On-Shell Methods in Field Theory

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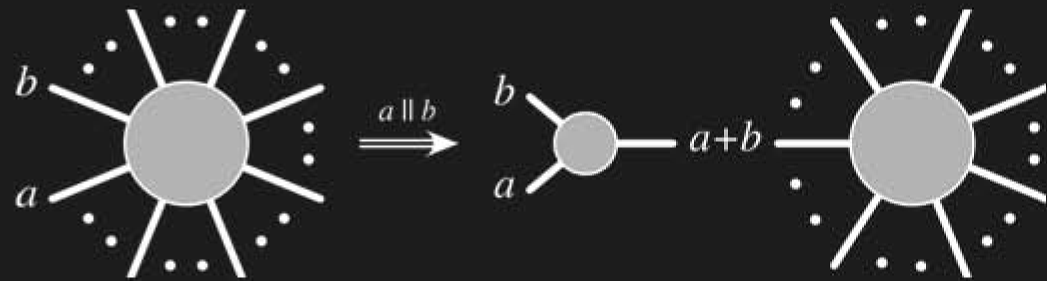
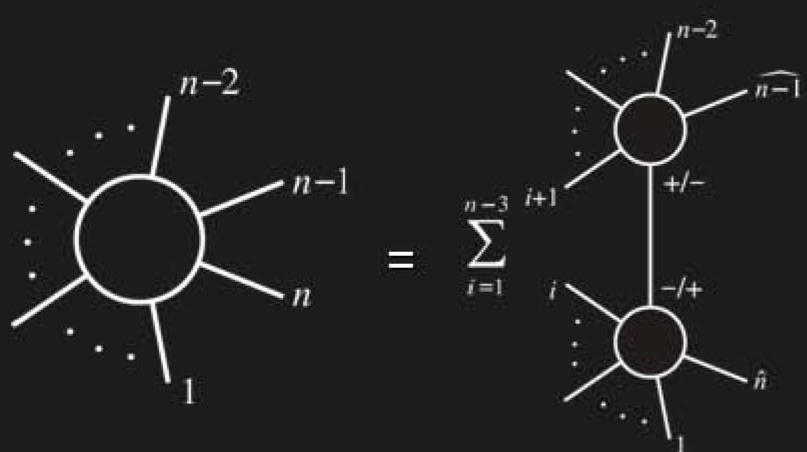
Lecture IV

- Property: tree-level factorization



$\Rightarrow$ Computational tool at tree level: on-shell recursion relations



Loop Calculations: Textbook Approach

- Sew together vertices and propagators into loop diagrams
- Obtain a sum over [2,n]-point [0,n]-tensor integrals, multiplied by coefficients which are functions of k and ε
- Reduce tensor integrals using Brown-Feynman & Passarino-Veltman brute-force reduction, or perhaps Vermaseren-van Neerven method
- Reduce higher-point integrals to bubbles, triangles, and boxes

- Can apply this to color-ordered amplitudes, using color-ordered Feynman rules
- Can use spinor-helicity method at the end to obtain helicity amplitudes

BUT

- This fails to take advantage of gauge cancellations early in the calculation, so a lot of calculational effort is just wasted.

Can We Take Advantage?

- Of tree-level techniques for reducing computational effort?
- Of any other property of the amplitude?

Unitarity

- Basic property of any quantum field theory: conservation of probability. In terms of the scattering matrix,

$$S^\dagger S = 1$$

In terms of the transfer matrix $iT = S - 1$ we get,

$$-i(T - T^\dagger) = T^\dagger T$$

or

$$2\ \text{“Im”}\ T_{fi} = (T^\dagger T)_{fi}$$

with the Feynman $i\varepsilon$

$$\text{Disc}\, T = T^\dagger T$$

This has a direct translation into Feynman diagrams, using the *Cutkosky* rules. If we have a Feynman integral,

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2 + i\delta} \cdots \frac{1}{(\ell - K)^2 + i\delta}$$

