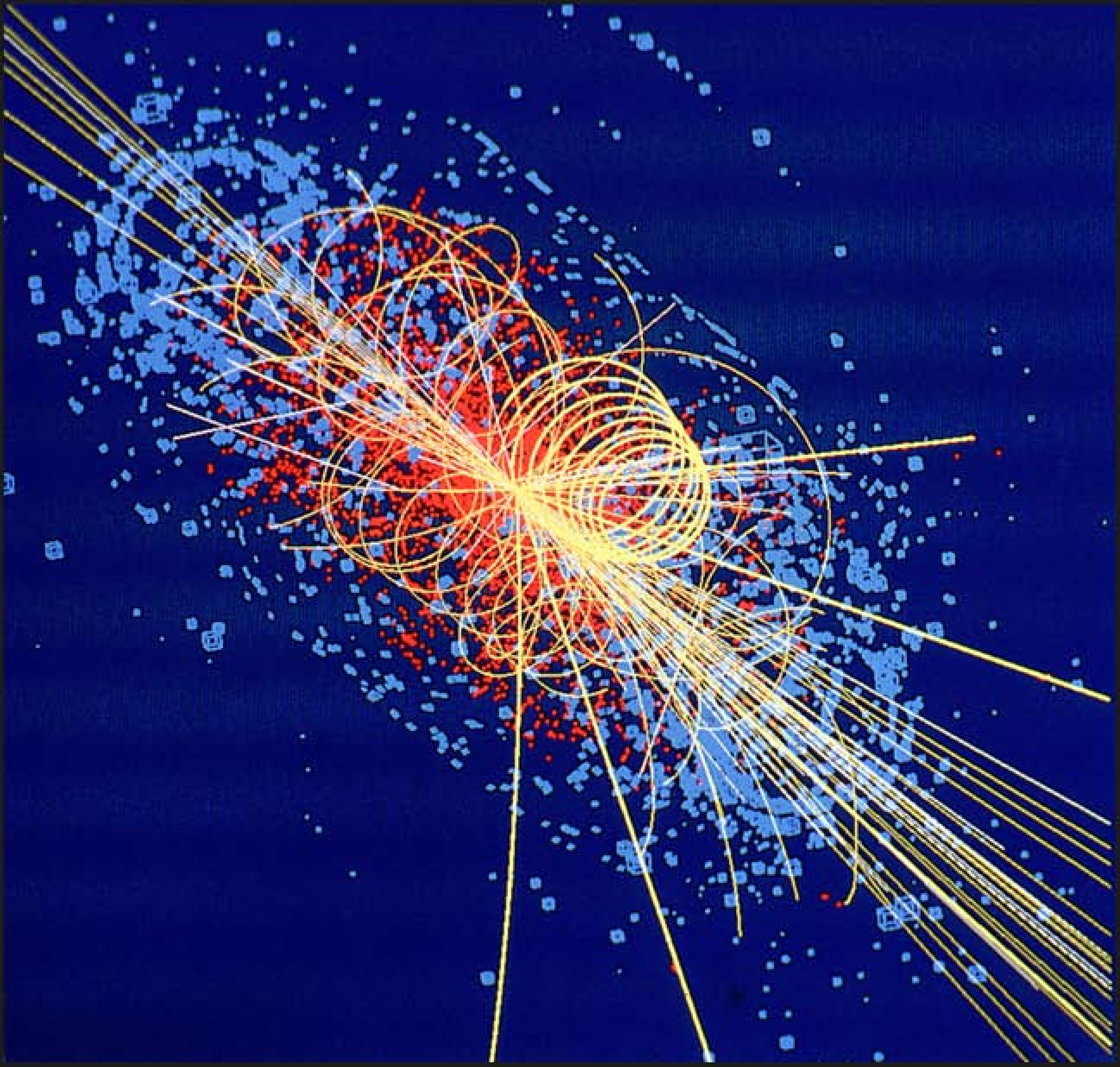



65t

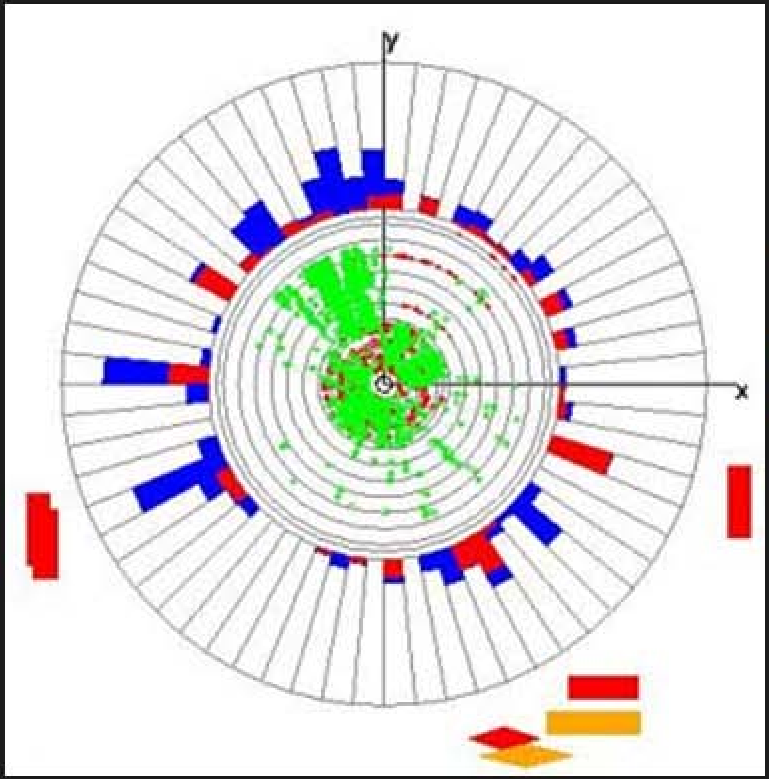

Detector characteristics
Width: 44m
Diameter: 22m
Weight: 7000t
ATLAS
CERN AC - ATLAS V1997
Muon Detectors
Electromagnetic Calorimeters
Solenoid
Forward Calorimeters
End Cap Toroid
Barrel Toroid
Inner Detector
Hadronic Calorimeters
Shielding

