On-Shell Methods in Field Theory

David A. Kosower

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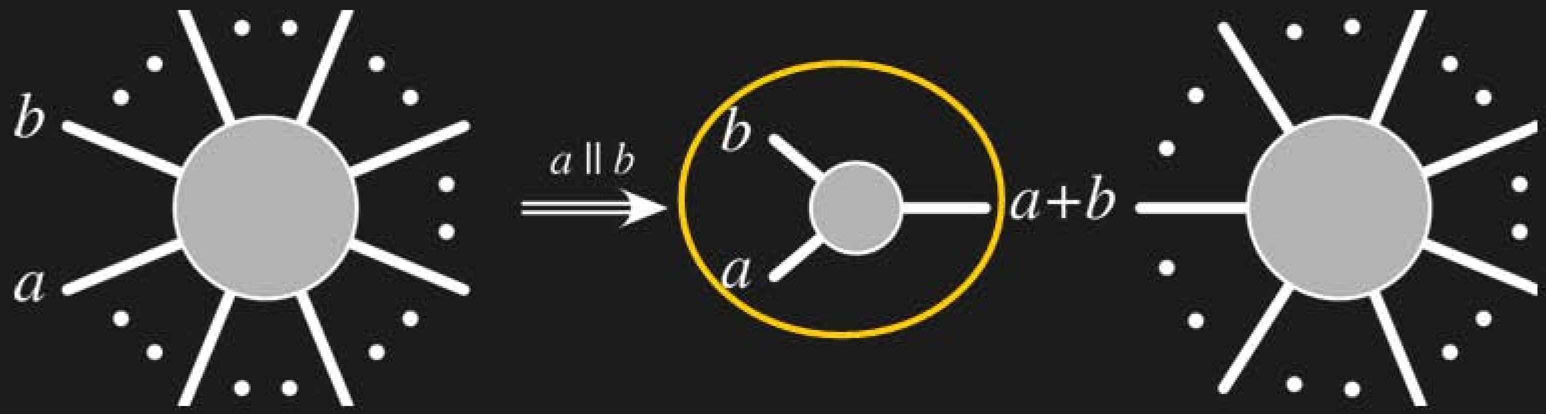
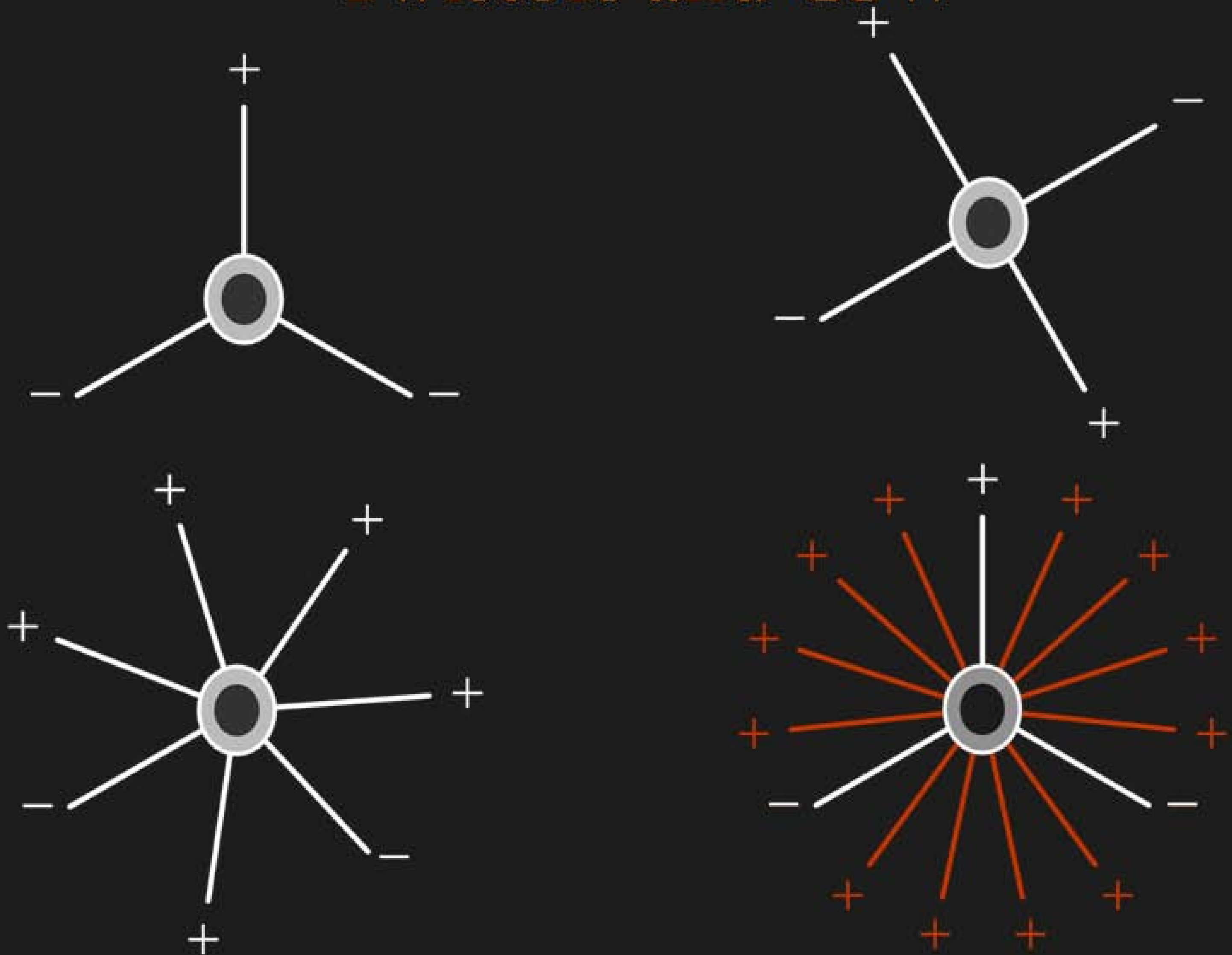
Lecture III