and we want the discontinuity in the K^2 channel, we should replace

$$\frac{1}{\ell^2 + i\delta} \longrightarrow -2\pi i \delta^{(+)}(\ell^2)$$

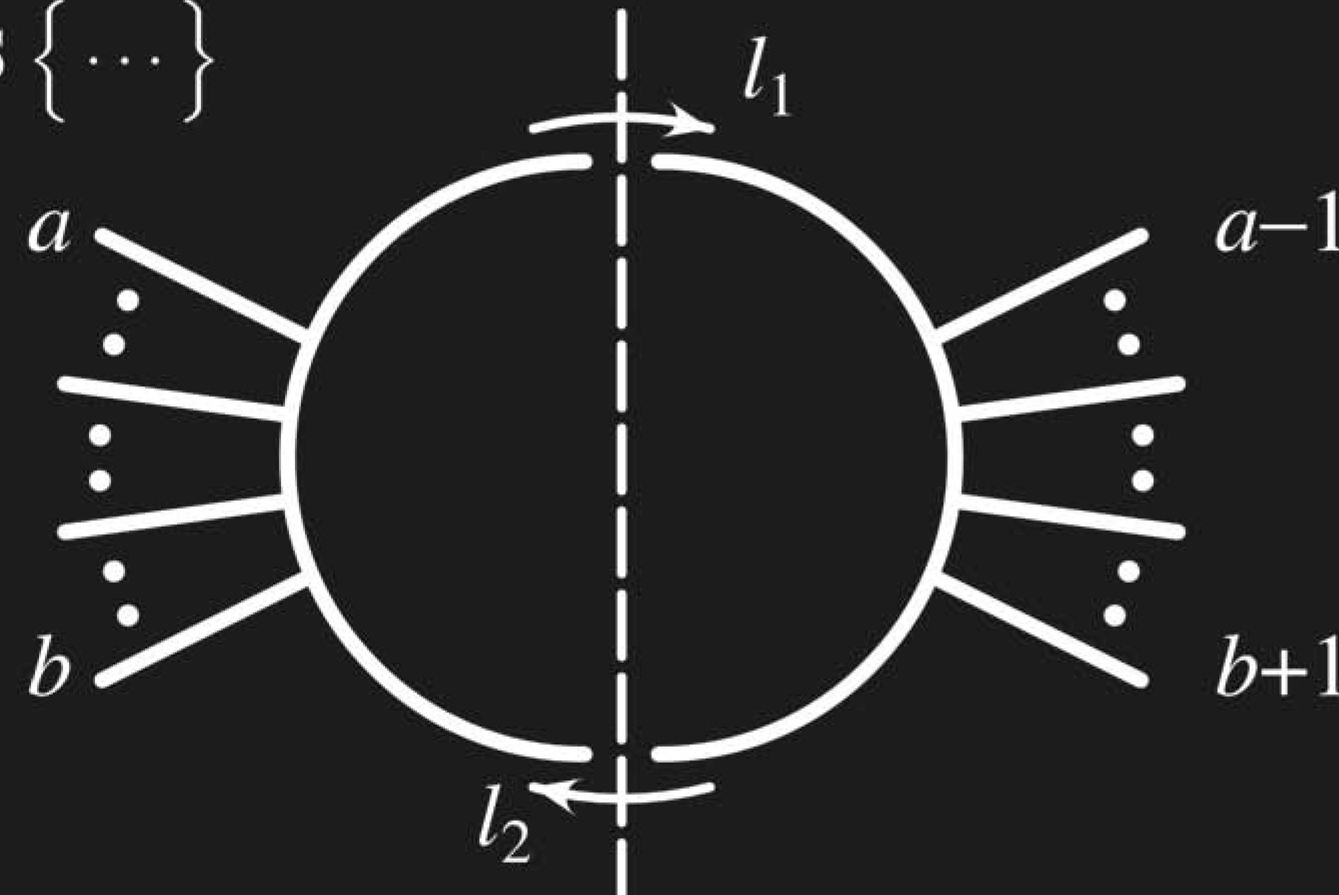
$$\frac{1}{(\ell - K)^2 + i\delta} \longrightarrow -2\pi i \delta^{(+)}((\ell - K)^2)$$

$$\delta^{(+)}(k^2) = \Theta(k^0)\delta(k^2)$$

- When we do this, we obtain a phase-space integral

$$\int \frac{d^D\ell}{(2\pi)^{D-1}}\, \delta^{(+)}(\ell^2)\delta^{(+)}((\ell-K)^2)\left\{\cdots\right\} =$$

$$\int d^D\mathrm{LIPS}\left\{\cdots\right\}$$



In the Bad Old Days of Dispersion Relations

- To recover the full integral, we could perform a dispersion integral

$$\mathrm{Re}\, f(s) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dw\, \frac{\mathrm{Im}\, f(w)}{w - s} + \mathrm{Re}\, C_\infty$$

in which $C_\infty = 0$ so long as $f(w) \to 0$ when $w \to \infty$

- If this condition isn't satisfied, there are 'subtraction' ambiguities corresponding to terms in the full amplitude which have no discontinuities

- But it's better to obtain the full integral by identifying which Feynman integral(s) the cut came from.

- Allows us to take advantage of sophisticated techniques for evaluating Feynman integrals: identities, modern reduction techniques, differential equations, reduction to master integrals, etc.