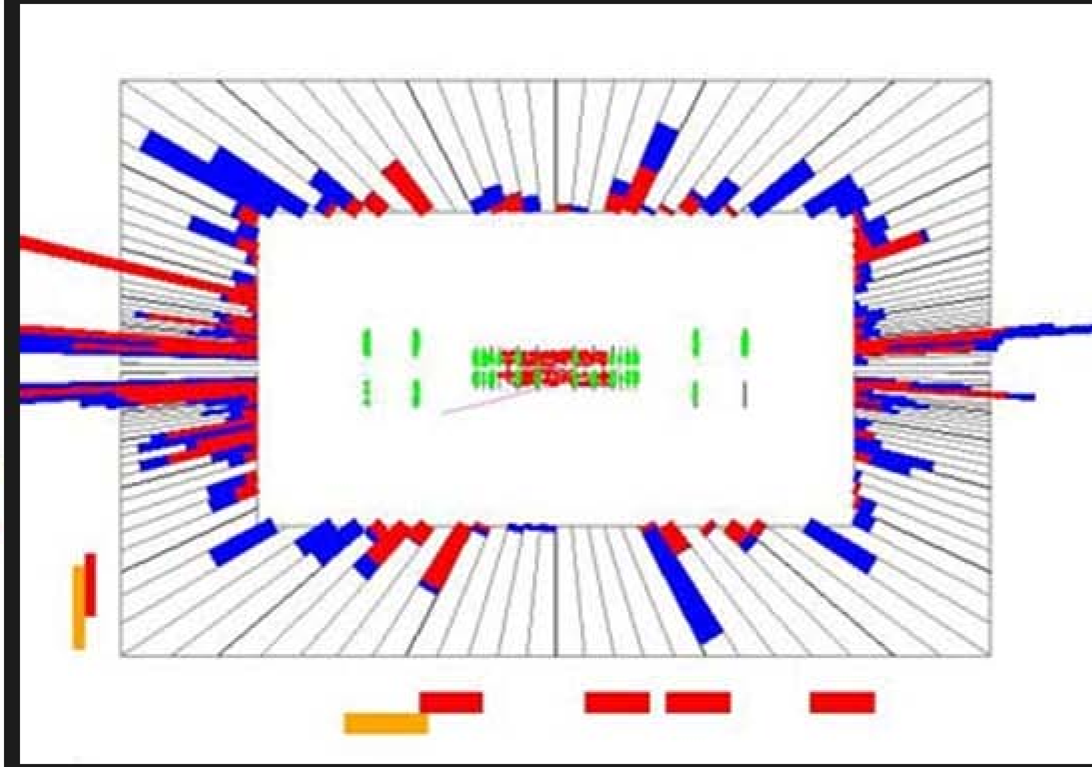
CDF event





CMS


y
x



$SU(3) \times SU(2) \times U(1)$ Standard Model

- Known physics, and background to new physics
- Hunting for new physics beyond the Standard Model
- Discovery of new physics
- Compare measurements to predictions — need to calculate signals
- Expect to confront backgrounds
- Backgrounds are large

Event rates

Event production rates at $L=10^{33}\ cm^{-2}\ s^{-1}$ and statistics to tape

Process	Events/s	Evts on tape, 10 fb^{-1}
$W \to e\nu$	15	10^8
$Z \to ee$	1	10^7
$t\bar{t}$	1	10^6
gluinos, m=1 TeV	0.001	10^3
Higgs, m=130 GeV	0.02	10^4
Minimum bias	10^8	10^7
$b\bar{b} \to \mu X$	10^3	10^7
QCD jets p_T>150 GeV/c	10^2	10^7

assuming 1% of trigger bandwidth (applies to the last three rows)

⇒ statistical error negligible after few days!

⇒ dominated by systematic errors (detector understanding, luminosity, theory)

Guenther Dissertori, ETH Zurich (KITP Collider Conf 1-12-04) Issues in

Hunting for New Physics

- Yesterday's new physics is tomorrow's background
- To measure new physics, need to understand backgrounds in detail
- Heavy particles decaying into SM or invisible states
  - Often high-multiplicity events
  - Low multiplicity signals overwhelmed by SM:

    Higgs $\longrightarrow b\bar{b} \longrightarrow$ 2 jets

- Predicting backgrounds requires precision calculations of known Standard Model physics

- Complexity is due to QCD

- Perturbative QCD:

  Gluons & quarks $\longrightarrow$ gluons & quarks

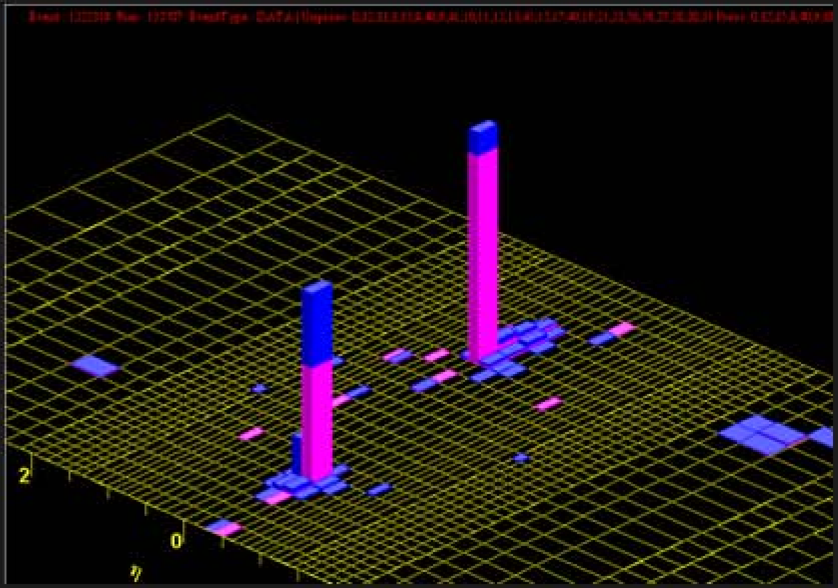
- Real world:

  Hadrons $\longrightarrow$ hadrons   with hard physics described by pQCD

- Hadrons $\longrightarrow$ jets   *narrow nearly collimated streams of hadrons*

Jets

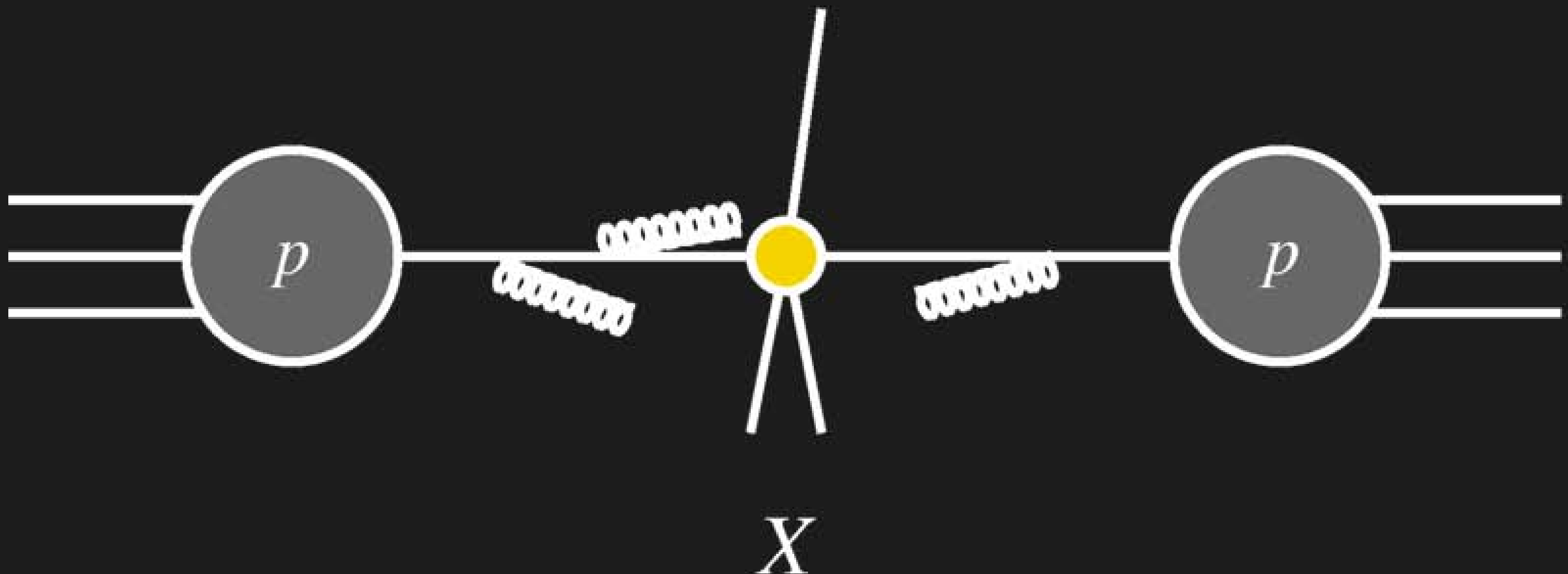
- Defined by an experimental resolution parameter
  - invariant mass in e^+e^-
  - cone algorithm in hadron colliders: cone size in $R = \sqrt{(\Delta\eta)^2 + (\Delta\phi)^2}$ and minimum E_T
  - k_T algorithm: essentially by a relative transverse momentum



CDF (Lefevre 2004)
1374 GeV

In theory, theory and practice are the same. In practice, they are different — Yogi Berra

QCD-Improved Parton Model



$$\int dx_a dx_b \, d\text{Phase} \; f_a f_b \, \sigma_{ab} \delta(v - \text{Observable})$$

The Challenge

- Everything at a hadron collider (signals, backgrounds, luminosity measurement) involves QCD
- Strong coupling is not small: $\alpha_s(M_Z) \approx 0.12$ and running is important
  - ⇒ events have high multiplicity of hard clusters (jets)
  - ⇒ each jet has a high multiplicity of hadrons
  - ⇒ higher-order perturbative corrections are important
- Processes can involve multiple scales: $p_T(W)$ & M_W
  - ⇒ need resummation of logarithms
- Confinement introduces further issues of mapping partons to hadrons, but for suitably-averaged quantities (infrared-safe) avoiding small E scales, this is not a problem (power corrections)