Factorization in Gauge Theory

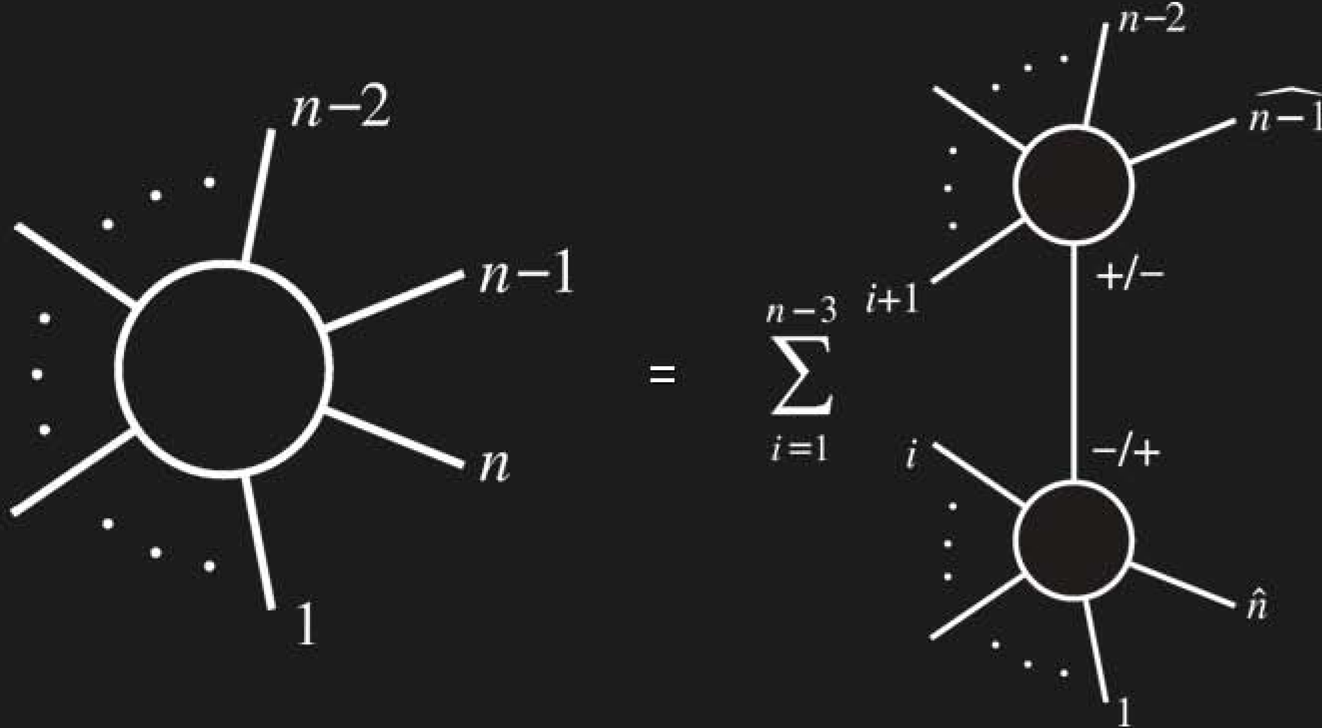


Collinear limits with splitting amplitudes

Twistors and CSW



On-Shell Recursion Relation



Three-Gluon Amplitude Revisited

Let's compute it with complex momenta chosen so that

$$|1^-\rangle \propto |2^-\rangle \propto |3^-\rangle$$

that is,

$$[1\,2] = [2\,3] = [3\,1] = 0$$

but

$$\langle 1\,2\rangle \neq 0,\ \langle 2\,3\rangle \neq 0,\ \langle 3\,1\rangle \neq 0$$

compute $A_3(1^-, 2^-, 3^+)$

$A_3(1^-, 2^-, 3^+)$

Choose common reference momentum q

$\varepsilon_1 \cdot \varepsilon_2 = 0$ so we have to compute

$$\sqrt{2}i\left[\varepsilon_2 \cdot \varepsilon_3\, k_2 \cdot \varepsilon_1 + \varepsilon_3 \cdot \varepsilon_1\, k_3 \cdot \varepsilon_2\right]$$

$$= i\frac{\langle q\,2\rangle\,[q\,3]}{[2\,q]\,\langle q\,3\rangle}\frac{[q\,2]\,\langle 2\,1\rangle}{[1\,q]} + i\frac{\langle q\,1\rangle\,[q\,3]}{[1\,q]\,\langle q\,3\rangle}\frac{[q\,3]\,\langle 3\,2\rangle}{[2\,q]}$$

$$= i\frac{[q\,3]}{\langle q\,3\rangle}\left[\frac{\langle q\,2\rangle\,\langle 2\,1\rangle}{[1\,q]} - \frac{\langle q\,1\rangle\,\langle 1\,2\rangle}{[2\,q]}\right]$$

$$= -2i\frac{\langle 1\,2\rangle\,[q\,3]\,q \cdot (k_1 + k_2)}{\langle q\,3\rangle\,[q\,1]\,[q\,2]}$$

$$= i\frac{\langle 1\,2\rangle\,[q\,3]^2}{[q\,1]\,[q\,2]}$$

Not manifestly gauge invariant

$$i\frac{\langle 1\,2\rangle\,[q\,3]^2}{[q\,1]\,[q\,2]}$$

$$= -i\frac{\langle 1\,2\rangle\,[q\,3]\,\langle 3\,2\rangle\,[q\,3]\,\langle 3\,1\rangle}{\langle 2\,3\rangle\,\langle 3\,1\rangle\,[q\,1]\,[q\,2]}$$