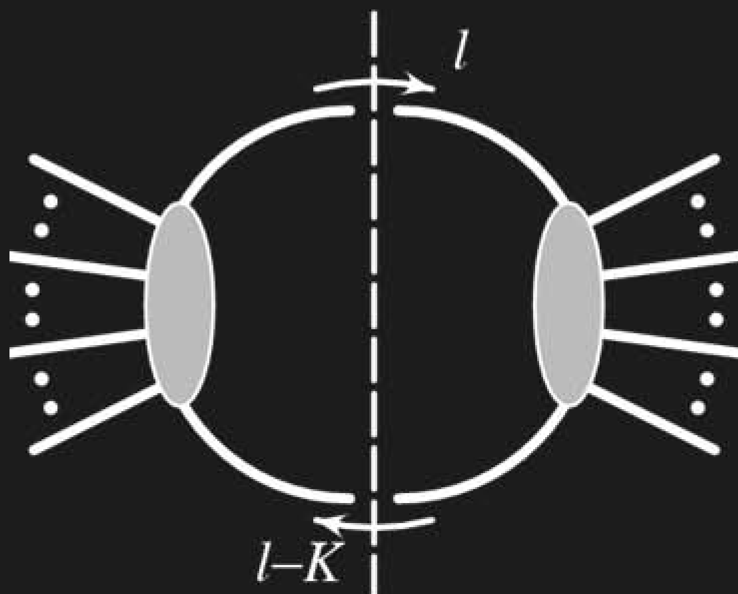
Computing Amplitudes *Not* Diagrams

- The cutting relation can also be applied to sums of diagrams, in addition to single diagrams
- Looking at the cut in a given channel s of the sum of all diagrams for a given process throws away diagrams with no cut — that is diagrams with one or both of the required propagators missing — and yields the sum of all diagrams on each side of the cut.
- Each of those sums is an on-shell tree amplitude, so we can take advantage of all the advanced techniques we've seen for computing them.

Unitarity-Based Method at One Loop

- Compute cuts in a set of channels
- Compute required tree amplitudes
- Form the phase-space integrals
- Reconstruct corresponding Feynman integrals
- Perform integral reductions to a set of master integrals
- Assemble the answer

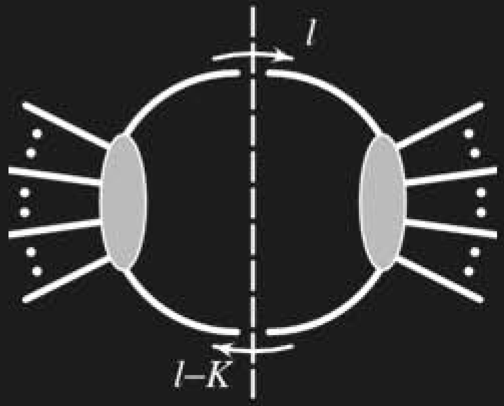
Unitarity-Based Calculations



Bern, Dixon, Dunbar, & DAK,
ph/9403226, ph/9409265

$$A^{\text{1-loop}} = \sum_{\text{cuts } K^2} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} A^{\text{tree}}_{\text{left}} \frac{i}{(\ell-K)^2} A^{\text{tree}}_{\text{right}}$$

