On-Shell Methods in Field Theory

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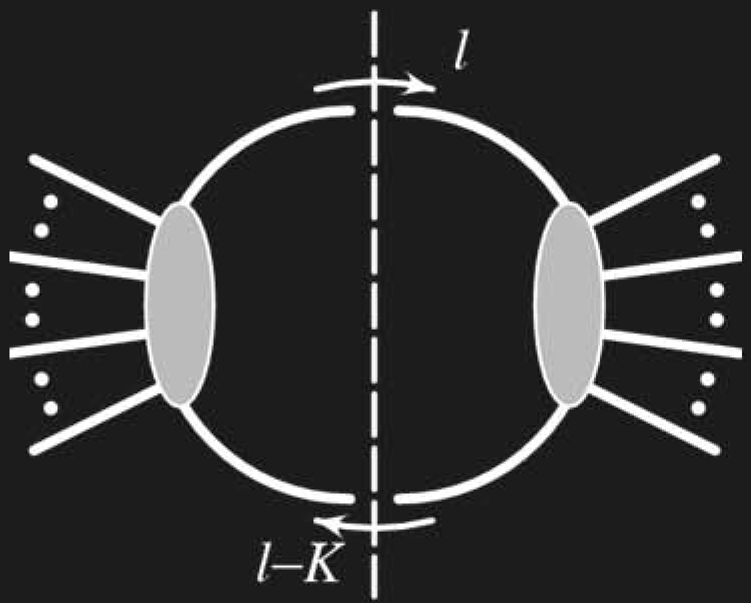
Lecture V