Approaches

- General parton-level fixed-order calculations
  - Numerical jet programs: general observables
  - Systematic to higher order/high multiplicity in perturbation theory
  - Parton-level, approximate jet algorithm; match detector events only statistically
- Parton showers
  - General observables
  - Leading- or next-to-leading logs only, approximate for higher order/high multiplicity
  - Can hadronize & look at detector response event-by-event
- Semi-analytic calculations/resummations
  - Specific observable, for high-value targets
  - Checks on general fixed-order calculations

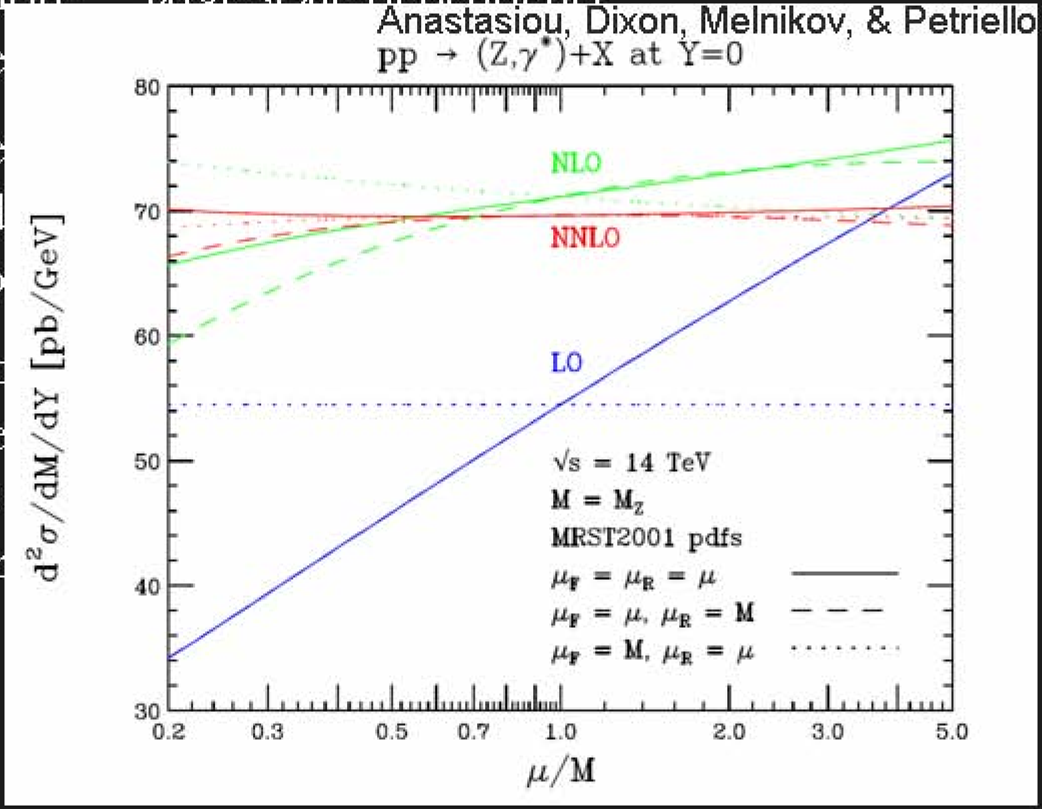
Precision Perturbative QCD

- Predictions of signals, signals+jets
- Predictions of backgrounds
- Measurement of luminosity
- Measurement of fundamental parameters (α_s, m_t)
- Measurement of electroweak parameters
- Extraction of parton distributions — ingredients in any theoretical prediction

Everything at a hadron collider involves QCD

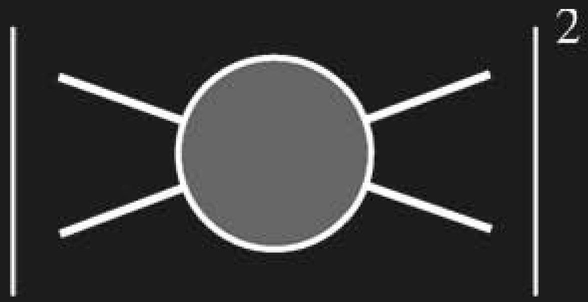
Leading-Order, Next-to-Leading Order

- LO: Basic shapes of distributions
  but: no quantitative prediction — large scale dependence
  missing sensitivity to je
- NLO: First quantitative pre
  improved scale depend
  basic approximation to
- NNLO: Precision predictic
  small scale dependence
  better correspondence
  understanding of theor

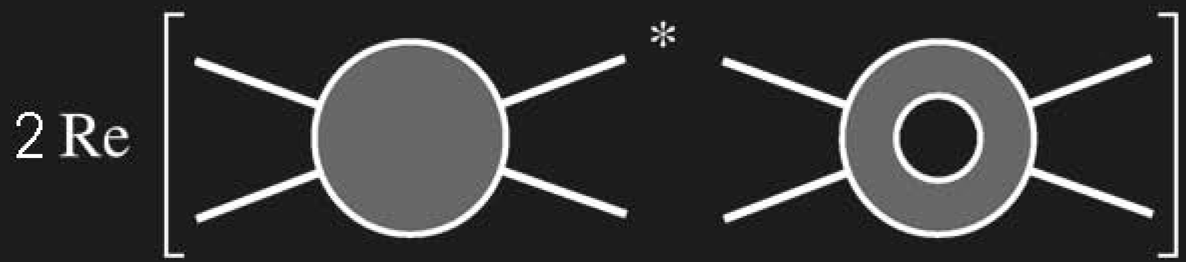


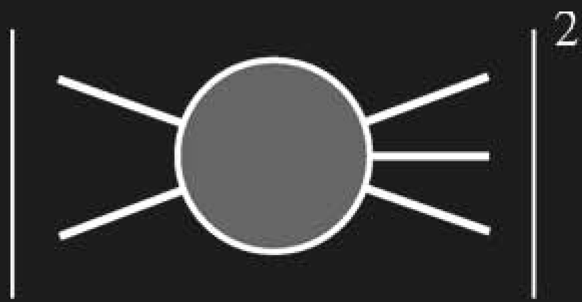
What Contributions Do We Need?