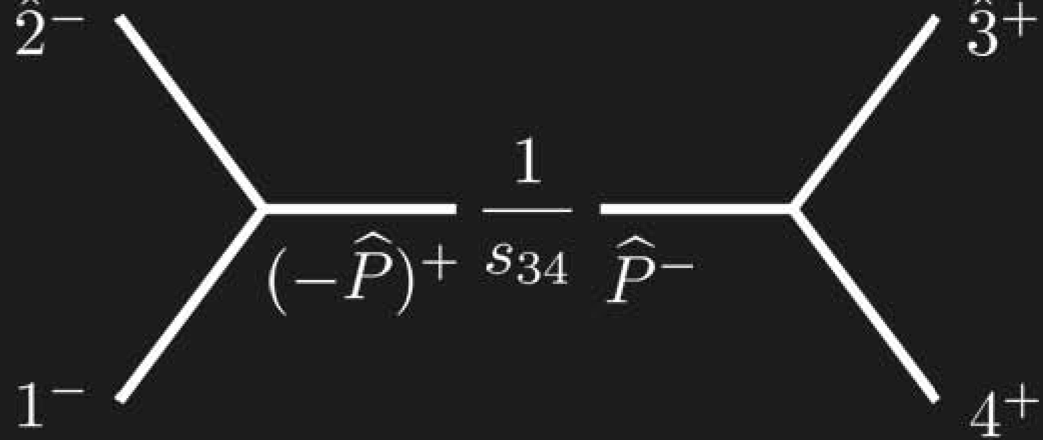
$$= i\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,1\rangle}$$

but gauge invariant nonetheless,

and exactly the $n=3$ case of the general Parke–Taylor formula!

Four-Point Example

Pick a $[2,3\rangle$ shift, giving one diagram



$$i\frac{\langle 1\,2\rangle^3}{\langle 2\,(-\widehat{P}\,)\rangle\,\langle(-\widehat{P}\,)\,1\rangle} \times \frac{i}{s_{34}} \times (-i)\frac{[3\,4]^3}{[\widehat{P}\;3]\,[4\,\widehat{P}\,]}$$

$$= -i \frac{\langle 1\,2\rangle^3\,[3\,4]^2}{\langle 4\,3\rangle \left\langle 1^- \middle| \widehat{P} \middle| 3^- \right\rangle \left\langle 2^- \middle| \widehat{P} \middle| 4^- \right\rangle}$$

$$= -i \frac{\langle 1\,2\rangle^3\,[3\,4]^2}{\langle 4\,3\rangle \left\langle 1^- \middle| P \middle| 3^- \right\rangle \left\langle 2^- \middle| P \middle| 4^- \right\rangle}$$

$$\left\langle 2^- \middle| \gamma^\mu \middle| 3^- \right\rangle$$

$$= i \frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}$$

Choosing Shift Momenta

- What are legitimate choices?
- Need to ensure that $A(z) \to 0$ as $z \to \infty$
- At tree level, legitimate choices

$$[-,+\rangle,\ [+,+\rangle,\ [-,-\rangle$$

- Power counting argument in Feynman diagrams for $[-,+\rangle$

$$\varepsilon_{\hat{j}}^{(-)} = \frac{\langle j^- | \gamma^\mu | q^- \rangle}{\sqrt{2}\left[\hat{j}\, q\right]} \sim \frac{1}{z}$$

$$\varepsilon_{\hat{l}}^{(+)} = \frac{\langle q^- | \gamma^\mu | l^- \rangle}{\sqrt{2}\left\langle q\, \hat{l}\right\rangle} \sim \frac{1}{z}$$

Three-point vertices with z-dependent momentum flow ~ z

Four-point vertices with z-dependent momentum flow ~ 1

Propagators with z-dependent momentum flow ~ $1/z$

⇒ Leading contributions from diagrams with only three-point vertices and propagators connecting j to l: ~ $1/z$

