On-Shell Methods in Field Theory

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Lecture II

Review of Lecture I

Color-ordered amplitude

$$\mathcal{A}_n^{\text{tree}}(\{k_i, \varepsilon_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}})$$
$$\times A_n^{\text{tree}}(k_{\sigma(1)}, \varepsilon_{\sigma(1)}; k_{\sigma(2)}, \varepsilon_{\sigma(2)}; k_{\sigma(n)}, \varepsilon_{\sigma(n)})$$

Color-ordered amplitude — function of momenta & polarizations alone; *not* Bose symmetric

Spinor-Helicity Representation for Gluons

Gauge bosons also have only ± physical polarizations

Elegant — and covariant — generalization of circular polarization

$$\varepsilon_\mu^+(k,q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2}\,\langle q\,k\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2}\,[k\,q]}$$

Xu, Zhang, Chang (preprint 1984); NPB291:392 (1987)

reference momentum q $\qquad q \cdot k \neq 0$

Transverse $\qquad k \cdot \varepsilon^\pm(k,q) = 0$

Normalized $\qquad \varepsilon^+ \cdot \varepsilon^- = -1, \qquad \varepsilon^+ \cdot \varepsilon^+ = 0$

Parke-Taylor Equations

For any number of external legs:

$$A_n^{\rm tree}(1^+, 2^+, \ldots, n^+) = 0\,,$$
$$A_n^{\rm tree}(1^-, 2^+, \ldots, n^+) = 0$$

Parke & Taylor, PRL 56:2459 (1986)

Mangano, Xu, & Parke, NPB298:653 (1986)

Maximally helicity-violating or 'MHV'

$$A_n^{\rm tree}(1^+, \ldots, m_1^-, (m_1+1)^+, \ldots, m_2^-, (m_2+1)^+, \ldots, n^+) =$$
$$i\frac{\langle m_1\, m_2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle (n-1)\, n\rangle\,\langle n\,1\rangle}$$

Proven using the Berends–Giele recurrence relations

Berends & Giele, NPB294:700 (1987)

Gauge-theory amplitude

↓ Color decomposition & stripping

Color-ordered amplitude: function of k_i and ε_i

↓ Spinor-helicity basis

Helicity amplitude: function of spinor products and helicities ±1

Spinor products → spinors

Spinor Variables

From Lorentz vectors to bi-spinors

$$p_\mu \quad \longleftrightarrow \quad p_{a\dot{a}} \equiv p \cdot \sigma = \begin{pmatrix} p^0 + p^3 & p^1 + ip^2 \\ p^1 - ip^2 & p^0 - p^3 \end{pmatrix}$$

$$p^2 \quad \longleftrightarrow \quad \det(p)$$

$$p' = \Lambda p \quad \longleftrightarrow \quad p' = upu^\dagger, \qquad u \in SL(2,C)$$

2×2 complex matrices
with det = 1

Null momenta $p^2 = 0 \Longrightarrow \det(p) = 0$

can write it as a bispinor $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$

phase ambiguity in $\lambda_a, \tilde{\lambda}_{\dot{a}}$ (same as seen in spinor products)

For real Minkowski p, take $\tilde{\lambda} = \text{sign}(p^0)\overline{\lambda}$

Invariant tensor ϵ_{ab}

$$u_{aa'}u_{bb'}\epsilon_{a'b'} = \det(u) = 1$$

gives spinor products

$$\langle \lambda_1, \lambda_2 \rangle = \epsilon_{ab}\lambda_1^a \lambda_2^b$$

$$[\tilde{\lambda}_1, \tilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}}\tilde{\lambda}_1^{\dot{a}}\tilde{\lambda}_2^{\dot{b}}$$

Connection to earlier spinor products

$$\langle\lambda_1, \lambda_2\rangle = \langle 1\,2\rangle$$
$$[\tilde\lambda_1, \tilde\lambda_2] = -[1\,2]$$

and spinor-helicity basis

$$+1: \qquad \varepsilon_{a\dot a} = \frac{\eta_a \tilde\lambda_{\dot a}}{\langle\eta, \lambda\rangle}$$
$$-1: \qquad \varepsilon_{a\dot a} = \frac{\lambda_a \tilde\eta_{\dot a}}{[\tilde\lambda, \tilde\eta]}$$

$\Rightarrow$ Amplitudes as functions of spinor variables $\lambda_a, \tilde\lambda_{\dot a}$ and helicities ± 1

Scaling of Amplitudes

Suppose we scale the spinors

also called 'phase weight'

$$\lambda_i \longmapsto \alpha_i \lambda_i,$$
$$\tilde{\lambda}_i \longmapsto \alpha_i^{-1} \tilde{\lambda}_i,$$

then by explicit computation we see that the MHV amplitude

$$A^{\text{MHV}} \longmapsto i \frac{\alpha_{m_1}^2 \alpha_{m_2}^2}{\Pi_{j \neq m_1, m_2} \alpha_j^2} \frac{\langle m_1\, m_2 \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle (n-1)\, n \rangle \langle n\, 1 \rangle}$$

and that more generally

$$A \longmapsto \Pi_j \alpha_j^{-2h_j} A$$

For the non-trivial parts of the amplitude, we might as well use uniformly rescaled spinors $\Rightarrow \mathbb{C}\mathrm{P}^1$ 'complex projective space'

Start with $\mathbb{C}^2$, and rescale all vectors by a common scale

$$\begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \equiv \tau \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}$$

the spinors are then 'homogeneous' coordinates on $\mathbb{C}\mathrm{P}^1$

If we look at each factor in the MHV amplitude,

$$\frac{1}{\langle \lambda_1, \lambda_2 \rangle} = \frac{1}{\lambda_1^1 \lambda_2^1 (w_1 - w_2)} \qquad w_i = \lambda_i^2 / \lambda_i^1$$

we see that it is just a free-field correlator (Green function) on $\mathbb{C}\mathrm{P}^1$

This is the essence of Nair's construction of MHV amplitudes as correlation functions on the 'line' = $\mathbb{C}\mathrm{P}^1$

Gauge-theory amplitude

↓ Color decomposition & stripping

Color-ordered amplitude: function of k_i and ε_i

↓ Spinor-helicity basis

Helicity amplitude: function of spinor products and helicities ±1

↓

Function of spinor variables and helicities ±1

↓ Half-Fourier transform

Conjectured support on simple curves in twistor space

Let's Travel to Twistor Space!

It turns out that the natural setting for amplitudes is not exactly spinor space, but something similar. The motivation comes from studying the representation of the conformal algebra.