- Short-distance matrix elements to 2-jet production at leading order: tree level



- Short-distance matrix elements to 2-jet production at next-to-leading order: tree level + one loop + real emission





Real-Emission Singularities

$$e^+e^- \longrightarrow q\bar{q}g$$

Matrix element

$$|\mathcal{M}|^2 \propto \frac{(2-y_{qg})^2+(2-y_{\bar{q}g})^2}{y_{qg}\,y_{\bar{q}g}}$$

Integrate

$$\int_{0\le y_{qg}+y_{\bar{q}g}\le 1} dy_{qg}dy_{\bar{q}g}\;|\mathcal{M}|^2$$

$$y_{qg} = 0 = y_{\bar{q}g} \qquad \text{soft}$$

$$y_{qg} = 0\,,\, y_{\bar{q}g} \neq 0$$

collinear

$$y_{qg} \neq 0\,,\, y_{\bar{q}g} = 0$$

- Physical quantities are finite

- Depend on resolution parameter

- Finiteness thanks to combination of Kinoshita–Lee–Nauenberg theorem and factorization

Scattering

Scattering matrix element

$$_{\text{out}}\langle p_1 p_2 \cdots \mid k_a k_b \rangle_{\text{in}} \equiv {}_{\text{asymp}}\langle p_1 p_2 \cdots \mid S \mid k_a k_b \rangle_{\text{asymp}}$$

Decompose it $S = 1 + iT$

Invariant matrix element $\mathscr{M}$

$$\langle p_1 p_2 \cdots \mid iT \mid k_a k_b \rangle = (2\pi)^4 \delta(k_a + k_b - P) i\mathcal{M}(k_a, k_b \to \{p_f\})$$

Differential cross section

$$d\sigma \text{ `='} \ \frac{1}{4E_a E_b |v_a - v_b|} \prod_f \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f}$$
$$\times (2\pi)^4 \delta^4(k_a + k_b - P) |\mathcal{M}(k_a, k_b \to \{p_f\})|^2$$

Lorentz-invariant phase-space measure

$$\int \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_i}\, F(p_i) = \int \frac{d^4 p_i}{(2\pi)^4}\,(2\pi)\,\Theta(p_i^0)\delta(p_i^2)$$

$$\delta^{(+)}(p_i^2)$$

Compute invariant matrix element by crossing

$$\begin{aligned}\mathcal{M}(k_a, k_b \to \{p_f\}) &= \mathcal{M}(0 \to -k_a, -k_b, \{p_f\}) \\ &= \mathcal{F}\,\langle \Omega \mid T\phi_a(x_a)\phi_b(x_b)\phi_1(x_1)\cdots \mid \Omega\rangle\end{aligned}$$

Lagrangian

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu\right)\left(\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu}\right)$$

$$\sum_f \overline{q}(i\not{\partial})q + \frac{g}{\sqrt{2}} G^a_\mu \sum_f \overline{q}\gamma^\mu T^a q$$

$$-\frac{g}{2} f^{abc}\left(\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu}\right) G^b_\mu G^c_\nu$$

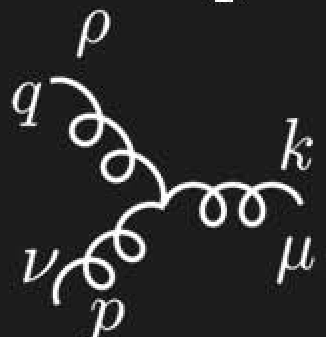
$$-\frac{g^2}{4} f^{abe} f^{cde} G^{a\mu} G^{b\nu} G^c_\mu G^d_\nu$$

Feynman Rules

Propagator (like QED)

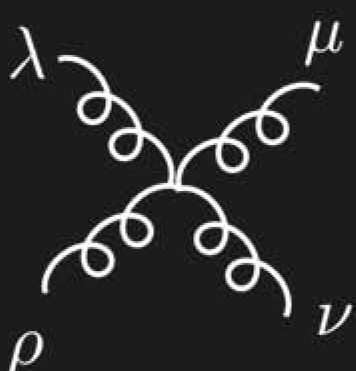
$$-\frac{i}{k^2+i\epsilon}\delta^{ab}\left(g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

Three-gluon vertex (unlike QED)



$$g f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right],$$

Four-gluon vertex (unlike QED)



$$-ig^2\left[f^{abe}f^{dce}(g^{\mu\rho}g^{\nu\lambda}-g^{\mu\lambda}g^{\nu\rho})\right.$$
$$+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\lambda}-g^{\mu\lambda}g^{\rho\nu})$$
$$\left.+f^{ace}f^{bde}(g^{\mu\nu}g^{\lambda\rho}-g^{\mu\rho}g^{\lambda\nu})\right]$$

From the Faddeev–Popov functional determinant

$$\det \partial \cdot D = \int [Dc][D\overline{c}] \exp\left\{ -i \int \overline{c}(\partial^\mu D_\mu) c \right\}$$

anticommuting scalars or *ghosts*

Propagator $\quad \dfrac{i}{k^2 + i\epsilon}\delta^{ab}$

coupling to gauge bosons $\quad g f^{abc} k^\mu$

So What's Wrong with Feynman Diagrams?