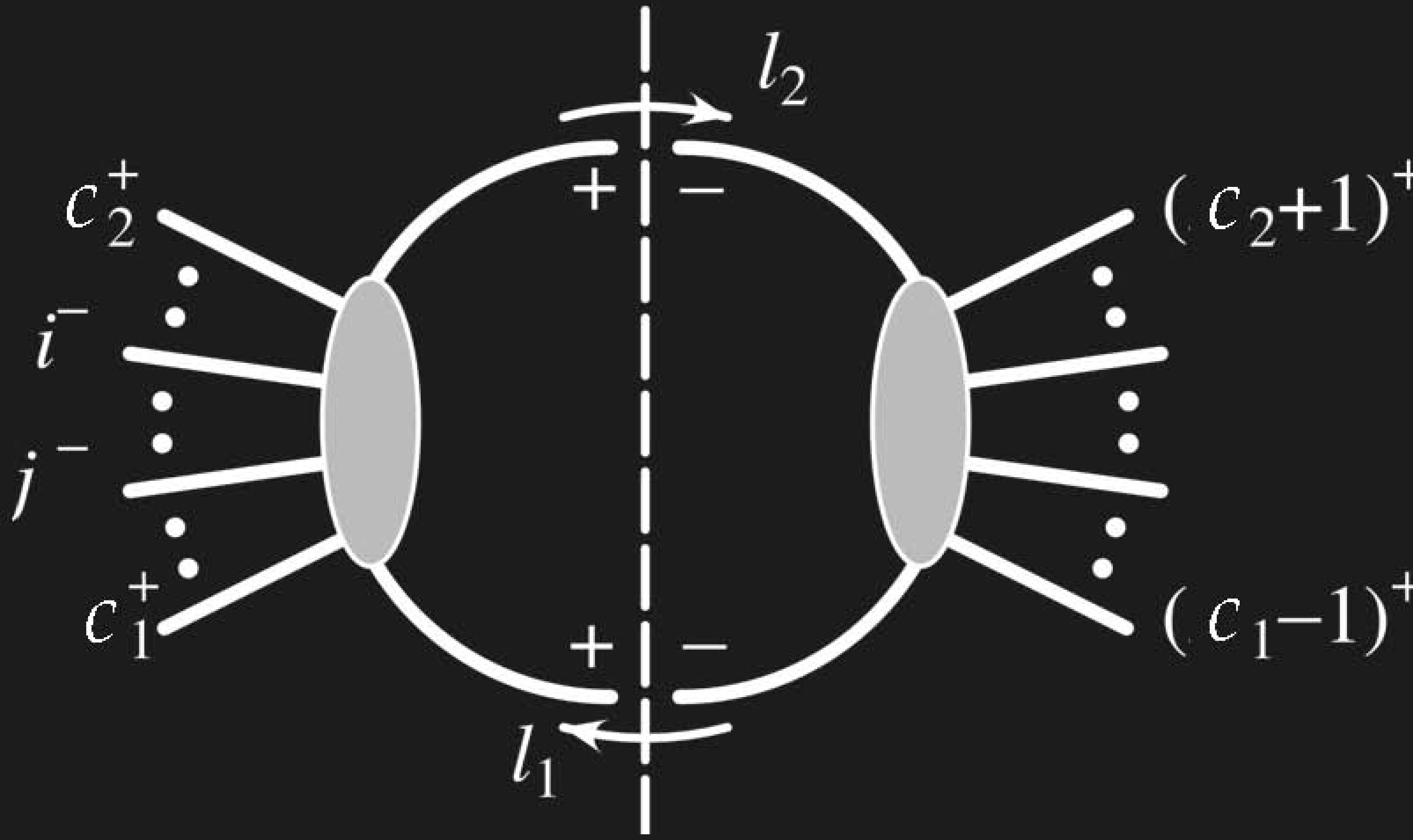
Unitarity-Based Method for Loops



Bern, Dixon, Dunbar, & DAK,
ph/9403226, ph/9409265

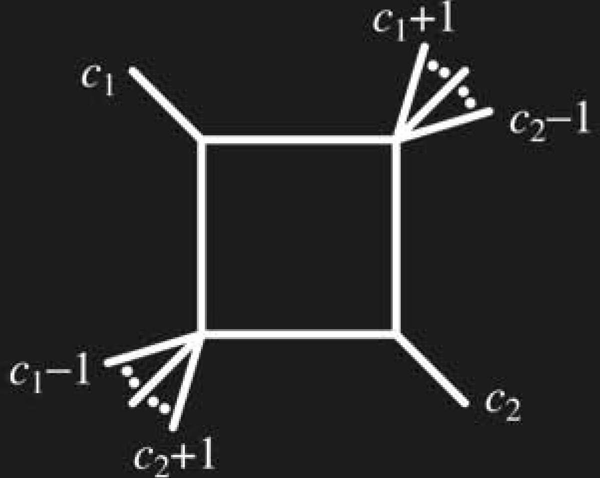
$$A^{\text{1-loop}} = \sum_{\text{cuts } K^2} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} A^{\text{tree}}_{\text{left}} \frac{i}{(\ell-K)^2} A^{\text{tree}}_{\text{right}}$$

Example: MHV at One Loop



The result,

$$-A^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$
$$\times \sum_{\text{easy 2 mass}} \text{Box} \cdot \tfrac{1}{2}(\text{its denominator})$$

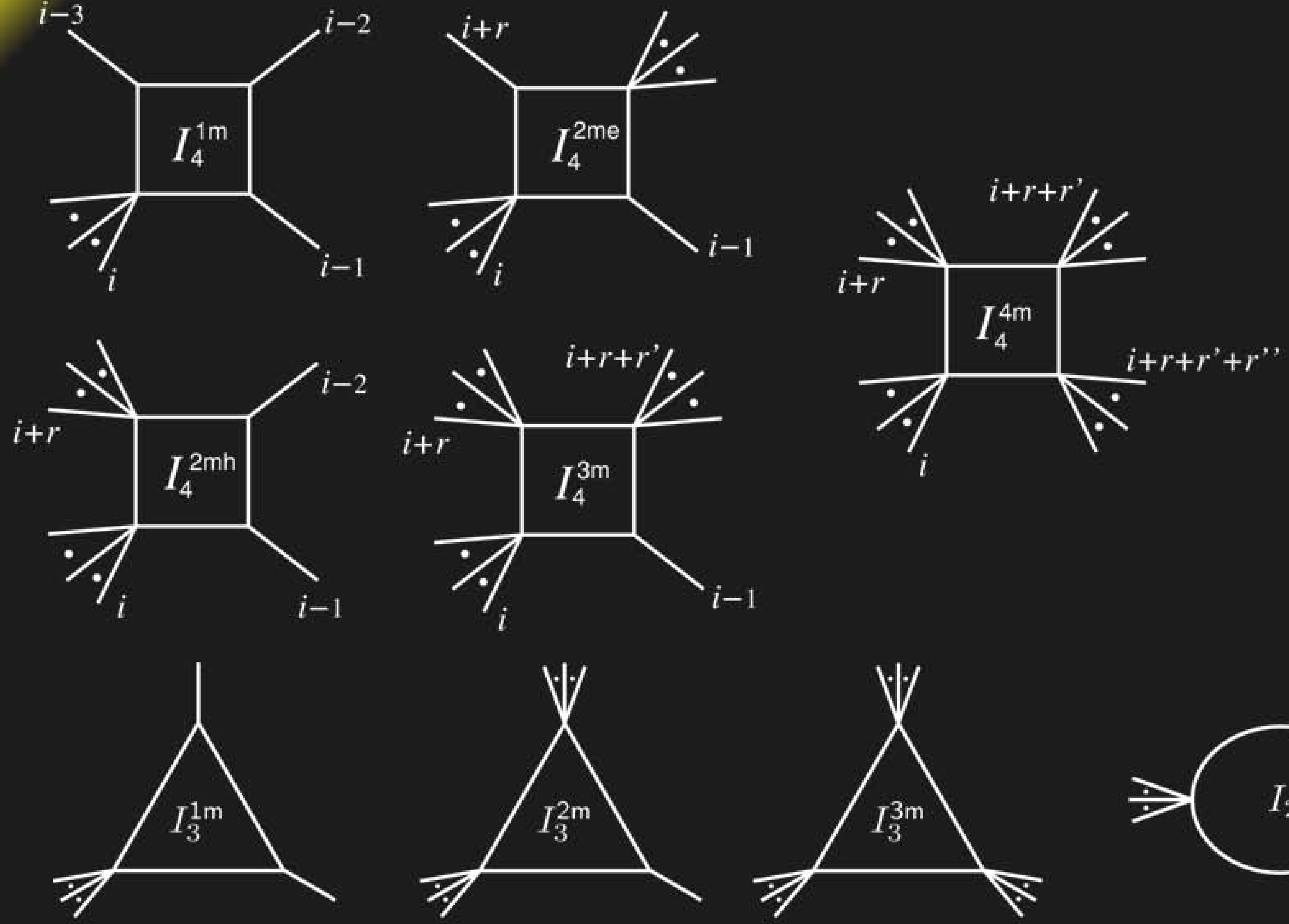


Have We Seen This Denominator Before?