Half-Fourier transform of spinors: transform $\tilde{\lambda}_{\dot{a}}$, leave alone λ_a

$\Rightarrow$ Penrose's original twistor space, real or complex

$$\tilde{\lambda}_{\dot{a}} \to i\frac{\partial}{\partial \mu^{\dot{a}}}, \qquad -i\frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \to \mu_{\dot{a}}$$

Study amplitudes of definite helicity: introduce homogeneous coordinates $Z_I = (\lambda_a, \mu_a)$

$\Rightarrow$ **CP^3 or RP^3 (projective) twistor space**

Back to momentum space by Fourier-transforming μ

MHV Amplitudes in Twistor Space

Write out the half-Fourier transform including the energy-momentum conserving δ function

$$\widetilde{A}(Z) = \int \prod_j \frac{d^2\tilde{\lambda}_j}{(2\pi)^2} \; \exp[i\mu_{j\dot{a}}\tilde{\lambda}_j^{\dot{a}}] \; (2\pi)^4\delta^4(\Sigma_r \lambda_r^a \tilde{\lambda}_r^{\dot{a}}) \; A^{\mathrm{MHV}}(\lambda_j)$$

$$\int d^4x \; \exp[\Sigma_j x_{a\dot{a}} \lambda_j^a \tilde{\lambda}_j^{\dot{a}}]$$

$$\widetilde{A}(Z) = \int d^4x \; \prod_j \frac{d^2\tilde{\lambda}_j}{(2\pi)^2} \; \exp[\Sigma_j i(\mu_{j\dot{a}} + x_{a\dot{a}}\lambda_j^a)\tilde{\lambda}_j^{\dot{a}}] \; A^{\mathrm{MHV}}(\lambda_j)$$

Result

$$\widetilde{A}(Z) = \int d^4x \prod_j \delta^2(\mu_{j\dot{a}} + x_{a\dot{a}}\lambda_j^a)\, A^{\mathrm{MHV}}(\lambda_j)$$

equation for a line

MHV amplitudes live on lines in twistor space

Value of the twistor-space amplitude is given by a correlation function on the line

Analyzing Amplitudes in Twistor Space

Amplitudes in twistor space turn out to be hard to compute directly. Even with computations in momentum space, the Fourier transforms are hard to compute explicitly.

We need other tools to analyze the amplitudes.

Simple 'algebraic' properties in twistor space — support on $\mathbb{CP}^1$s or $\mathbb{CP}^2$s — become differential properties in momentum space.

Construct differential operators.

Equation for a line ($\mathbb{CP}^1$): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K = 0$

gives us a differential ('line') operator in terms of momentum-space spinors

$$F_{123} = \langle \lambda_1\, \lambda_2 \rangle \frac{\partial}{\partial \tilde\lambda_3} + \langle \lambda_2\, \lambda_3 \rangle \frac{\partial}{\partial \tilde\lambda_1} + \langle \lambda_3\, \lambda_1 \rangle \frac{\partial}{\partial \tilde\lambda_2}.$$

Equation for a plane ($\mathbb{CP}^2$): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K Z_4^L = 0$

also gives us a differential ('plane') operator

$$K_{1234} = \langle \lambda_1\, \lambda_2 \rangle \frac{\partial}{\partial \tilde\lambda_{3\dot a}} \frac{\partial}{\partial \tilde\lambda_4^{\dot a}} + \text{ perms}$$

Properties

$$F_{ijl}\, f(p_i + p_j + p_l) = 0$$
$$F_{ijl}\, f(\{\lambda_r\}) = 0$$

$$K_{ijlm}\, f(\{\lambda_r\}) = 0$$

Thus for example

$$F_{ijl} \frac{\langle m_1\, m_2 \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle (n-1)\, n \rangle \langle n\, 1 \rangle} = 0$$

Beyond MHV

Witten's proposal: ph/0312171

- Each external particle represented by a point in twistor space
- Amplitudes non-vanishing only when points lie on a curve of degree d and genus g, where
  - d = # negative helicities – 1 + # loops
  - $g \leq$ # loops; $g = 0$ for tree amplitudes
- Integrand on curve supplied by a topological string theory
- Obtain amplitudes by integrating over all possible curves $\Rightarrow$ moduli space of curves
- Can be interpreted as D_1-instantons

Strings in Twistor Space

- String theory can be defined by a two-dimensional field theory whose fields take values in target space:
  - n-dimensional flat space
  - 5-dimensional Anti-de Sitter × 5-sphere
  - twistor space: intrinsically four-dimensional ⇒ Topological String Theory
- Spectrum in Twistor space is $\mathcal{N} = 4$ supersymmetric multiplet (gluon, four fermions, six real scalars)
- Gluons and fermions each have two helicity states

A New Duality

- String Theory ⟷ Gauge Theory

  Topological *B*-model on $CP^{3|4}$ ⟷ $\mathcal{N}=4$ SUSY

'Twistor space'

Witten (2003); Berkovits & Motl; Neitzke & Vafa; Siegel (2004)

weak–weak

Simple Cases

Amplitudes with all helicities '+' $\Rightarrow$ degree -1 curves.
No such curves exist, so the amplitudes should vanish.
Corresponds to the first Parke–Taylor equation.

Amplitudes with one '–' helicity $\Rightarrow$ degree-0 curves: points.
Generic external momenta, all external points won't coincide (singular configuration, all collinear), $\Rightarrow$ amplitudes must vanish.
Corresponds to the second Parke–Taylor equation.

Amplitudes with two '–' helicities (MHV) $\Rightarrow$ degree-1 curves: lines.
All F operators should annihilate them, and they do.

Other Cases

Amplitudes with three negative helicities (next-to-MHV) live on conic sections (quadratic curves)

Amplitudes with four negative helicities (next-to-next-to-MHV) live on twisted cubics

Fourier transform back to spinors $\Rightarrow$ differential equations in conjugate spinors

Even String Theorists Can Do Experiments

- Apply F operators to NMHV (3 –) amplitudes: products annihilate them! K annihilates them;
- Apply F operators to N^{2}MHV (4 –) amplitudes: longer products annihilate them! Products of K annihilate them;