- Huge number of diagrams in calculations of interest
- But answers often turn out to be very simple
- Vertices and propagators involve gauge-variant off-shell states
- Each diagram is not gauge invariant — huge cancellations of gauge-noninvariant, redundant, parts in the sum over diagrams

Simple results should have a simple derivation — attr to Feynman

- Want approach in terms of physical states only

Light-Cone Gauge

Only physical (transverse) degrees of freedom propagate

$$-\frac{i}{k^2 + i\epsilon}\delta^{ab}\left(g_{\mu\nu} - \frac{n^\mu k^\nu + n^\nu k^\mu}{n \cdot k}\right)$$

$$n^2 = 0$$

physical projector — two degrees of freedom

Color Decomposition

Standard Feynman rules ⇒ function of momenta, polarization vectors ε, and color indices

Color structure is predictable. Use representation

$$f^{abc} = -\frac{i}{\sqrt{2}}\,\mathrm{Tr}([T^a, T^b]T^c)$$

$$\mathrm{Tr}(T^a T^b) = \delta^{ab}$$

to represent each term as a product of traces,

and the Fierz identity

$$(T^a)_{i_1}{}^{\bar{\imath}_1}(T^a)_{i_2}{}^{\bar{\imath}_2} = \delta_{i_1}{}^{\bar{\imath}_2}\delta_{i_2}{}^{\bar{\imath}_1} - \frac{1}{N}\delta_{i_1}{}^{\bar{\imath}_1}\delta_{i_2}{}^{\bar{\imath}_2}$$

To unwind traces

$$f^{abc} f^{cde} = -\frac{1}{2} \operatorname{Tr}([T^a, T^b] T^c) \operatorname{Tr}(T^c [T^d, T^e])$$
$$= -\frac{1}{2} \operatorname{Tr}([T^a, T^b][T^d, T^e]) + \frac{1}{2N} \operatorname{Tr}([T^a, T^b]) \operatorname{Tr}([T^d, T^e])$$

Leads to tree-level representation in terms of single traces

$$\mathcal{A}_n^{\text{tree}}(\{k_i, \varepsilon_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \operatorname{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}})$$
$$\times A_n^{\text{tree}}(k_{\sigma(1)}, \varepsilon_{\sigma(1)}; k_{\sigma(2)}, \varepsilon_{\sigma(2)}; k_{\sigma(n)}, \varepsilon_{\sigma(n)})$$

Color-ordered amplitude — function of momenta & polarizations alone; *not* Bose symmetric

Symmetry properties

- Cyclic symmetry

$$A_n^{\text{tree}}(1,\ldots,n) = A_n(2,\ldots,n,1)$$

- Reflection identity

$$A_n^{\text{tree}}(n,\ldots,1) = (-1)^n A_n(1,\ldots,n)$$

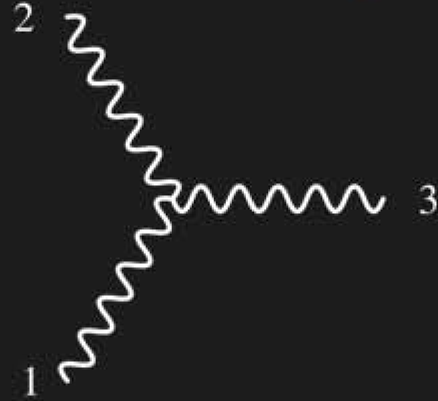
- Parity flips helicities

$$A_n^{\text{tree}}(1^{-\lambda_1},\ldots,n^{-\lambda_n}) = \left[A_n^{\text{tree}}(1^{\lambda_1},\ldots,n^{\lambda_n})\right]^\dagger$$

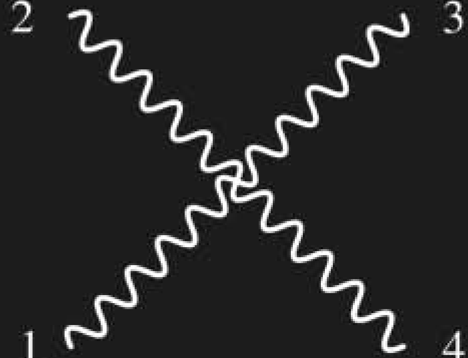
- Decoupling equation

$$A_{n+1}^{\text{tree}}(p,1,2,\ldots,n) + A_{n+1}^{\text{tree}}(1,p,2,\ldots,n) + A_{n+1}^{\text{tree}}(1,2,p,\ldots,n) + \cdots + A_{n+1}^{\text{tree}}(1,2,\ldots,p,n) = 0$$

Color-Ordered Feynman Rules



$$\frac{i}{\sqrt{2}}\left[\varepsilon_1\cdot\varepsilon_2\,(k_1-k_2)\cdot\varepsilon_3+\varepsilon_2\cdot\varepsilon_3\,(k_2-k_3)\cdot\varepsilon_1+\varepsilon_3\cdot\varepsilon_1\,(k_3-k_1)\cdot\varepsilon_2\right]$$



$$i\varepsilon_1\cdot\varepsilon_3\,\varepsilon_2\cdot\varepsilon_4-\frac{i}{2}(\varepsilon_1\cdot\varepsilon_2\,\varepsilon_3\cdot\varepsilon_4+\varepsilon_2\cdot\varepsilon_3\,\varepsilon_4\cdot\varepsilon_1)$$

Amplitudes

Functions of momenta k, polarization vectors ε for gluons; momenta k, spinor wavefunctions u for fermions