$$\langle 5^- | 3 + 4 | 2^- \rangle$$

Consider

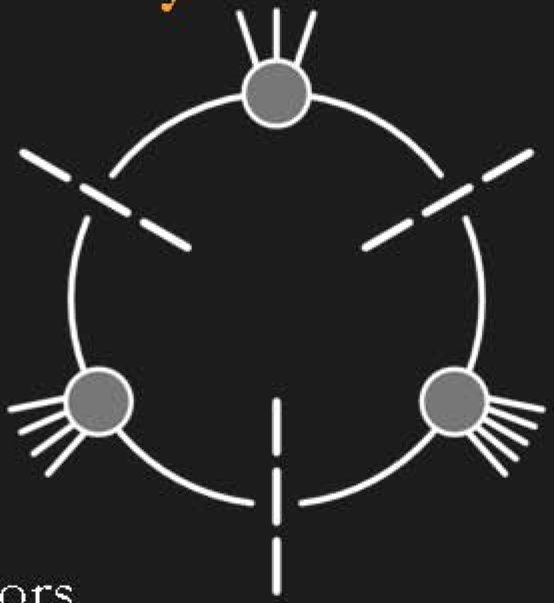
$$\langle c_1{}^- | \not{K}_{c_1 \ldots c_2} | c_2{}^- \rangle \langle c_2{}^- | \not{K}_{c_1 \ldots c_2} | c_1{}^- \rangle =$$


i−3
i−2
i
i−1
I_4^{1m}
i+r
i
i−1
I_4^{2me}
i+r+r'
i+r
i
i+r+r'+r''
I_4^{4m}
i+r
i−2
i
i−1
I_4^{2mh}
i+r+r'
i+r
i
i−1
I_4^{3m}
I_3^{1m}
I_3^{2m}
I_3^{3m}
I_2

$$A = \sum_j c_j I_j$$

Generalized Unitarity

- Can sew together more than two tree amplitudes
- Corresponds to 'leading singularities'

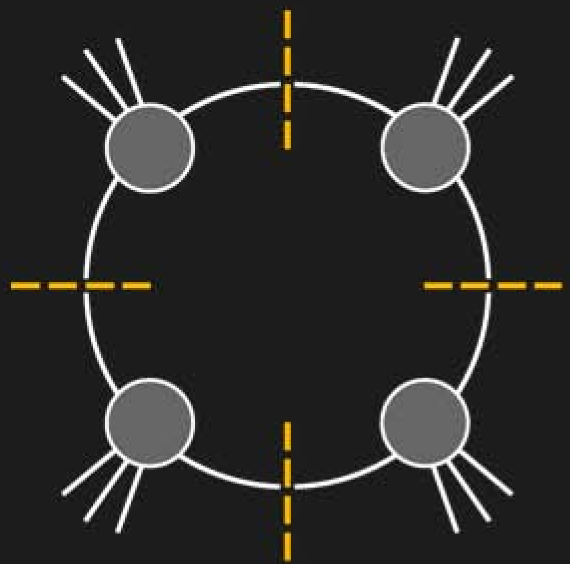


- Isolates contributions of a smaller set of integrals: only integrals with propagators corresponding to cuts will show up

Bern, Dixon, DAK (1997)

- Example: in triple cut, only boxes and triangles will contribute

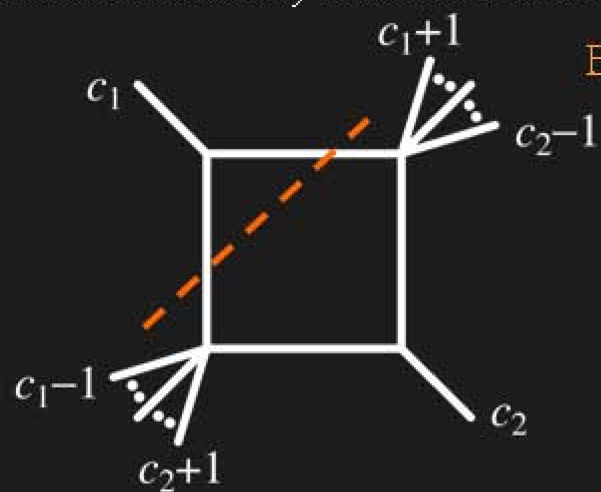
- Can we isolate a single integral?
- Quadruple cuts would isolate a single box
- Can't do this for one-mass, two-mass, or three-mass boxes because that would isolate a three-point amplitude
- Unless…



Cuts in Massless Channels

- With complex momenta, can form cuts using three-vertices too

Britto, Cachazo, & Feng, th/0412103



⇒ all box coefficients can be computed directly and algebraically, with no reduction or integration

- $\mathcal{N}=1$ and non-supersymmetric theories need triangles and bubbles, for which integration is still needed

Quadruple Cuts

Work in D=4 for the algebra

$$\int \frac{d^4\ell}{(2\pi)^4}\, \delta^{(+)}(\ell^2)\,\delta^{(+)}((\ell-k_1)^2)\,\delta^{(+)}((\ell-K_{12})^2)\,\delta^{(+)}((\ell-K_{123})^2)$$

$$\int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2(\ell-k_1)^2(\ell-K_{12})^2(\ell-K_{123})^2}$$

Four degrees of freedom & four delta functions

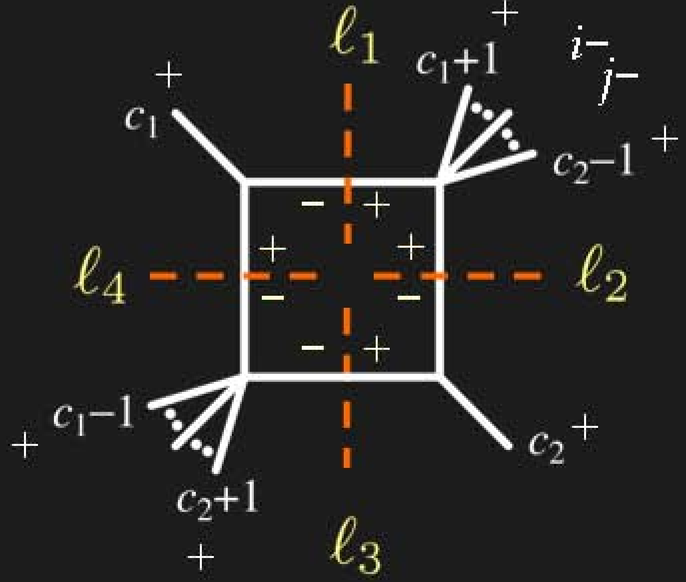
$\Rightarrow$ no integrals left, only algebra

$$\frac{1}{\#\ \text{solutions}} \sum_{\substack{\text{solutions}\\ \text{helicities}}} A_a^{\text{tree}} A_b^{\text{tree}} A_c^{\text{tree}} A_d^{\text{tree}}$$