$$F_{512}F_{234}F_{345}F_{451}A_5(1^-,2^-,3^-,4^+,5^+) =$$

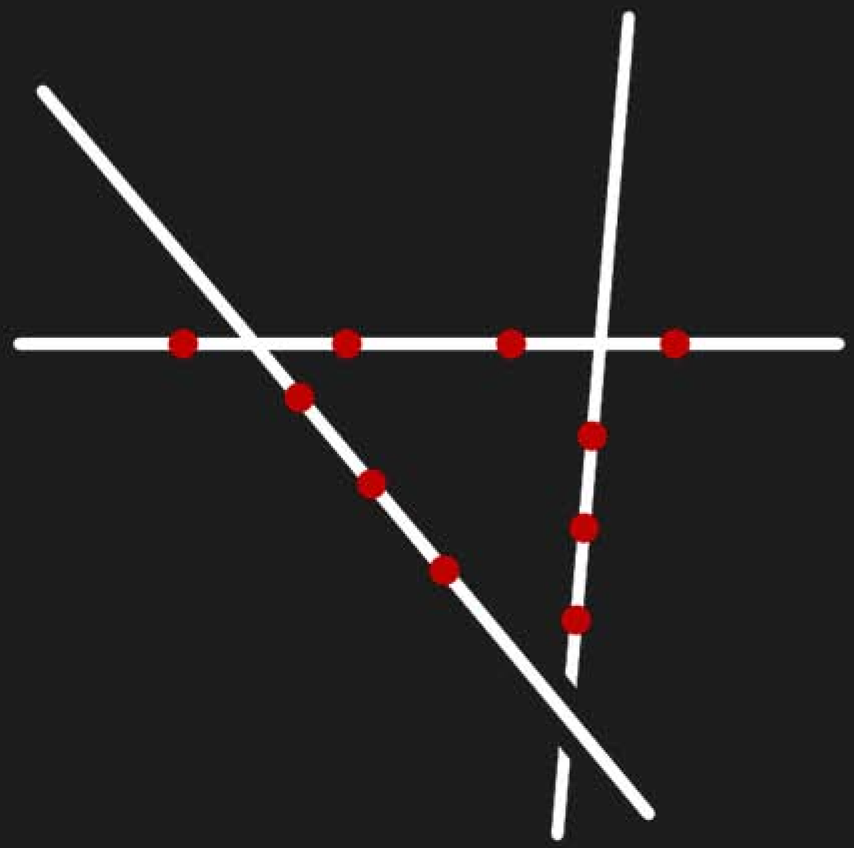
$$F_{512}F_{234}F_{345}F_{451}\frac{[4\,5]^4}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,1]} = 0$$

A more involved example

$$F_{612}F_{234}F_{345}F_{561}A_6(1^-,2^-,3^-,4^+,5^+,6^+) = 0$$

Don’t try this at home!

Interpretation: twistor-string amplitudes are supported on intersecting line segments

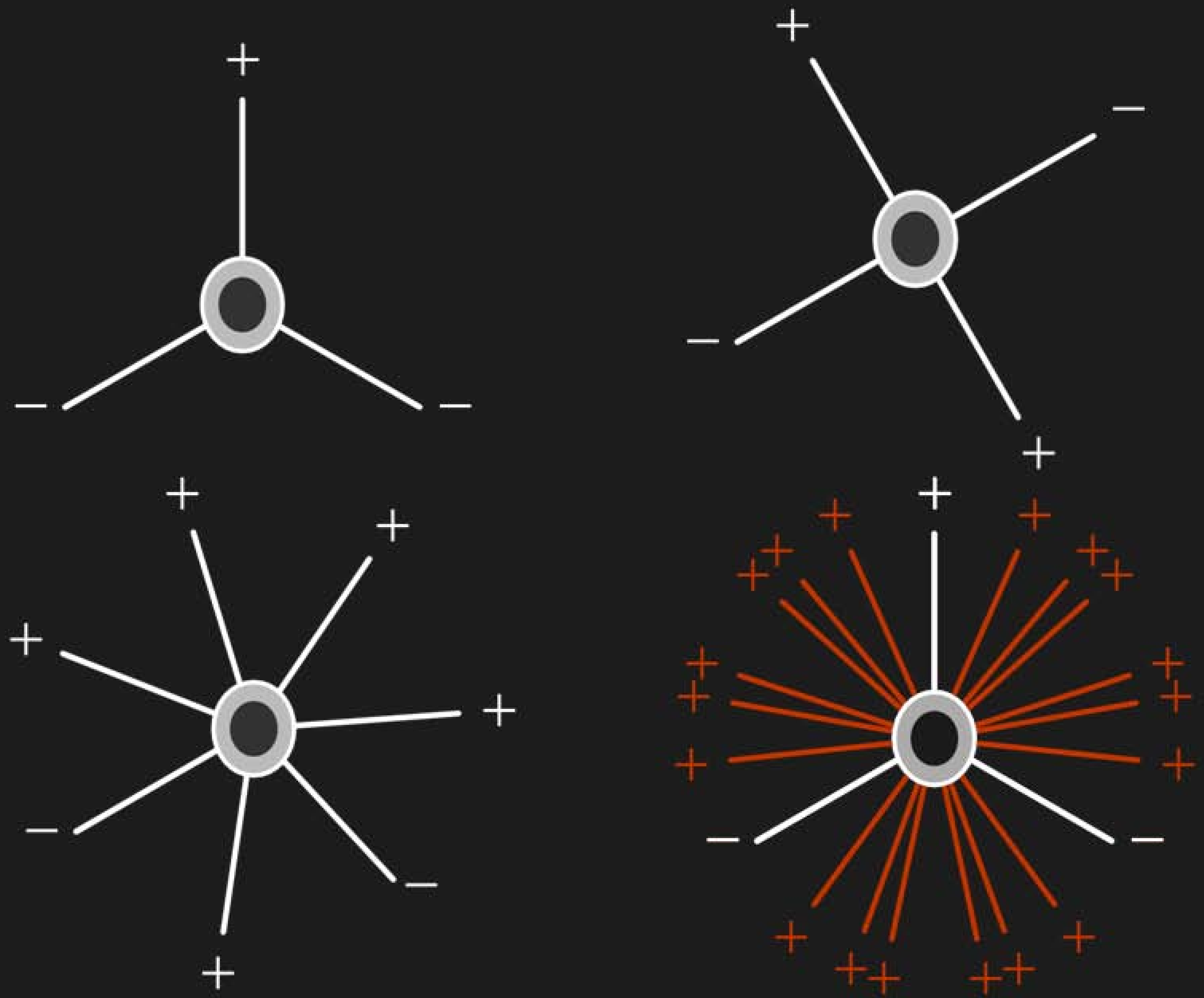


Simpler than expected: what does this mean in field theory?