(one more vertex than propagators & two εs)

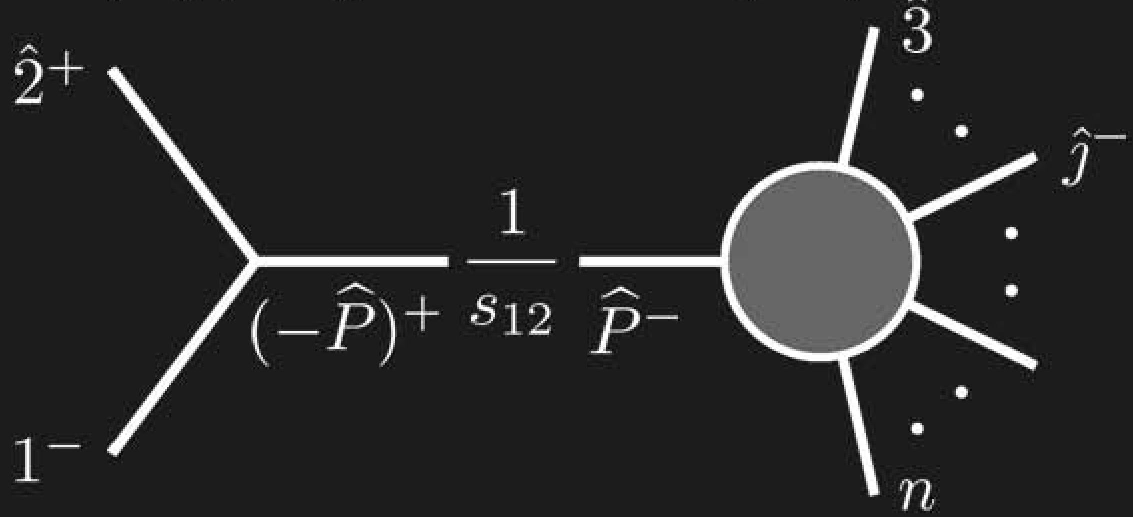
Factorization in Complex Momenta

- Factorization theorems derived for real momenta
- For multiparticle poles, hold for complex momenta as well

- At tree level, collinear factorization holds for complex momenta as well, because splitting amplitudes only involve 1/spinor product, so we only get pure single poles
- Double poles cannot arise because each propagator can only give rise to a single invariant in the denominator

MHV Amplitudes

Compute the $(1^-, j^-)$ amplitude: choose $[3,2\rangle$ shift



Other diagrams vanish because $A_n(+\cdots+\pm)=0$ or $A_3(\hat{3}^+, 4^+, -\widehat{P}^{\pm})=0$

$$A_3(1^-, \hat{2}^+, -\widehat{P}^+) \frac{i}{s_{12}} A_{n-1}(\widehat{P}^-, \hat{3}^+, \ldots, j^-, \ldots, n^+)$$

$$= i \frac{[2\,(-\widehat{P}\,)]^3}{[(-\widehat{P}\,)\,1]\,[1\,2]} \frac{1}{s_{12}} \frac{\langle \widehat{P}\; j\rangle^4}{\langle \widehat{P}\; 3\rangle\,\langle 3\,4\rangle \cdots \langle (n-1)\,n\rangle\,\langle n\,\widehat{P}\,\rangle}$$

$$= i \frac{\left\langle j^- \right| \widehat{P} \left| 2^- \right\rangle^4 \langle 2\,3\rangle\,\langle n\,1\rangle}{[1\,2]^2\,\left\langle 3^- \right| \widehat{P} \left| 1^- \right\rangle \left\langle n^- \right| \widehat{P} \left| 2^- \right\rangle} \frac{1}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle (n-1)\,n\rangle\,\langle n\,1\rangle}$$

$$= i \frac{\langle 1\,j\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle (n-1)\,n\rangle\,\langle n\,1\rangle}$$