2

MHV

- Coefficient of a specific easy two-mass box: only one solution will contribute



$$\langle \ell_1\, c_1\rangle = 0 = \langle \ell_4\, c_1\rangle\,,$$
$$\langle \ell_2\, c_2\rangle = 0 = \langle \ell_3\, c_2\rangle$$

$$\tfrac{1}{2}A_3((-\ell_4)^+, c_1^+, \ell_1^-)A((-\ell_1)^+, c_1{+}1, \ldots, i^-, \ldots, j^-, \ldots, c_2{-}1, \ell_2^+)$$
$$\times A_3((-\ell_2)^-, c_2^+, \ell_3^+)A((-\ell_3)^-, c_2{+}1, \ldots, c_1{-}1, \ell_4^-)$$
$$= \frac{1}{2}\left(\frac{[\ell_4\, c_1]^3}{[c_1\,\ell_1]\,[\ell_1\,\ell_4]}\right)\left(\frac{\langle i\, j\rangle^4}{\langle \ell_1\,(c_1{+}1)\rangle\, \langle\!\langle (c_1{+}1)\cdots(c_2{-}1)\rangle\!\rangle\, \langle (c_2{-}1)\,\ell_2\rangle\,\langle \ell_2\,\ell_1\rangle}\right)$$
$$\times\left(\frac{[c_2\,\ell_3]^3}{[\ell_2\,c_2]\,[\ell_3\,\ell_2]}\right)\left(\frac{\langle \ell_4\,\ell_3\rangle^3}{\langle \ell_3\,(c_2{+}1)\rangle\,\langle\!\langle (c_2{+}1)\cdots(c_1{-}1)\rangle\!\rangle\,\langle (c_1{-}1)\,\ell_4\rangle}\right)$$
$$= \frac{1}{2}A_n^{\rm tree}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)$$
$$\times\frac{\langle (c_1{-}1)\,c_1\rangle\,\langle c_1\,(c_1{+}1)\rangle\,\langle (c_2{-}1)\,c_2\rangle\,\langle c_2\,(c_2{+}1)\rangle}{\langle (c_1{-}1)\,\ell_4\rangle\,\langle \ell_1\,(c_1{+}1)\rangle\,\langle (c_2{-}1)\,\ell_2\rangle\,\langle \ell_3\,(c_2{+}1)\rangle}$$
$$\times\frac{\left\langle c_1^{\,+}\right|\not{\ell}_4\not{\ell}_3\left|c_2^{\,-}\right\rangle^3}{\left\langle c_1^{\,+}\right|\not{\ell}_1\not{\ell}_2\left|c_2^{\,-}\right\rangle\,[\ell_1\,\ell_4]\,[\ell_2\,\ell_3]}$$

Using momentum conservation

$$\langle c_1{}^+|\,\not{\ell}_4\not{\ell}_3\,|c_2{}^-\rangle = \langle c_1{}^+|\,\not{\ell}_1\not{\ell}_2\,|c_2{}^-\rangle$$

$$\frac{\langle (c_1-1)\, c_1\rangle}{\langle (c_1-1)\,\ell_4\rangle\,[\ell_1\,\ell_4]} = \frac{1}{[\ell_1\,c_1]}$$

our expression becomes

$$\frac{1}{2}A_n^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)\frac{\langle c_1{}^+|\,\not{\ell}_4\not{\ell}_3\,|c_2{}^-\rangle^2\,[\ell_1\,\ell_4]\,[\ell_2\,\ell_3]}{[c_1\,\ell_1]\,[c_1\,\ell_4]\,[\ell_3\,c_2]\,[\ell_2\,c_2]}$$

$$= \frac{1}{2}A_n^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)\,[\ell_1\,\ell_4]\,\langle \ell_4\,\ell_3\rangle\,[\ell_3\,\ell_2]\,\langle \ell_2\,\ell_1\rangle$$

$$= A_n^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$
$$\times\,\langle c_1{}^-|\,\ell_1+\ell_2\,|c_2{}^-\rangle\langle c_2{}^-|\,\ell_3+\ell_4\,|c_1{}^-\rangle$$

$$= -A_n^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$
$$\times\,\langle c_1{}^-|\,K_{(c_1+1)\cdots(c_2-1)}\,|c_2{}^-\rangle\langle c_2{}^-|\,K_{(c_1+1)\cdots(c_2-1)}\,|c_1{}^-\rangle$$