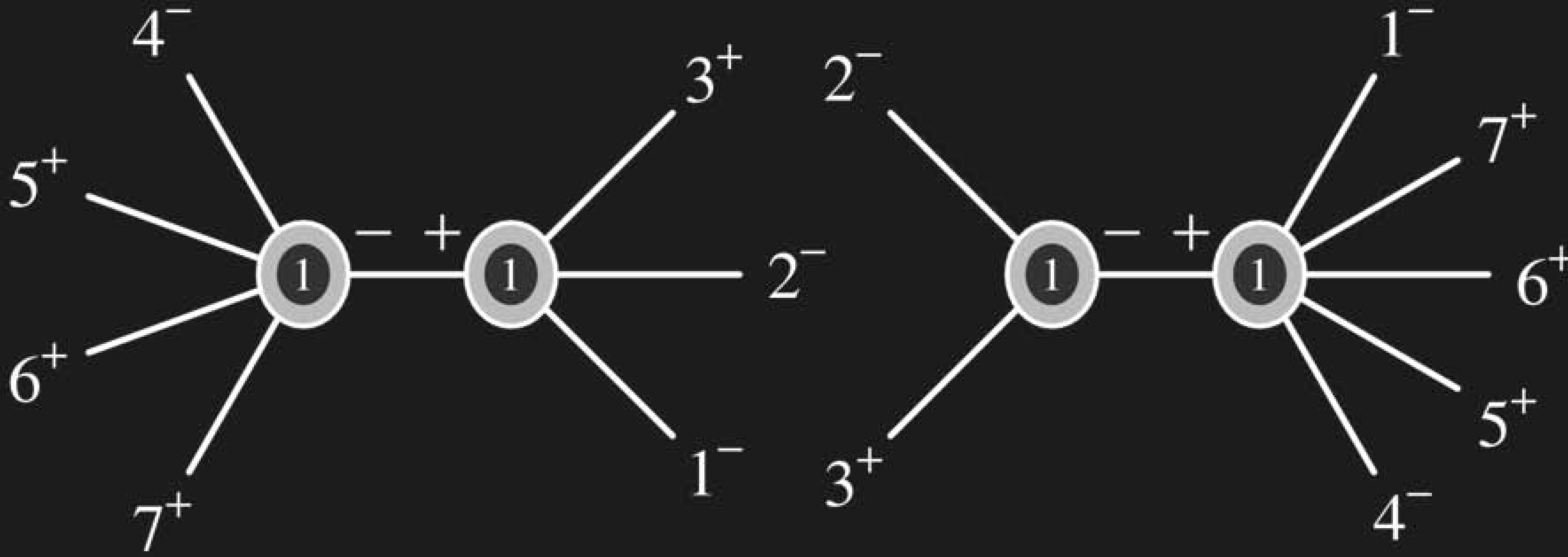
Cachazo–Svrček–Witten Construction

Cachazo, Svrček, & Witten, th/0403047

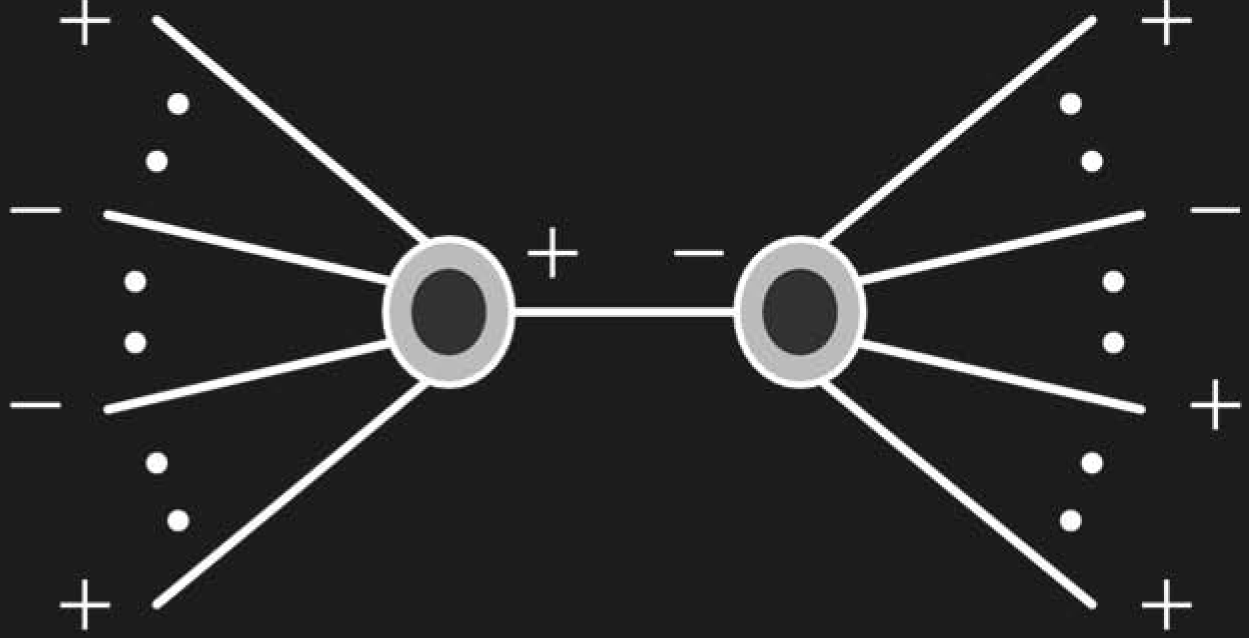
- Vertices are off-shell continuations of MHV amplitudes: every vertex has two '–' helicities, and one or more '+' helicities
- Includes a three-point vertex
- Propagators are scalar ones: i/K^2; helicity projector is in the vertices
- Draw all tree diagrams with these vertices and propagator
- Different sets of diagrams for different helicity configurations
- Corresponds to all multiparticle factorizations



- Seven-point example with three negative helicities



Next-to-MHV



Factorization Properties of Amplitudes

- As sums of external momenta approach poles,

$$p^2 = (k_1 + k_2)^2 \to m_X^2$$

- amplitudes factorize

$$A(1{+}2 \to \ldots) \to A_L(1{+}2 \to X)\frac{i}{p^2 - m^2}A_R(X \to \ldots)$$

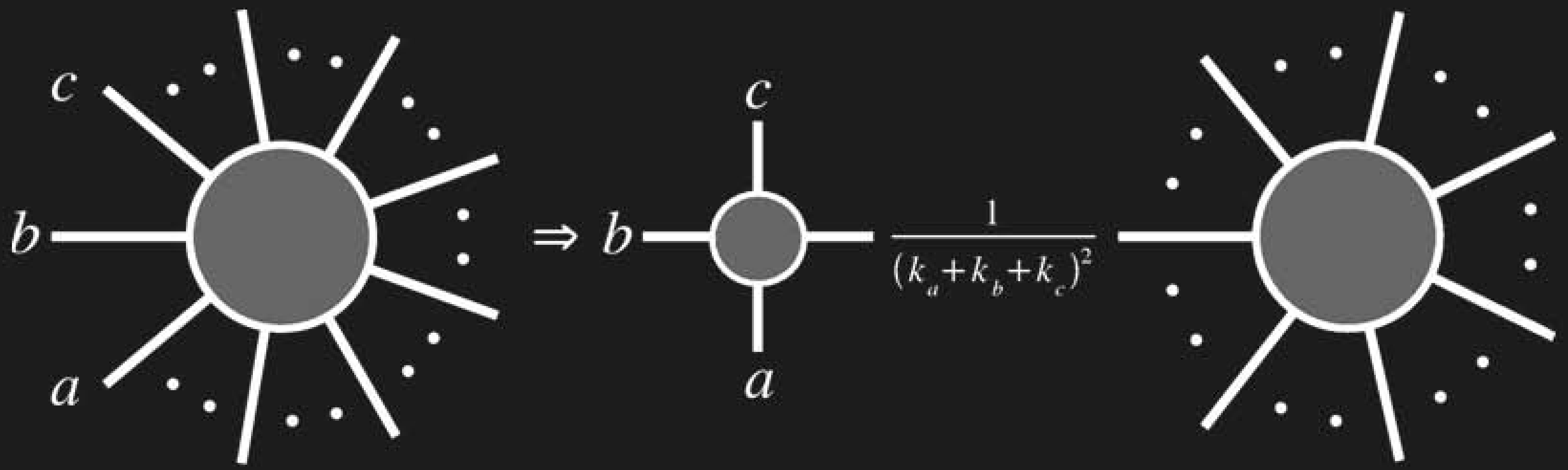
- More generally as

$$p^2 = (k_1 + \cdots + k_n)^2 \to m_X^2$$

$$A(1{+}\cdots{+}n \to \ldots) \to A_L(1{+}\cdots{+}n \to X)\frac{i}{p^2 - m^2}A_R(X \to \ldots)$$