Unitarity-Based Calculations



- In general, work in D=4-2ε ⇒ full answer

  van Neerven (1986): dispersion relations converge

- At one loop in D=4 for SUSY ⇒ full answer

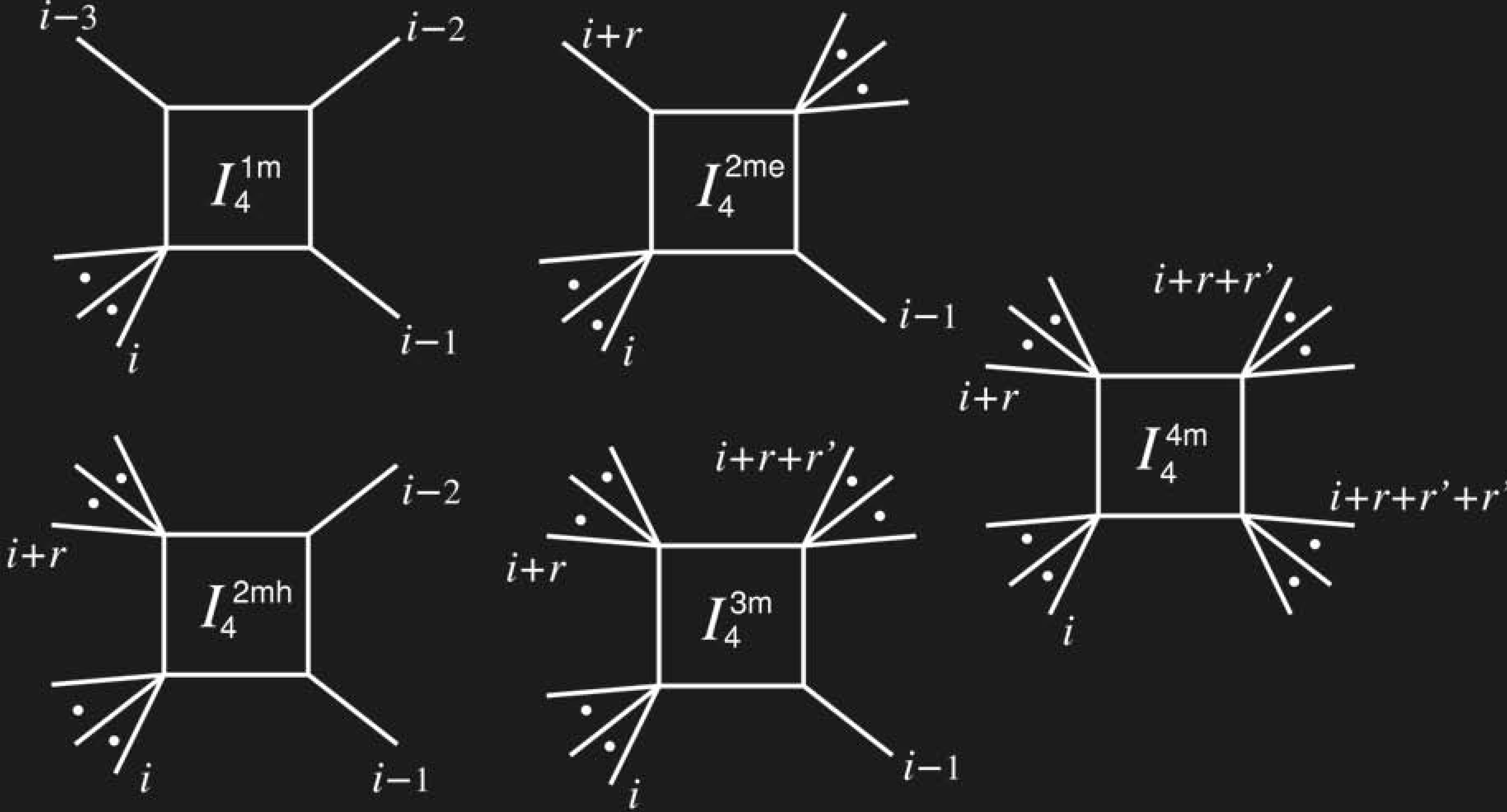
- Merge channels rather than blindly summing: find function w/given cuts in all channels

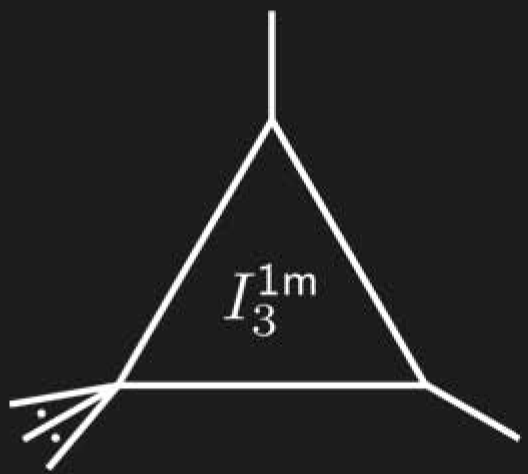
The Three Roles of Dimensional Regularization

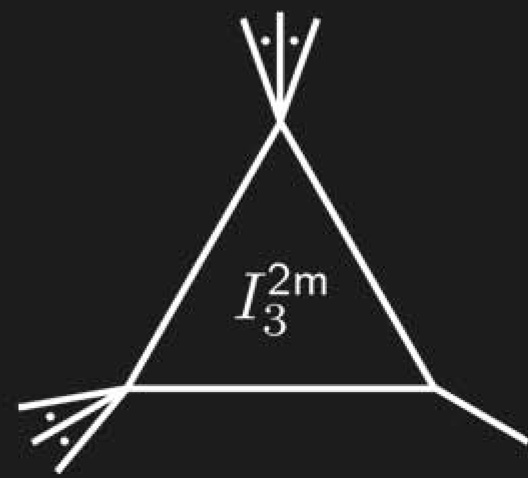
- Ultraviolet regulator;
- Infrared regulator;
- Handle on rational terms.
- Dimensional regularization effectively removes the ultraviolet divergence, rendering integrals convergent, and so removing the need for a subtraction in the dispersion relation
- Pedestrian viewpoint: dimensionally, there is always a factor of $(-s)^{-\epsilon}$, so at higher order in ϵ, even rational terms will have a factor of $\ln(-s)$, which has a discontinuity