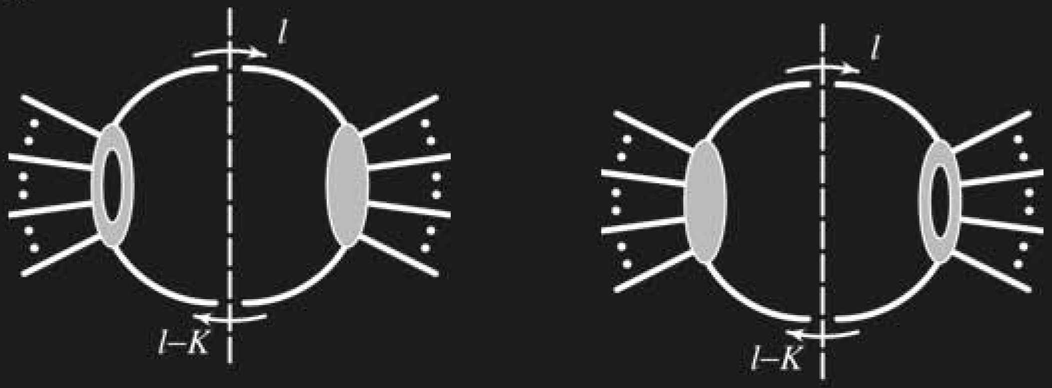
$$= \frac{1}{2}A_n^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)\,(m_2^2m_4^2 - st)$$

Higher Loops

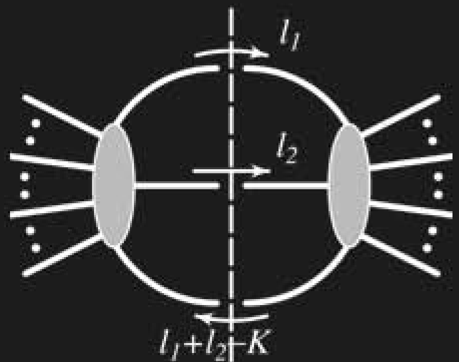
- Technology for finding a set of *master* integrals for any given process: "integration by parts", solved using Laporta algorithm
- But no general basis is known
- So we may have to reconstruct the integrals in addition to computing their coefficients

Unitarity-Based Method at Higher Loops

- Loop amplitudes on either side of the cut



- Multi-particle cuts in addition to two-particle cuts



- Find integrand/integral with given cuts in all channels

Generalized Cuts

- In practice, replace loop amplitudes by their cuts too

Computing QCD Amplitudes

$\mathcal{N}=4$ = pure QCD + 4 fermions + 3 complex scalars

QCD = $\mathcal{N}=4$ + $\delta\mathcal{N}=1$ + $\delta\mathcal{N}=0$

	chiral multiplet	scalar	
cuts + rational	cuts + rational	cuts	+ rational
⇑	⇑	⇑	⇑ (⇖ D=4−2ε unitarity)
D=4 unitarity	D=4 unitarity	D=4 unitarity	bootstrap *or* on-shell recursion relations

Rational Terms

- At tree level, we used on-shell recursion relations
- We want to do the same thing here
- Need to confront
  - Presence of branch cuts
  - Structure of factorization

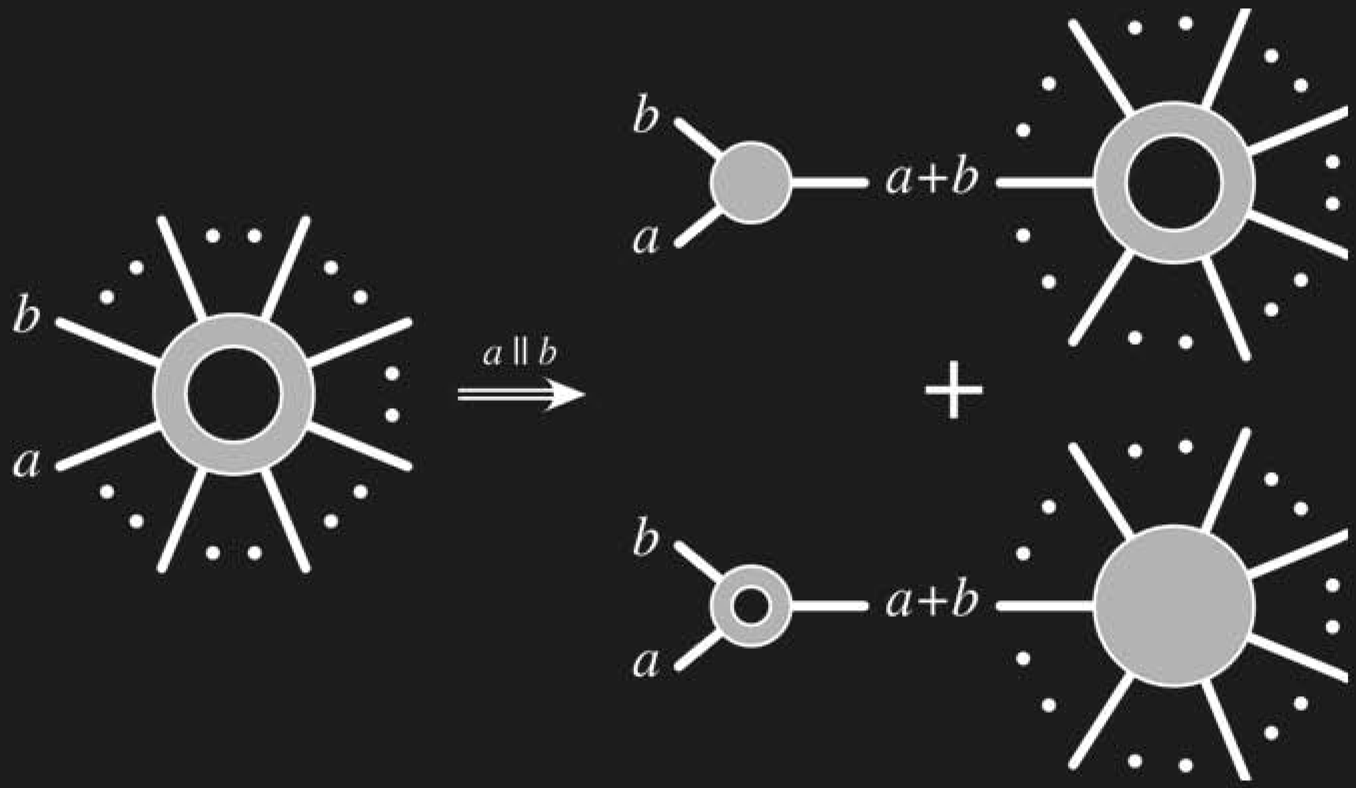
- At tree level,

$$A_n(1^\pm, 2^+, \ldots, n^+) = 0$$

- True in supersymmetric theories at all loop orders
- Non-vanishing at one loop in QCD
- but finite: no possible UV or IR singularities
- Separate V and F terms