Factorization in Gauge Theories

Tree level



As $(k_a + k_b + k_c)^2 \to 0$ but $s_{ab},\ s_{bc},\ s_{ac} \not\to 0$

Sum over helicities of intermediate leg

In massless theories beyond tree level, the situation is more complicated but at tree level it works in a standard way

What Happens in the Two-Particle Case?

We would get a three-gluon amplitude on the left-hand side

But $k_3^2 = 0 = (k_1 + k_2)^2 = 2k_1 \cdot k_2$

so all invariants vanish,

$$k_1 \cdot k_2 = k_2 \cdot k_3 = k_3 \cdot k_1 = 0$$

hence all spinor products vanish

$$\langle 12 \rangle = 0, \quad \langle 23 \rangle = 0, \quad \langle 31 \rangle = 0$$

$$[12] = 0, \quad [23] = 0, \quad [31] = 0$$

hence the three-point amplitude vanishes

$$A_3(1, 2, 3) = 0$$

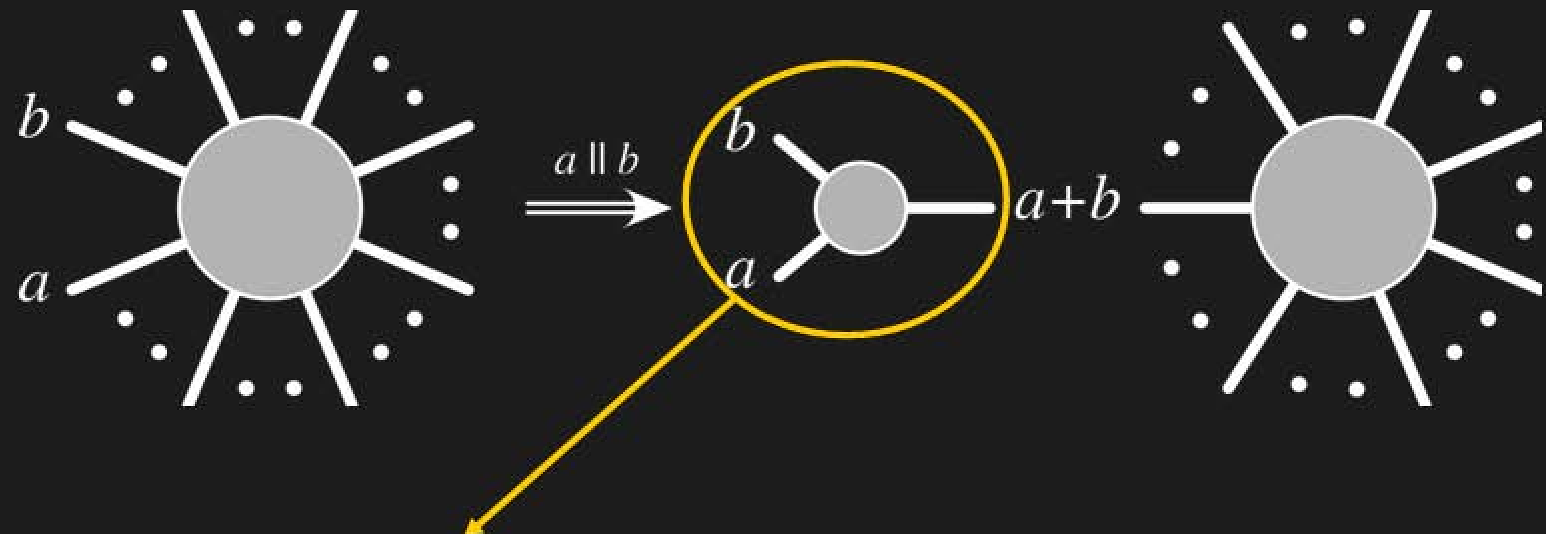
- In gauge theories, it holds (at tree level) for $n \geq 3$ but breaks down for $n = 2$: $A_3 = 0$ so we get $0/0$
- However A_3 only vanishes linearly, so the amplitude is not finite in this limit, but should $\sim 1/k$, that is $1/\sqrt{s_{12}}$
- This is a *collinear* limit

$$(k_1 + k_2)^2 = 2k_1 \cdot k_2 \to 0$$

$$\Longrightarrow k_1 \propto k_2, \;\; i.e., \; k_1 \parallel k_2$$

- Combine amplitude with propagator to get a non-vanishing object

Two-Particle Case



Collinear limit: *splitting* amplitude

Universal Factorization

- Amplitudes have a universal behavior in this limit

$$A_n^{\text{tree}}(\ldots, a^{h_a}, b^{h_b}, \ldots) \xrightarrow{k_a \| k_b}$$

$$\sum_{h=\pm} \text{Split}_{-h}^{\text{tree}}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{tree}}(\ldots, (k_a+k_b)^h, \ldots)$$

$$+ \;\; \text{non-singular}$$

- Depend on a collinear momentum fraction z

$$k_a = z(k_a + k_b), \qquad k_b = (1-z)(k_a + k_b)$$

- In this form, a powerful tool for checking calculations

- As expressed in on-shell recursion relations, a powerful tool for computing amplitudes

Example: Three-Particle Factorization

Consider

$$-iA_6(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{(\langle 2\,3\rangle\,[5\,6]\,\langle 1^-|\,1+2+3\,|4^-\rangle)^2}{s_{234}s_{23}s_{34}s_{56}s_{61}} + \frac{(\langle 1\,2\rangle\,[4\,5]\,\langle 3^-|\,1+2+3\,|6^-\rangle)^2}{s_{345}s_{34}s_{45}s_{61}s_{12}} + \frac{s_{123}\,\langle 2\,3\rangle\,[5\,6]\,\langle 1^-|\,1+2+3\,|4^-\rangle\,\langle 1\,2\rangle\,[4\,5]\,\langle 3^-|\,1+2+3\,|6^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}$$