- Prove Parke–Taylor equation by induction

CSW From Recursion

Risager, th/0508206

Consider NMHV amplitude: 3 negative helicities m_1, m_2, m_3, any number of positive helicities

Choose shift

$$\begin{aligned} |m_1^-\rangle &\rightarrow |m_1^-\rangle + z\,\langle m_2\,m_3\rangle\,|q^-\rangle \\ |m_2^-\rangle &\rightarrow |m_2^-\rangle + z\,\langle m_3\,m_1\rangle\,|q^-\rangle \\ |m_3^-\rangle &\rightarrow |m_3^-\rangle + z\,\langle m_1\,m_2\rangle\,|q^-\rangle \end{aligned}$$

Momenta are still on shell, and

$$\delta(k_1 + k_2 + k_3)^\mu =$$

$$\left(\langle m_2\,m_3\rangle\,\langle m_1^-| + \langle m_3\,m_1\rangle\,\langle m_2^-| + \langle m_1\,m_2\rangle\,\langle m_3^-|\right)\gamma^\mu|q^-\rangle = 0$$

because of the Schouten identity

- **z-dependent** momentum flow comes from configurations with one minus helicity on one amplitude, two on the other

$$A(1^+,\ldots,m_1^-,\ldots,m_2^-,\ldots,-\widehat{P}^{\,+})\frac{1}{P^2}A(\widehat{P}^{\,-},\ldots,m_3^-,\ldots,n^+)$$

- MHV × MHV

- For more negative helicities, proceed recursively or solve globally for shifts using Schouten identity that yield a complete factorization ⇒ CSW construction

- Can be applied to gravity too!

Bjerrum-Bohr, Dunbar, Ita, Perkins & Risager, th/0509016

Singularity Structure

- On-shell recursion relations lead to compact analytic expression
- Different form than Feynman-diagram computation
- Appearance of spurious singularities

$$A_6(1^-,2^-,3^-,4^+,5^+,6^+) =$$

$$\frac{1}{\langle 5^- | 3+4 | 2^- \rangle}\left(\frac{\langle 1^- | 2+3 | 4^- \rangle^3}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\, s_{234}} + \frac{\langle 3^- | 4+5 | 6^- \rangle^3}{[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\, s_{345}}\right)$$

unphysical singularity — cancels between terms

physical singularities

Review of Supersymmetry

- Equal number of bosonic and fermionic degrees of freedom
- Only local extension possible of Poincaré invariance
- Extended supersymmetry: only way to combine Poincaré invariance with internal symmetry
- Poincaré algebra

$$[P_\mu, P_\nu] = 0$$
$$[P_\rho, M_{\mu\nu}] = \eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu$$
$$[M_{\mu\nu}, M_{\rho\lambda}] = -\eta_{\mu\rho} M_{\nu\lambda} - \eta_{\nu\lambda} M_{\mu\rho} + \eta_{\mu\lambda} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\lambda}$$

- Supersymmetry algebra is graded, that is uses both commutators and anticommutators. For **N**=1, there is one supercharge Q, in a spin-½ representation (and its conjugate)

$$\{Q_a, \overline{Q}_{\dot{a}}\} = 2\sigma^{\mu}_{a\dot{a}} P_\mu$$
$$[P_\mu, Q_a] = [P_\mu, \overline{Q}_{\dot{a}}] = 0$$
$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu})_a{}^b Q_b$$
$$[\overline{Q}_{\dot{a}}, M_{\mu\nu}] = (\overline{\sigma}_{\mu\nu})_{\dot{a}}{}^{\dot{b}} \overline{Q}_{\dot{b}}$$
$$\{Q_a, Q_b\} = \{\overline{Q}_{\dot{a}}, \overline{Q}_{\dot{b}}\} = 0$$

- There is also an R symmetry, a U(1) charge that distinguishes between particles and superpartners