$$A_n = V_n A_n^{\text{tree}} + F_n$$

Factorization at One Loop



Collinear Factorization at One Loop

- Most general form we can get is antisymmetric + nonsingular: two independent tensors for splitting amplitude

$$f_1(s_{ab},z)\,\frac{1}{s_{ab}}\left(\varepsilon_a^\mu \varepsilon_b\cdot k_a - \varepsilon_b^\mu \varepsilon_a\cdot k_b + \tfrac{1}{2}(k_b^\mu - k_a^\mu)\varepsilon_a\cdot\varepsilon_b\right)$$

$$f_2(s_{ab},z)\,\frac{(k_a^\mu - k_b^\mu)}{s_{ab}}\left(\varepsilon_a\cdot\varepsilon_b - \frac{\varepsilon_a\cdot k_b\varepsilon_b\cdot k_a}{k_a\cdot k_b}\right)$$

- Second tensor arises only beyond tree level, and only for like helicities

- Explicit form of +++ splitting amplitude

$$\mathrm{Split}_{+}^{1-\mathrm{loop,\ scalar}}(z;a^{+},b^{+}) = -\frac{1}{48\pi^2}\sqrt{z(1-z)}\,\frac{[a\,b]}{\langle a\,b\rangle^2}$$

- No general theorems about factorization in complex momenta
- Just proceed
- Look at $-+\ldots++$

- Amplitudes contain factors like $\frac{[a\,b]}{\langle a\,b\rangle^2}$ known from collinear limits
- Expect also $\frac{[a\,b]}{\langle a\,b\rangle}$ as 'subleading' contributions, seen in explicit results

'unreal' poles

- Double poles with vertex $V_3(+++)$
- Non-conventional single pole: one finds the double-pole, multiplied by

$$s_{ab}\ \mathrm{Soft}(\hat{a},(-\hat{P}_{ab})^-,b)\ \mathrm{Soft}(b+1,\hat{P}_{ab}^+,a-1)$$

On-Shell Recursion at Loop Level

Bern, Dixon, DAK (1–7/2005)

- Finite amplitudes are purely rational
- We can obtain simpler forms for known finite amplitudes (Chalmers, Bern, Dixon, DAK; Mahlon)
- These again involve spurious singularities
- Obtained last of the finite amplitudes: $f^- f^+ g^+ \ldots g^+$

On-Shell Recursion at Loop Level

Bern, Dixon, DAK (1–7/2005)

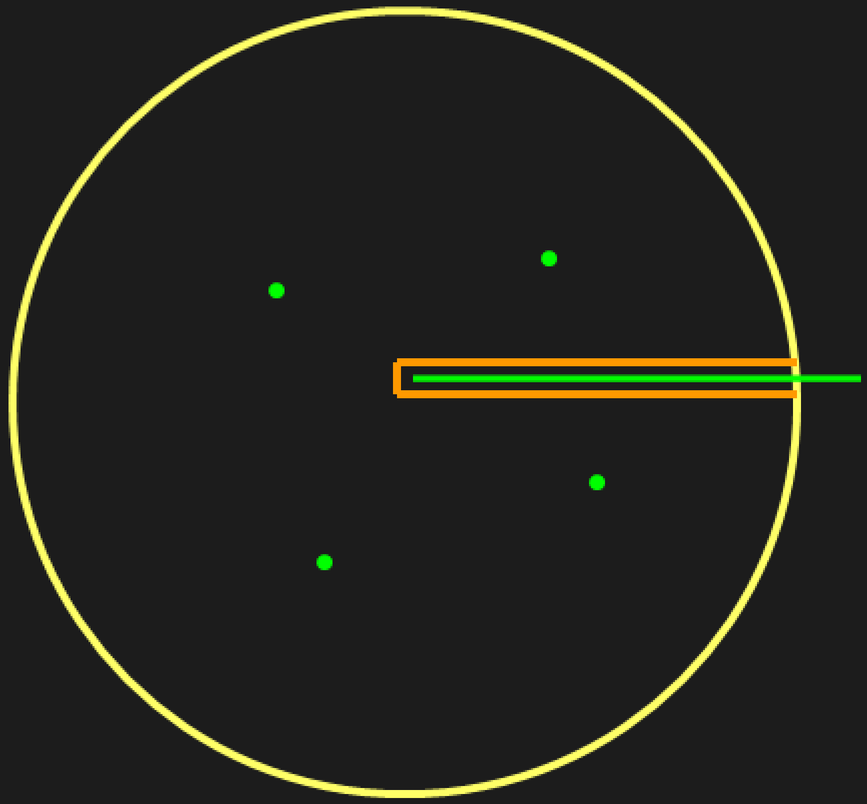
- Complex shift of momenta $[j,l\rangle$

$$p_j^\mu \to p_j^\mu(z) = p_j^\mu - \frac{z}{2}\langle j^- |\, \gamma^\mu \,| l^- \rangle,$$
$$p_l^\mu \to p_l^\mu(z) = p_l^\mu + \frac{z}{2}\langle j^- |\, \gamma^\mu \,| l^- \rangle.$$