As $s_{123} \to 0$, it's finite: expected because $A_4(1^-,2^-,3^-,X^\pm) = 0$

As $s_{234} \to 0$, pick up the first term; with $K = k_2 + k_3 + k_4$

$$\frac{\langle 2\,3\rangle\,[K\,4]^2}{[2\,3]\,s_{34}} \times \frac{1}{s_{234}} \times \frac{[5\,6]\,\langle 1\,K\rangle^2}{\langle 5\,6\rangle\,s_{61}} =$$

$$\frac{[4\,K]^2\,[K\,2]\,\langle 2\,3\rangle}{[2\,3]\,[3\,4]\,\langle 4\,3\rangle\,[K\,2]} \times \frac{1}{s_{234}} \times \frac{\langle 1\,K\rangle^2\,\langle K\,5\rangle\,[5\,6]}{\langle K\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,[1\,6]}$$

$$= \frac{[4\,K]^3}{[2\,3]\,[3\,4]\,[K\,2]} \times \frac{1}{s_{234}} \times \frac{\langle 1\,K\rangle^3}{\langle K\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle}$$

$$= A_4(2^-,3^-,4^+,K^+) \times \frac{1}{s_{234}} \times A_4(1^-,(-K)^-,5^+,6^+)$$

Splitting Amplitudes

Compute it from the three-point vertex

$$\begin{aligned}
\mathrm{Split}^{\mathrm{tree}}_{-}(a^+, b^+) &= -\frac{\sqrt{2}}{s_{ab}}\left[k_b\cdot\varepsilon_a\,\varepsilon_b\cdot\varepsilon_{a+b} - k_a\cdot\varepsilon_b\,\varepsilon_a\cdot\varepsilon_{a+b}\right] \\
&= -\frac{1}{s_{ab}}\left[\frac{\langle q\,b\rangle\,[b\,a]}{\langle q\,a\rangle}\,\frac{\langle q\,(a+b)\rangle\,[q\,b]}{\langle q\,b\rangle\,[(a+b)\,q]}\right. \\
&\qquad\left.-\frac{\langle q\,a\rangle\,[a\,b]}{\langle q\,b\rangle}\,\frac{\langle q\,(a+b)\rangle\,[q\,a]}{\langle q\,a\rangle\,[(a+b)\,q]}\right] \\
&= \frac{1}{\langle a\,b\rangle}\left[\sqrt{\frac{1-z}{z}}+\sqrt{\frac{z}{1-z}}\right] \\
&= \frac{1}{\sqrt{z(1-z)}\;\langle a\,b\rangle}\,.
\end{aligned}$$

Explicit Values

$$\text{Split}_{-}^{\text{tree}}(a^-, b^-) = 0$$

$$\text{Split}_{-}^{\text{tree}}(a^+, b^+) = \frac{1}{\sqrt{z(1-z)}\,\langle a\,b\rangle}$$

$$\text{Split}_{-}^{\text{tree}}(a^+, b^-) = -\frac{z^2}{\sqrt{z(1-z)}\,[a\,b]}$$

$$\text{Split}_{-}^{\text{tree}}(a^-, b^+) = -\frac{(1-z)^2}{\sqrt{z(1-z)}\,[a\,b]}$$

Collinear Factorization at One Loop

$$A_n^{\text{1-loop; LC}}(\ldots, a^{h_a}, b^{h_b}, \ldots) \xrightarrow{k_a \| k_b}$$

$$\sum_{h=\pm} \Big(\text{Split}^{\text{tree}}_{-h}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{1-loop;LC}}(\ldots, (k_a + k_b)^h, \ldots)$$

$$+ \text{Split}^{\text{1-loop}}_{-h}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{tree}}(\ldots, (k_a + k_b)^h, \ldots) \Big)$$

$$+ \ \text{non-singular}$$

Anomalous Dimensions & Amplitudes

- In QCD, one-loop anomalous dimensions of twist-2 operators in the OPE are related to the tree-level Altarelli-Parisi function,

Twist-2 Anomalous Dimension	Mellin Transform $\Leftrightarrow$	Altarelli-Parisi function	=	Helicity-summed splitting amplitude

- Relation understood between two-loop anomalous dimensions & one-loop splitting amplitudes

DAK & Uwer (2003)

Recursion Relations

Considered color-ordered amplitude with one leg off-shell, amputate its polarization vector

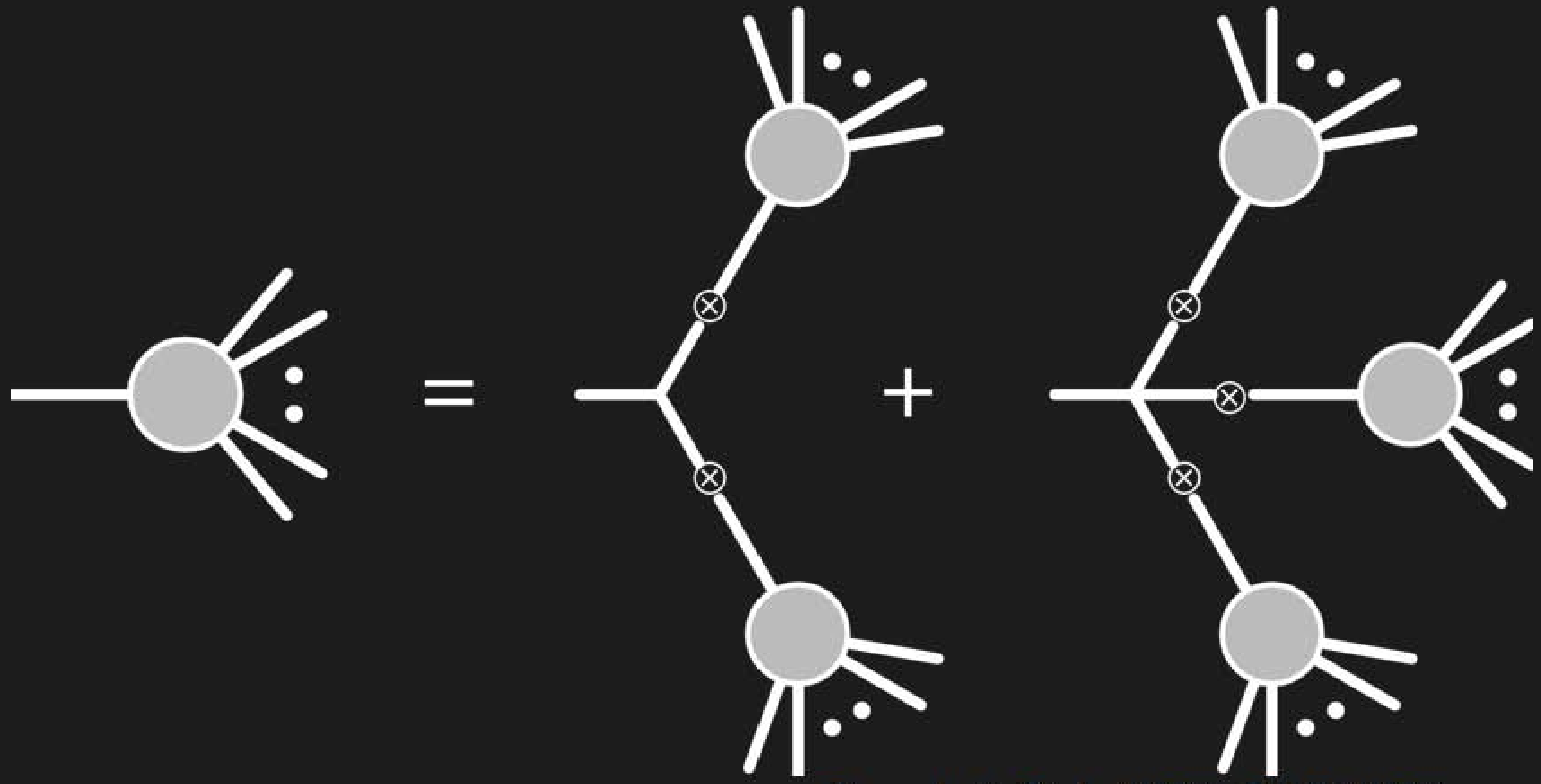
This is the Berends–Giele current $J^\mu(1,\ldots,n)$