Gauge invariance implies this is a redundant representation:

$\varepsilon \to k$: $A = 0$

Spinor Helicity

Spinor wavefunctions $|j^{\pm}\rangle \equiv u_{\pm}(k_j), \qquad \langle j^{\pm}| \equiv \overline{u_{\pm}}(k_j)\ .$

Introduce *spinor products*

$$\langle i\, j\rangle \equiv \langle i^-|j^+\rangle = \overline{u_-}(k_i)u_+(k_j)\,,$$
$$[i\, j] \equiv \langle i^+|j^-\rangle = \overline{u_+}(k_i)u_-(k_j)$$

Explicit representation

$$u_+(k) = \begin{pmatrix} \sqrt{k_+} \\ \sqrt{k_-}e^{i\phi_k} \end{pmatrix}, \quad u_-(k) = \begin{pmatrix} \sqrt{k_-}e^{-i\phi_k} \\ -\sqrt{k_+} \end{pmatrix}$$

where

$$e^{\pm i\phi_k} = \frac{k^1 \pm ik^2}{\sqrt{k_+k_-}}, \qquad k_{\pm} = k^0 \pm k^3$$

We then obtain the explicit formulæ

$$\langle i\, j\rangle = \sqrt{k_{i-}k_{j+}}e^{i\phi_{k_i}} - \sqrt{k_{i+}k_{j-}}e^{i\phi_{k_j}}\ ,$$

$$[i\, j] = \langle j\, i\rangle^* = \sqrt{k_{i+}k_{j-}}e^{-i\phi_{k_j}} - \sqrt{k_{i-}k_{j+}}e^{-i\phi_{k_i}} \qquad (k^0_{i,j} > 0)$$

otherwise $[j\, i] = \mathrm{sign}(k_i^0 k_j^0)\, \langle i\, j\rangle^*$

so that the identity $\langle i\, j\rangle\, [j\, i] = 2k_i \cdot k_j$ always holds

Introduce four-component representation

$$\begin{bmatrix} u_+(k) \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ u_-(k) \end{bmatrix}$$

corresponding to γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

in order to define spinor strings

$$\langle i^\pm|\gamma^\mu|j^\pm\rangle \leftrightarrow \langle i^\pm|\sigma^\mu|j^\pm\rangle \qquad \sigma^\mu = (1, \sigma^i)$$

Properties of the Spinor Product

- Antisymmetry $\langle j\, i\rangle = -\langle i\, j\rangle\,, \qquad [j\, i] = -[i\, j]$
- Gordon identity $\langle i^{\pm}|\gamma^{\mu}|i^{\pm}\rangle = 2k_i^{\mu}$
- Charge conjugation $\langle i^{-}|\,\gamma^{\mu}\,|j^{-}\rangle = \langle j^{+}|\,\gamma^{\mu}\,|i^{+}\rangle\,,$
- Fierz identity $\langle i^{-}|\,\gamma^{\mu}\,|j^{-}\rangle\,\langle p^{+}|\,\gamma_{\mu}\,|q^{+}\rangle = 2\,\langle i\, q\rangle\,[p\, j]$
- Projector representation $|i^{\pm}\rangle\langle i^{\pm}| = \frac{1}{2}(1 \pm \gamma_5)\not{k}_i$
- Schouten identity $\langle i\, j\rangle\,\langle p\, q\rangle = \langle i\, q\rangle\,\langle p\, j\rangle + \langle i\, p\rangle\,\langle j\, q\rangle\,.$

Spinor-Helicity Representation for Gluons

Gauge bosons also have only ± physical polarizations

Elegant — and covariant — generalization of circular polarization

$$\varepsilon_\mu^+(k,q) = \frac{\langle q^- |\, \gamma_\mu \,| k^- \rangle}{\sqrt{2}\,\langle q\, k\rangle}\,, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q^+ |\, \gamma_\mu \,| k^+ \rangle}{\sqrt{2}\,[k\, q]}$$

Xu, Zhang, Chang (1984)

reference momentum q $\quad q\cdot k \neq 0$

Transverse $\quad k\cdot\varepsilon^\pm(k,q) = 0$

Normalized $\quad \varepsilon^+\cdot\varepsilon^- = -1, \qquad \varepsilon^+\cdot\varepsilon^+ = 0$

What is the significance of q?

$$\varepsilon_\mu^+(k,q') = \frac{\langle q'^- |\, \gamma_\mu \,| k^- \rangle}{\sqrt{2}\,\langle q'\,k\rangle} = \frac{\langle q'^- |\, \gamma_\mu \not{k} \,| q^+ \rangle}{\sqrt{2}\,\langle q'\,k\rangle\,\langle k\,q\rangle}$$

$$= -\frac{\langle q'^- |\, \not{k}\gamma_\mu \,| q^+ \rangle}{\sqrt{2}\,\langle q'\,k\rangle\,\langle k\,q\rangle} + \frac{\sqrt{2}\,\langle q\,q'\rangle}{\langle q'\,k\rangle\,\langle k\,q\rangle}\,k_\mu$$

$$= \varepsilon_\mu^+(k,q) + \frac{\sqrt{2}\,\langle q\,q'\rangle}{\langle q'\,k\rangle\,\langle k\,q\rangle}\,k_\mu$$

Properties of the Spinor-Helicity Basis

Physical-state projector

$$\sum_{\sigma=\pm} \varepsilon_\mu^\sigma(k,q)\varepsilon_\nu^{\sigma *}(k,q) = \sum_{\sigma=\pm} \varepsilon_\mu^\sigma(k,q)\varepsilon_\nu^{-\sigma}(k,q) = -g_{\mu\nu} + \frac{q_\mu k_\nu + k_\mu q_\nu}{q \cdot k}$$