Supersymmetric Gauge Theories

- $\mathcal{N}=1$: gauge bosons + Majorana fermions, all transforming under the adjoint representation
- $\mathcal{N}=4$: gauge bosons + 4 Majorana fermions + 6 real scalars, all transforming under the adjoint representation

Supersymmetry Ward Identities

- Color-ordered amplitudes don't distinguish between quarks and gluinos ⇒ same for QCD and **N**=1 SUSY
- Supersymmetry should relate amplitudes for different particles in a supermultiplet, such as gluons and gluinos
- Supercharge annihilates vacuum

$$\langle 0|[Q, \Phi_1 \Phi_2 \cdots \Phi_n]|0\rangle = 0 = \sum_{i=1}^{n} \langle 0|\Phi_1 \cdots [Q, \Phi_i] \cdots \Phi_n|0\rangle$$

Grisaru, Pendleton & van Nieuwenhuizen (1977)

- Use a practical representation of the action of supersymmetry on the fields. Multiply by a spinor wavefunction & Grassman parameter θ

$$[Q, G^{\pm}(k)] = \pm\Gamma^{\pm}(k,q)\Lambda^{\pm}(k)$$
$$[Q, \Lambda^{\pm}(k)] = \mp\Gamma^{\mp}(k,q)G^{\pm}(k)$$

- where $\Gamma^{-} = \theta\,[q\,k]\,,\ \Gamma^{+} = \theta\,\langle q\,k\rangle$
- With explicit helicity choices, we can use this to obtain equations relating different amplitudes
- Typically start with Q acting on an 'amplitude' with an *odd* number of fermion lines (overall a bosonic object)

Supersymmetry WI in Action

- All helicities positive:

$$
\begin{aligned}
0 &= \langle 0|[Q_q, \Lambda_1^+ G_2^+ \cdots G_n^+]|0\rangle \\
&= -\Gamma^-(k_1,q) A_n^{\rm tree}(1^+,\ldots,n^+) \\
&\qquad + \Gamma^+(k_2,q) A_n^{\rm tree}(1_\Lambda^+, 2_\Lambda^+, 3^+,\ldots,n^+) \\
&\qquad + \cdots + \Gamma^+(k_n,q) A_n^{\rm tree}(1_\Lambda^+, 2^+,\ldots,(n-1)^+, n_\Lambda^+)
\end{aligned}
$$

- Helicity conservation implies that the fermionic amplitudes vanish

$$
0 = -\Gamma^-(k_1,q) A_n^{\rm tree}(1^+,\ldots,n^+)
$$

- so that we obtain the first Parke–Taylor equation

- With two negative helicity legs, we get a non-vanishing relation

$$0 = \langle 0|[Q_q, \Lambda_1^+ G_2^- G_3^+ \cdots G_j^- G_{j+1}^+ \cdots G_n^+]|0\rangle$$
$$= -\Gamma^-(k_1, q) A_n^{\rm tree}(1^+, 2^-, \ldots, n^+)$$
$$- \Gamma^+(k_2, q) A_n^{\rm tree}(1_\Lambda^+, 2_\Lambda^-, 3^+, \ldots, j^-, \ldots, n^+)$$
$$- \Gamma^+(k_j, q) A_n^{\rm tree}(1_\Lambda^+, 2^-, 3^+, \ldots, j_\Lambda^-, \ldots, n^+)$$

- Choosing $q = k_2$

$$A_n^{\rm tree}(1_\Lambda^+, 2^-, 3^+, \ldots, j_\Lambda^-, \ldots, n^+) =$$
$$-\frac{\langle 1\, 2\rangle}{\langle 2\, j\rangle} A_n^{\rm tree}(1^+, 2^-, 3^+, \ldots j^-, \ldots, n^+)$$

- Tree-level amplitudes with external gluons or one external fermion pair are given by supersymmetry even in QCD.
- Beyond tree level, there are additional contributions, but the Ward identities are still useful.
- For supersymmetric theories, they hold to all orders in perturbation theory