Given by the sum of all (n+1)-point color-ordered diagrams with legs 1… n on shell

Follow the off-shell line into the sum of diagrams. It is attached to either a three- or four-point vertex.

Other lines attaching to that vertex are also sums of diagrams with one leg off-shell and other on shell, that is currents

Recursion Relations



Berends & Giele (1988); DAK (1989)

⇒ Polynomial complexity per helicity

$$J^\mu(1,\ldots,n) = -\frac{i}{K_{1,n}^2}\left[\sum_{j=1}^{n-1} V_3^{\mu\nu\lambda}\, J_\nu(1,\ldots,j) J_\lambda(j+1,\ldots,n) + \sum_{j=1}^{n-2}\sum_{l=j+1}^{n-1} V_4^{\mu\nu\lambda\rho}\, J_\nu(1,\ldots,j) \times J_\lambda(j+1,\ldots,l) J_\rho(l+1,\ldots,n)\right]$$

Properties of the Current

- Decoupling identity
- Reflection identity
- Conservation $K_{1,n}^{\mu} J_{\mu}(1,\ldots,n) = 0$

Complex Momenta

For real momenta, $|k^+\rangle = \pm|k^-\rangle^*$

but we can choose these two spinors independently and still have $k^2 = 0$

Recall the polarization vector: $\varepsilon^+ \propto \langle q^-|\gamma^\mu|k^-\rangle$
but $\varepsilon \cdot \varepsilon = 0$

Now when two momenta are collinear $k \cdot k' = 0$
only one of the spinors has to be collinear

$\langle k\, k'\rangle = 0$ or $[k\, k'] = 0$ but not necessarily both

On-Shell Recursion Relations

Britto, Cachazo, Feng th/0412308; & Witten th/0501052

- Ingredients
  - Structure of factorization
  - Cauchy's theorem

Introducing Complex Momenta

- Define a shift $[j, l\rangle$ of spinors by a complex parameter z

$$
\begin{aligned}
|j^-\rangle &\rightarrow |j^-\rangle - z|l^-\rangle, \\
|l^+\rangle &\rightarrow |l^+\rangle + z|j^+\rangle
\end{aligned}
$$

- which induces a shift of the external momenta

$$
\begin{aligned}
k_j^\mu &\rightarrow k_j^\mu(z) = k_j^\mu - \frac{z}{2}\langle j^-|\,\gamma^\mu\,|l^-\rangle, \\
k_l^\mu &\rightarrow k_l^\mu(z) = k_l^\mu + \frac{z}{2}\langle j^-|\,\gamma^\mu\,|l^-\rangle
\end{aligned}
$$

- and defines a z-dependent continuation of the amplitude $A(z)$
- Assume that $A(z) \rightarrow 0$ as $z \rightarrow \infty$

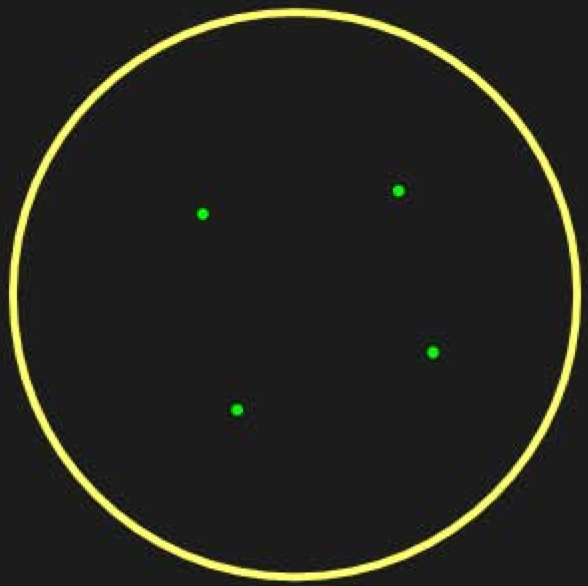
A Contour Integral

Consider the contour integral

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z}\,A(z)$$

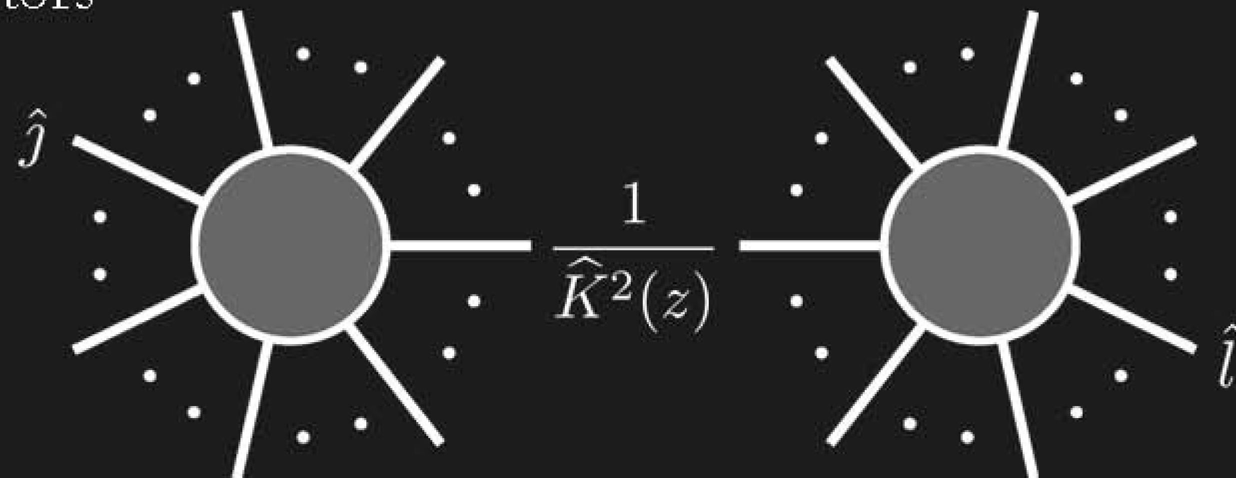
Determine A(0) in terms of other residues

$$A(0) = -\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{A(z)}{z}$$



Using Factorization

Other poles in z come from zeros of z-shifted propagator denominators



Splits diagram into two parts with z-dependent momentum flow

$$\longrightarrow \sum_{\text{partitions}} \begin{array}{c}\text{shifted legs on} \\ \text{opposite sides}\end{array}$$