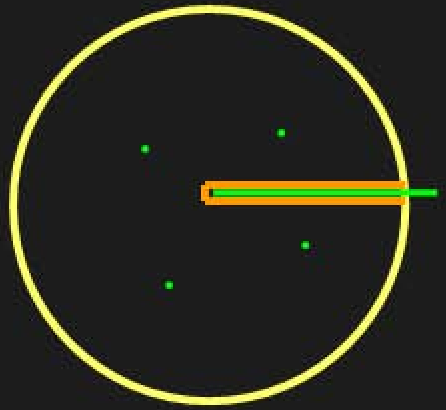
- Behavior as z → ∞: require $A(z) \to 0$
- Basic complex analysis: treat branch cuts
- Knowledge of *complex* factorization:
  - at tree level, tracks known factorization for real momenta
  - at loop level, same for multiparticle channels; and $- \to -+$
  - Avoid $\pm \to ++$

Rational Parts of QCD Amplitudes

- Start with cut-containing parts obtained from unitarity method, consider same contour integral



Derivation



- Consider the contour integral

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z}\,A(z)$$

- Determine A(0) in terms of other poles and branch cuts

$$A(0) = -\sum_{\text{poles }\alpha}\ \operatorname*{Res}_{z=z_\alpha}\frac{A(z)}{z} - \int_{\text{Branch}}\frac{dz}{z}\ \text{Disc}_{\,B}A(z)$$

Rational terms   Cut terms

- Cut terms have spurious singularities $\Rightarrow$ rational terms do too

$$\frac{\ln(s_1/s_2)}{(s_1 - s_2)^2}$$

- $\Rightarrow$ the sum over residues includes spurious singularities, for which there is no factorization theorem at all

Completing the Cut

- To solve this problem, define a modified 'completed' cut, adding in rational functions to cancel spurious singularities

$$\frac{\ln(s_1/s_2)}{(s_1 - s_2)^2} + \frac{1}{s_1 - s_2}$$

- We know these have to be there, because they are generated together by integral reductions
- Spurious singularity is unique
- Rational term is not, but difference is free of spurious singularities

- This eliminates residues of spurious poles

- $\widehat{C}$ entirely known from four-dimensional unitarity method
- Assume $\widehat{C}(z) \to 0$ as $z \to \infty$
- Modified separation

$$A_n(z) = c_\Gamma \left[\hat{C}(z) + \hat{R}(z)\right]$$

so

$$A(0) \quad = \quad - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{C}(z)}{z} - \int_{\text{Branch}} \frac{dz}{z} \text{ Disc}_B \hat{C}(z)$$

$$- \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{R}(z)}{z}$$

- Perform integral & residue sum for $\widehat{C}$

$$\hat{C}(0) \quad = \quad -\sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{C}(z)}{z} - \int_{\text{Branch}} \frac{dz}{z} \text{ Disc}_B \hat{C}(z)$$

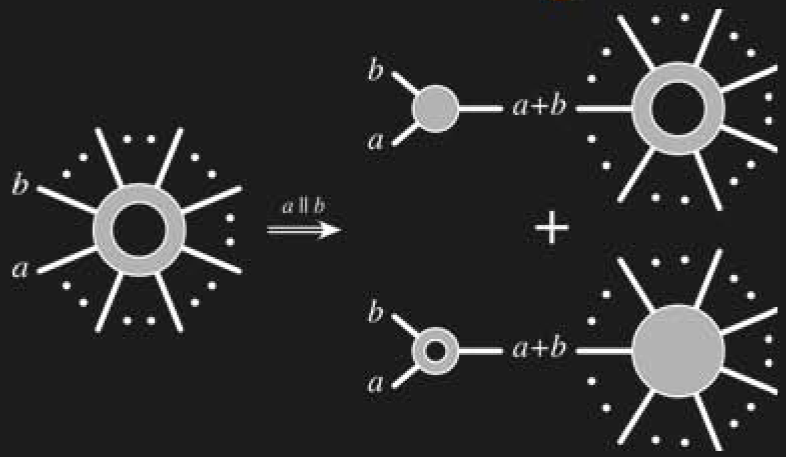
so

$$A(0) = c_\Gamma \left[\hat{C}(0) - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{R}(z)}{z} \right]$$

Unitarity Method

???

A Closer Look at Loop Factorization



- Only single poles in splitting amplitudes with cuts (like tree)
- Cut terms → cut terms
- Rational terms → rational terms
- Build up the latter using recursion, analogous to tree level

- Recursion on rational pieces would build up rational terms R, not $\hat{R}$

- Recursion gives

$$\text{Recursive} = -\sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\text{Rational}[\hat{C}(z)]}{z} - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{R}(z)}{z}$$

Double-counted: 'overlap'

- Subtract off overlap terms

$$A(0) = c_\Gamma \left[\hat{C}(0) + \text{Recursive} + \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\text{Rational}[\hat{C}(z)]}{z} \right]$$

Compute explicitly from known $\hat{C}$:
also have a diagrammatic expression

Tree-level On-Shell Recursion Relations

- Partition P: two or more cyclicly-consecutive momenta containing j, such that complementary set $\overline{P}$ contains l,

$$
\begin{aligned}
P &\equiv \{P_1, P_2, \ldots, j, \ldots, P_{-1}\}, \\
\overline{P} &\equiv \{\overline{P}_1, \overline{P}_2, \ldots, l, \ldots, \overline{P}_{-1}\}, \\
P \cup \overline{P} &= \{1, 2, \ldots, n\}
\end{aligned}
$$