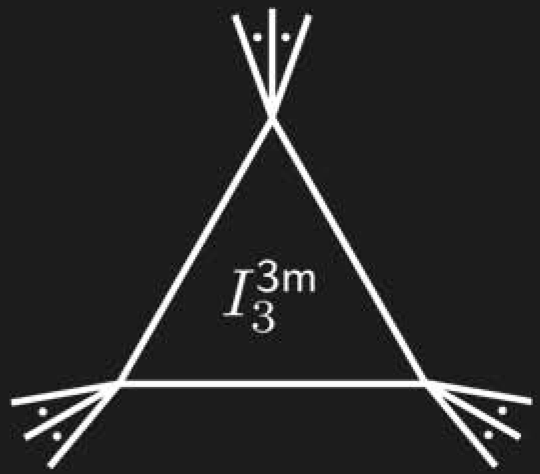
Integral Reductions

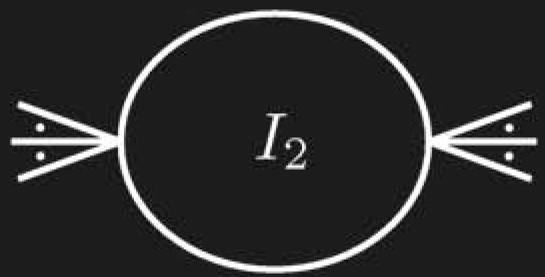
- At one loop, all $n \geq 5$-point amplitudes in a massless theory can be written in terms of nine different types of scalar integrals:
- boxes (one-mass, 'easy' two-mass, 'hard' two-mass, three-mass, and four-mass);
- triangles (one-mass, two-mass, and three-mass);
- bubbles
- In an $\mathcal{N}=4$ supersymmetric theory, only boxes are needed.


i−3
i−2
I_4^{1m}
i
i−1
i+r
I_4^{2me}
i
i−1
i+r+r'
i+r
I_4^{4m}
i+r+r'+r''
i
i−2
i+r
I_4^{2mh}
i
i−1
i+r+r'
i+r
I_4^{3m}
i
i−1


I_3^{1m}


I_3^{2m}


I_3^{3m}


I_2

The Easy Two-Mass Box

$$\int \frac{d^D\ell}{(2\pi)^D}\, \frac{1}{\ell^2(\ell-k_1)^2(\ell-K_{12})^2(\ell-K_{123})^2} =$$

$$\frac{c_\Gamma(\epsilon)}{st - m_2^2 m_4^2}\left\{\frac{2}{\epsilon^2}\left[(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-m_2^2)^{-\epsilon} - (-m_4^2)^{-\epsilon}\right]\right.$$

$$-2\,\mathrm{Li}_2\left(1-\frac{m_2^2}{s}\right) - 2\,\mathrm{Li}_2\left(1-\frac{m_2^2}{t}\right)$$

$$-2\,\mathrm{Li}_2\left(1-\frac{m_4^2}{s}\right) - 2\,\mathrm{Li}_2\left(1-\frac{m_4^2}{t}\right)$$