Exactly factorization limit of z-dependent amplitude poles from zeros of

$$K^2_{a\cdots j\cdots b}(z) = K^2_{a\cdots b} - z\,\langle j^-|\,\not{K}_{a\cdots b}\,|l^-\rangle$$

That is, a pole at

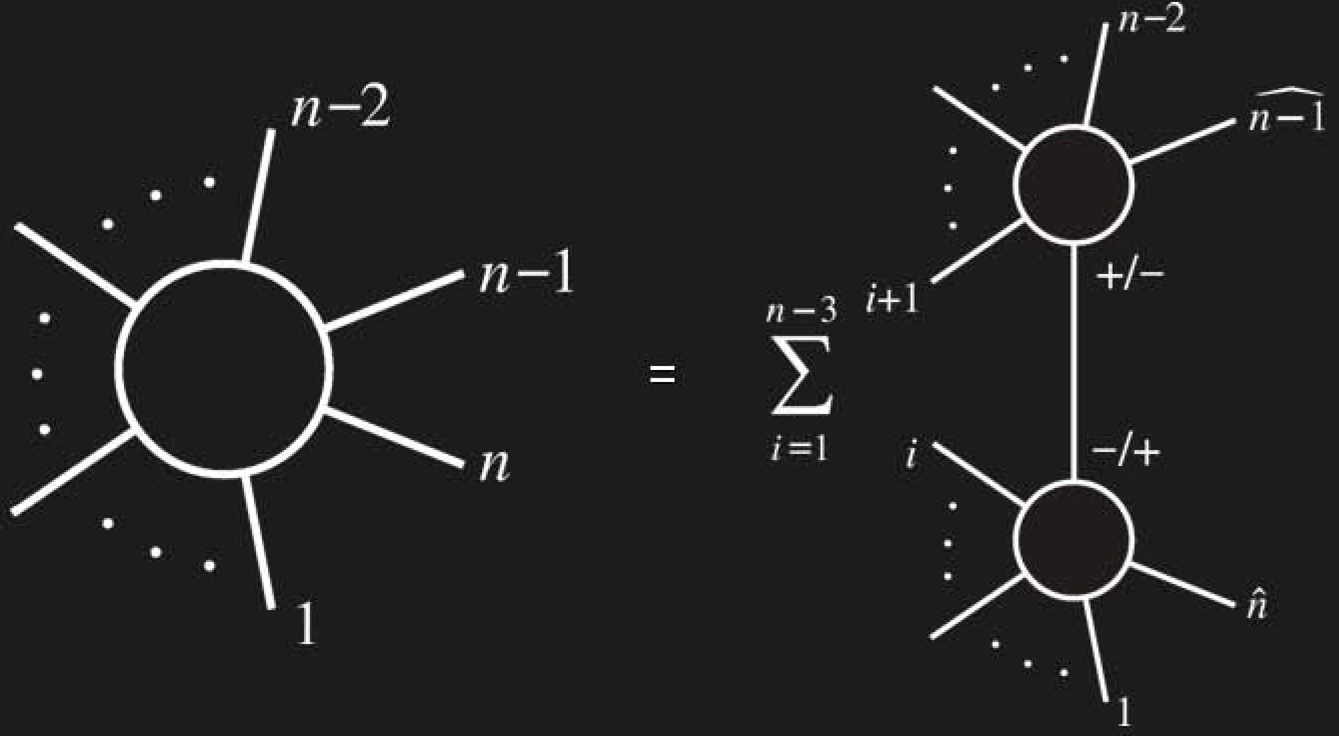
$$z_{ab} = \frac{K^2_{a\cdots b}}{\langle j^-|\,\not{K}_{a\cdots b}\,|l^-\rangle}$$

Residue

$$\operatorname*{Res}_{z=z_{ab}} \frac{f(z)}{z\,K^2_{a\cdots b}(z)} = A_L(z_{ab}) \times \frac{i}{K^2_{a\cdots b}} \times A_R(z_{ab})$$

Notation $\hat{k} = k(z_{ab})$

On-Shell Recursion Relation



- Partition P: two or more cyclicly-consecutive momenta containing j, such that complementary set $\overline{P}$ contains l,

$$
\begin{aligned}
P &\equiv \{P_1, P_2, \ldots, j, \ldots, P_{-1}\}, \\
\overline{P} &\equiv \{\overline{P}_1, \overline{P}_2, \ldots, l, \ldots, \overline{P}_{-1}\}, \\
P \cup \overline{P} &= \{1, 2, \ldots, n\}
\end{aligned}
$$

- The recursion relations are then

On shell

$$
\begin{aligned}
A_n(1, \ldots, n) &= \sum_{\substack{\text{partitions P} \\ \text{h}=\pm}} A_{\#P+1}(k_{P_1}, \ldots, \hat{j}, \ldots, k_{P_{-1}}, -\widehat{P}^{\,h}) \\
&\quad \times \frac{i}{P^2} \times A_{\#\overline{P}+1}(k_{\overline{P}_1}, \ldots, \hat{l}, \ldots, k_{\overline{P}_{-1}}, \widehat{P}^{\,-h})
\end{aligned}
$$

Number of terms ~ $|l-j| \times (n-3)$

so best to choose l and j nearby

Complexity still exponential, because shift changes as we descend the recursion

Applications

- Very general: relies only on complex analysis + factorization
- Fermionic amplitudes
- Applied to gravity

  Bedford, Brandhuber, Spence, & Travaglini (2/2005)

  Cachazo & Svrček (2/2005)
- Massive amplitudes

  Badger, Glover, Khoze, Svrček (4/2005, 7/2005)

  Forde & DAK (7/2005)
- Other rational functions

  Bern, Bjerrum-Bohr, Dunbar, & Ita (7/2005)
- Connection to Cachazo–Svrček–Witten construction

  Risager (8/2005)
- CSW construction for gravity

  Bjerrum-Bohr, Dunbar, Ita, Perkins, & Risager (9/2005)