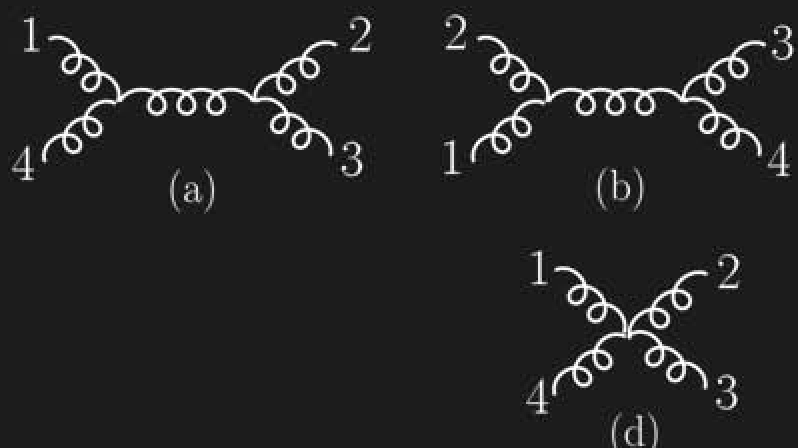
Simplifications

$$q \cdot \varepsilon^\pm(k,q) = 0\,,$$
$$\varepsilon^+(k_1,q)\cdot\varepsilon^+(k_2,q) = \varepsilon^-(k_1,q)\cdot\varepsilon^-(k_2,q) = 0\,,$$
$$\varepsilon^+(k_1,q)\cdot\varepsilon^-(k_2,k_1) = 0$$

Examples

By explicit calculation (or other arguments), every term in the gluon tree-level amplitude has at least one factor of $\varepsilon_i \cdot \varepsilon_j$

Look at four-point amplitude



Recall three-point color-ordered vertex

$$\frac{i}{\sqrt{2}}\left[\varepsilon_1 \cdot \varepsilon_2 \,(k_1 - k_2)\cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 \,(k_2 - k_3)\cdot \varepsilon_1 + \varepsilon_3 \cdot \varepsilon_1 \,(k_3 - k_1)\cdot \varepsilon_2\right]$$

Calculate $A_4^{\text{tree}}(1^+, 2^+, 3^+, 4^+)$

choose identical reference momenta for all legs $\Rightarrow$ all $\varepsilon \cdot \varepsilon$ vanish

$\Rightarrow$ amplitude vanishes

Calculate $A_4^{\text{tree}}(1^-, 2^+, 3^+, 4^+)$

choose reference momenta 4,1,1,1 $\Rightarrow$ all $\varepsilon \cdot \varepsilon$ vanish

$\Rightarrow$ amplitude vanishes

Calculate $A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$

choose reference momenta 3,3,2,2

$\Rightarrow$ only nonvanishing $\varepsilon \cdot \varepsilon$ is $\varepsilon_1 \cdot \varepsilon_4$

$\Rightarrow$ only s_{12} channel contributes

$$\left(\frac{i}{\sqrt{2}}\right)^2 \left(-\frac{ig^{\mu\nu}}{s_{12}}\right) \left[-2k_1 \cdot \varepsilon_2^- \varepsilon_{1\mu}^-\right] \left[2k_4 \cdot \varepsilon_3^+ \varepsilon_{4\nu}^+\right]$$
$$= -\frac{2i}{s_{12}} k_1 \cdot \varepsilon_2^- k_4 \cdot \varepsilon_3^+ \varepsilon_1^- \cdot \varepsilon_4^+$$
$$= -\frac{i}{s_{12}} \left(\frac{[3\,1]\,\langle 1\,2\rangle}{[3\,2]}\right) \left(\frac{\langle 2\,4\rangle\,[4\,3]}{\langle 2\,3\rangle}\right) \left(\frac{\langle 2\,1\rangle\,[3\,4]}{\langle 2\,4\rangle\,[3\,1]}\right)$$
$$= -i \frac{\langle 1\,2\rangle^2\,[3\,4]^2}{s_{12}s_{23}}$$
$$= i \frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}$$

No diagrammatic calculation required for the last helicity amplitude,

$$A_4^{\text{tree}}(1^-,2^+,3^-,4^+)$$

Obtain it from the decoupling identity

$$- A_4^{\text{tree}}(3^-,1^-,2^+,4^+) - A_4^{\text{tree}}(1^-,3^-,2^+,4^+)$$
$$= i\frac{\langle 1\,3\rangle^3}{\langle 2\,4\rangle}\left(-\frac{1}{\langle 1\,2\rangle\langle 4\,3\rangle}+\frac{1}{\langle 3\,2\rangle\langle 4\,1\rangle}\right)$$
$$= i\frac{\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}$$

These forms hold more generally, for larger numbers of external legs:

$$A_n^{\rm tree}(1^+,2^+,\ldots,n^+) = 0\,,$$
$$A_n^{\rm tree}(1^-,2^+,\ldots,n^+) = 0$$

Parke-Taylor equations

Mangano, Xu, Parke (1986)

Maximally helicity-violating or 'MHV'

$$A_n^{\rm tree}(1^+,\ldots,m_1^-,(m_1+1)^+,\ldots,m_2^-,(m_2+1)^+,\ldots,n^+) =$$
$$i\frac{\langle m_1\, m_2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle n\,1\rangle}$$

Proven using the Berends–Giele recurrence relations