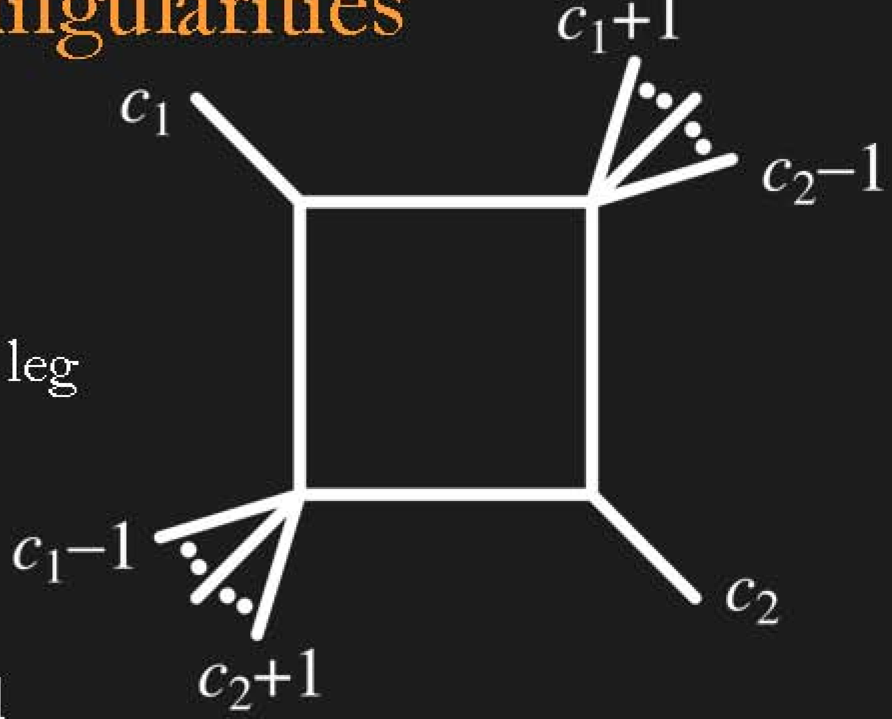
$$\left.+2\,\mathrm{Li}_2\left(1-\frac{m_2^2 m_4^2}{st}\right) - \ln^2\left(\frac{s}{t}\right)\right\} + \mathcal{O}(\epsilon)$$

Dilogarithm $\mathrm{Li}_2(x) = -\int_0^x dt\, \frac{\ln(1-t)}{t}$

Infrared Singularities

Loop momentum nearly on shell
and soft
or collinear with massless external leg
or both

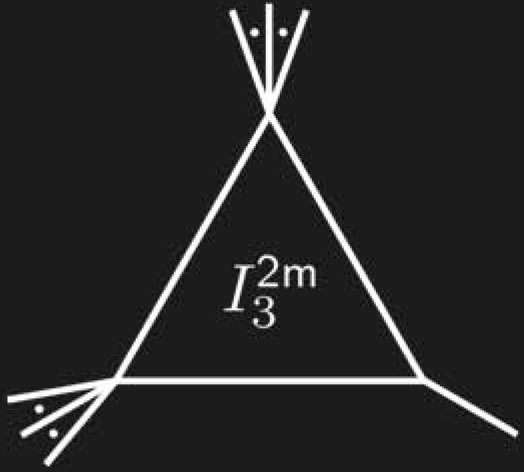
Coefficients of infrared poles and double poles must be proportional to the tree amplitude for cancellations to happen



Spurious Singularities

- When evaluating the two-mass triangle, we will obtain functions like

$$\frac{\ln(s_1/s_2)}{s_1 - s_2}$$



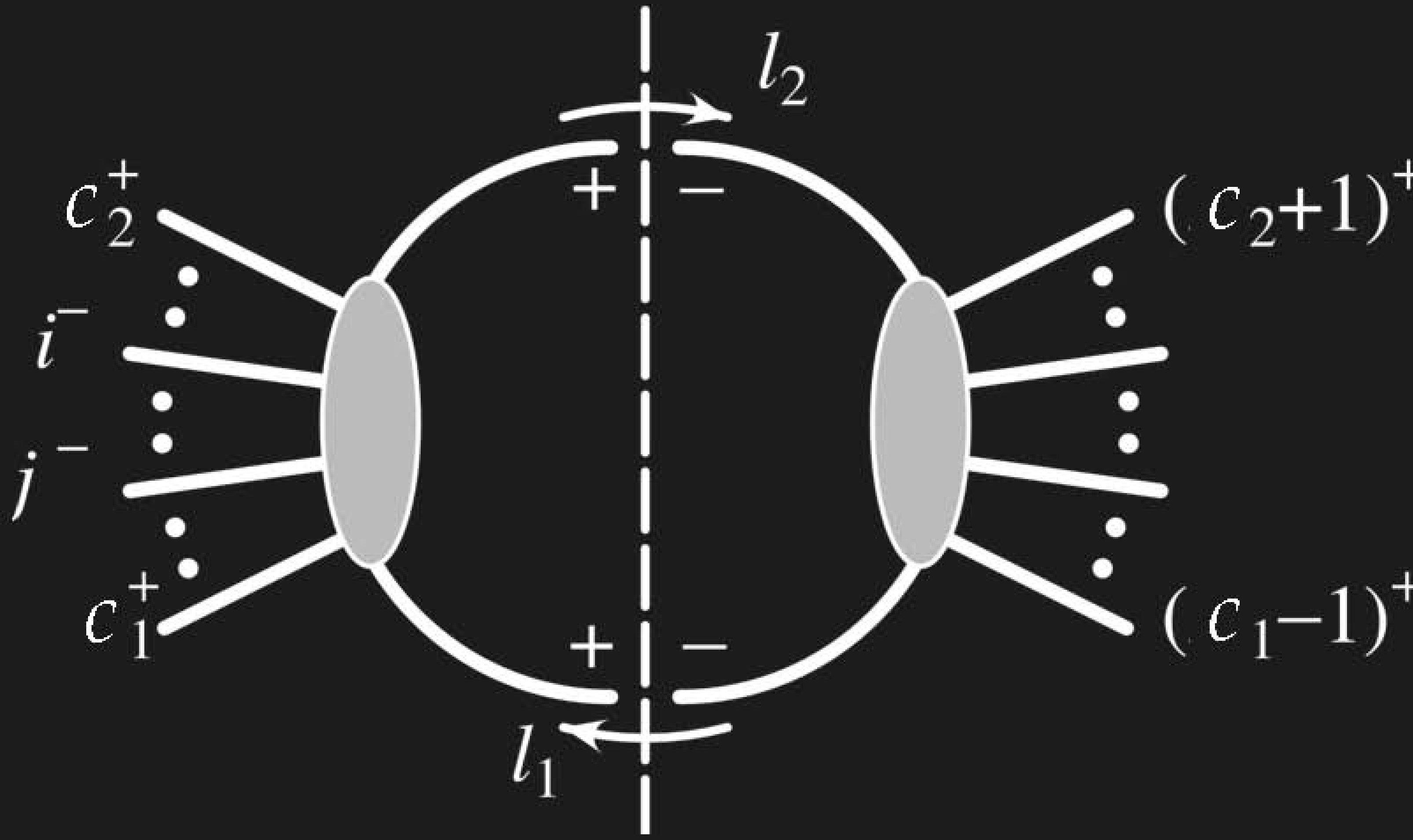
and $\frac{\ln(s_1/s_2)}{(s_1 - s_2)^2} + \frac{1}{s_1 - s_2}$

There can be no physical singularity as $s_1 \to s_2$

and there isn't

but cancellation happens non-trivially

Example: MHV at One Loop



- Start with the cut

$$\int d^4\,\mathrm{LIPS}(\ell_1,-\ell_2)\;A^{\mathrm{tree}}(-\ell_2,c_2{+}1,\ldots,c_1{-}1,\ell_1)$$
$$\times\,A^{\mathrm{tree}}(-\ell_1,c_1,\ldots,c_2,\ell_2)$$

- Use the known expressions for the MHV amplitudes

$$-\int d^4\,\mathrm{LIPS}(\ell_1,-\ell_2)\;\frac{\langle(-\ell_1)\,\ell_2\rangle^3}{\langle(-\ell_1)\,c_1\rangle\,\langle\langle c_1\cdots c_2\rangle\rangle\,\langle c_2\,\ell_2\rangle}$$
$$\times\,\frac{\langle i\,j\rangle^4}{\langle(-\ell_2)\,(c_2{+}1)\rangle\,\langle\langle(c_2{+}1)\cdots(c_1{-}1)\rangle\rangle\,\langle(c_1{-}1)\,\ell_1\rangle\,\langle\ell_1\,(-\ell_2)\rangle}$$

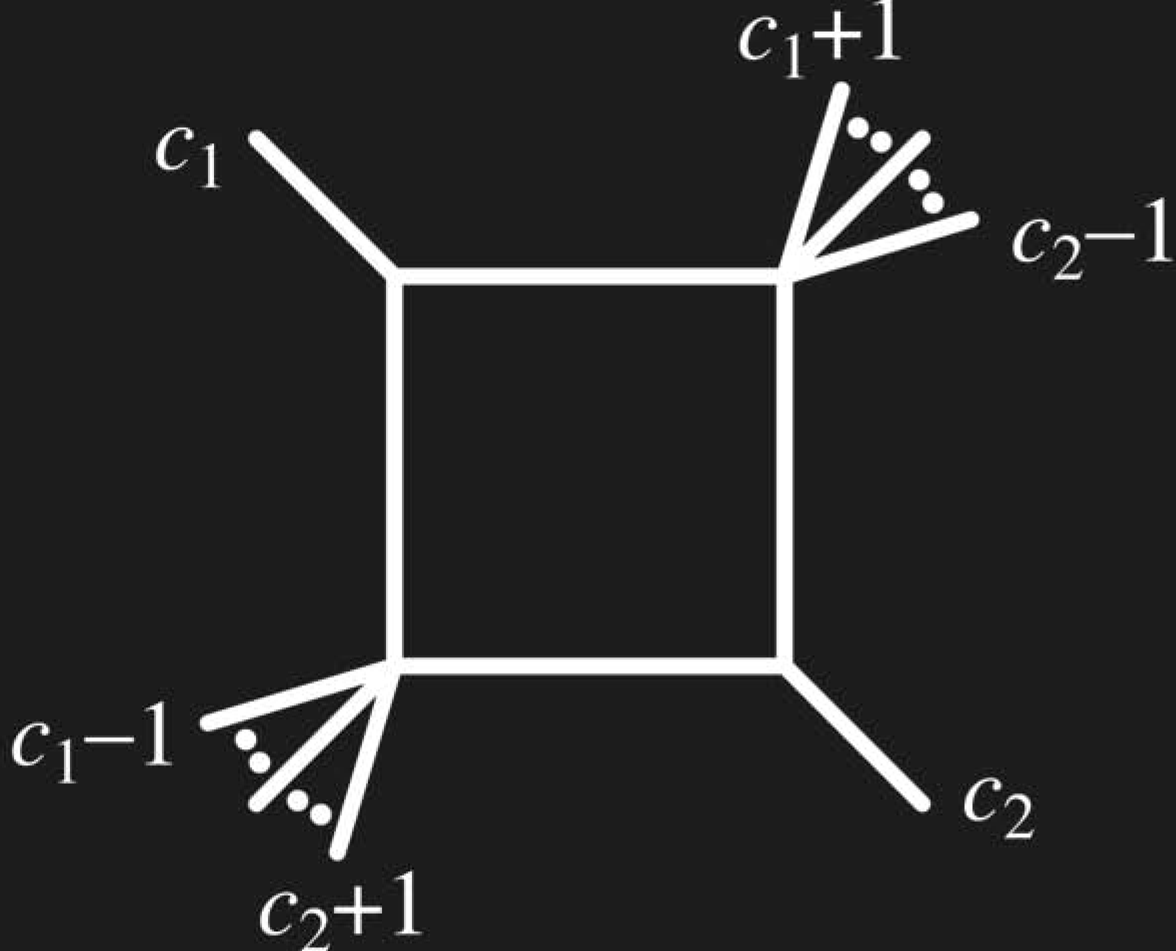
- Most factors are independent of the integration momentum

$$iA^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$
$$\times \int d^4\,{\rm LIPS}(\ell_1,-\ell_2)\;\frac{\langle (c_1\!-\!1)\,c_1\rangle\,\langle c_2\,(c_2\!+\!1)\rangle\,\langle \ell_1\,\ell_2\rangle^2}{\langle \ell_1\,c_1\rangle\,\langle c_2\,\ell_2\rangle\,\langle (c_1\!-\!1)\,\ell_1\rangle\,\langle \ell_2\,(c_2\!+\!1)\rangle}$$
$$= iA^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$
$$\times \int d^4\,{\rm LIPS}(\ell_1,-\ell_2)\;\langle (c_1\!-\!1)\,c_1\rangle\,\langle \ell_1\,\ell_2\rangle^2\,\langle c_2\,(c_2\!+\!1)\rangle$$
$$\times\;\frac{[c_1\,\ell_1]\,[\ell_2\,c_2]\,[(c_1\!-\!1)\,\ell_1]\,[\ell_2\,(c_2\!+\!1)]}{(\ell_1-k_{c_1})^2(\ell_2+k_{c_2})^2(\ell_1+k_{c_1-1})^2(\ell_2-k_{c_2+1})^2}$$

- We can use the Schouten identity to rewrite the remaining parts of the integrand,

$$(\ell_1 + k_{c_1-1})^2(\ell_2 - k_{c_2+1})^2\,\tfrac{1}{2}\,\mathrm{Tr}\big[(1+\gamma_5)\not{\ell}_1 \not{k}_{c_2} \not{\ell}_2 \not{k}_{c_1}\big]$$
$$-\{k_{c_1-1} \leftrightarrow -k_{c_1}\} - \{k_{c_2+1} \leftrightarrow -k_{c_2}\}$$
$$+\{k_{c_1-1} \leftrightarrow -k_{c_1}, k_{c_2+1} \leftrightarrow -k_{c_2}\}$$

- Two propagators cancel, so after a lot of algebra, and cancellation of triangles, we're left with a box — the γ_5 leads to a Levi-Civita tensor which vanishes upon integration
- What's left over is the same function which appears in the denominator of the box: $-st + m_2^2 m_4^2$


c_1+1
c_1
c_2-1
c_1-1
c_2
c_2+1

- We obtain the result,

$$-A^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$
$$\times \sum_{\text{easy 2 mass}} \text{Box} \cdot \tfrac{1}{2}(\text{its denominator})$$