- The recursion relations are

On shell

$$
\begin{aligned}
A_n(1, \ldots, n) &= \sum_{\substack{\text{partitions } P \\ h=\pm}} A_{\#P+1}(k_{P_1}, \ldots, \hat{j}, \ldots, k_{P_{-1}}, -\widehat{P}^{h}) \\
&\quad \times \frac{i}{P^2} \times A_{\#\overline{P}+1}(k_{\overline{P}_1}, \ldots, \hat{l}, \ldots, k_{\overline{P}_{-1}}, \widehat{P}^{-h})
\end{aligned}
$$

Recursive Diagrams

$$-\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{R_n(z)}{z} \equiv \text{Recursive Diagrams}$$

$$\begin{aligned}
= \sum_{\substack{\text{partitions } P \\ h=\pm}} \Big\{ & R_n^{(1)}(k_{P_1},\ldots,\hat{k}_j,\ldots,k_{P_{-1}},-\widehat{P}^h) \\
& \times \frac{i}{P^2} \times A_n^{\text{tree}}(k_{\overline{P}_1},\ldots,\hat{k}_l,\ldots,k_{\overline{P}_{-1}},\widehat{P}^{-h}) \\
& + A_n^{\text{tree}}(k_{P_1},\ldots,\hat{k}_j,\ldots,k_{P_{-1}},-\widehat{P}^h) \\
& \times \frac{i}{P^2} \times R_n^{(1)}(k_{\overline{P}_1},\ldots,\hat{k}_l,\ldots,k_{\overline{P}_{-1}},\widehat{P}^{-h}) \\
& + A_n^{\text{tree}}(k_{P_1},\ldots,\hat{k}_j,\ldots,k_{P_{-1}},-\widehat{P}^h) \\
& \times \frac{iR^{\text{Fact}}}{P^2} \times A_n^{\text{tree}}(k_{\overline{P}_1},\ldots,\hat{k}_l,\ldots,k_{\overline{P}_{-1}},\widehat{P}^{-h}) \Big\}
\end{aligned}$$

Five-Point Example

- Look at $F_5^s(1^-,2^-,3^+,4^+,5^+)$

- Cut terms

$$\frac{1}{6}\frac{\langle 1\,2\rangle^2\left(\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle+\langle 2\,3\rangle\,[4\,5]\,\langle 5\,1\rangle\right)}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\frac{\mathrm{L}_0\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}}$$

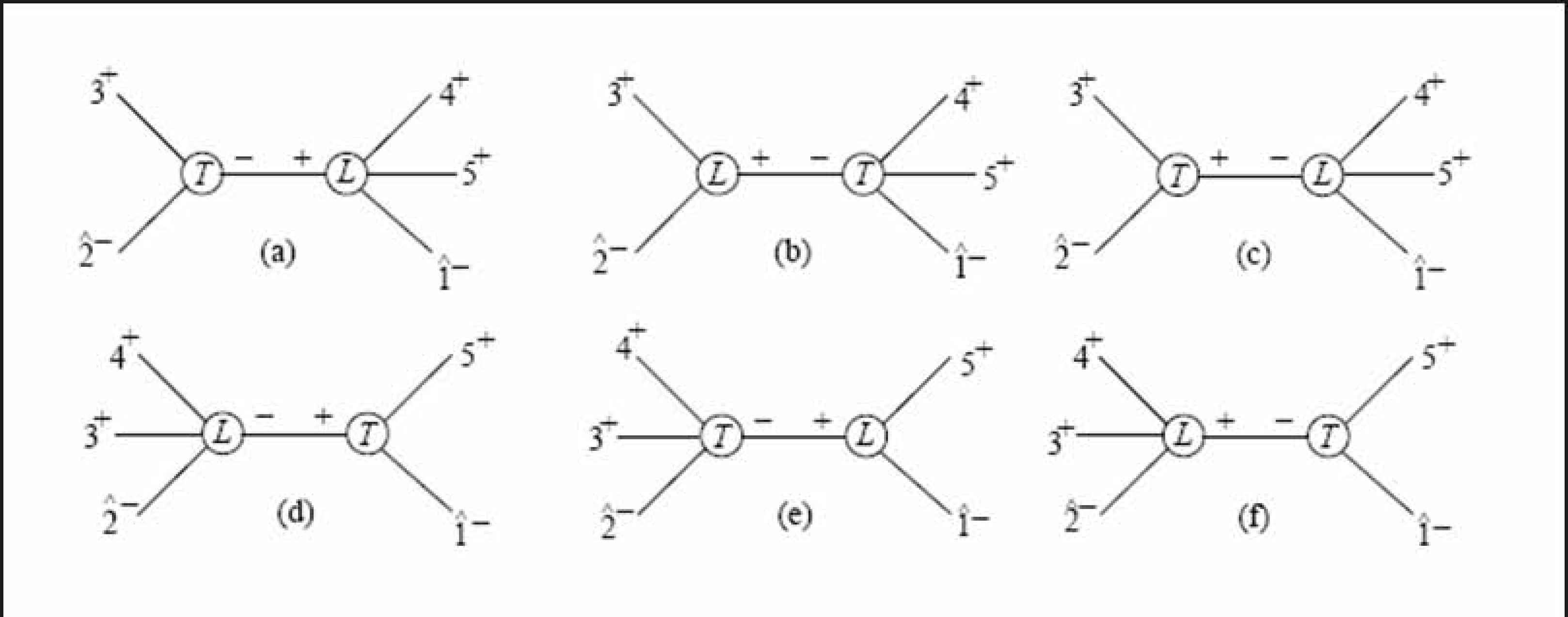
$$-\frac{1}{3}\frac{[3\,4]\,\langle 4\,1\rangle\,\langle 2\,4\rangle\,[4\,5]\left(\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle+\langle 2\,3\rangle\,[4\,5]\,\langle 5\,1\rangle\right)}{\langle 3\,4\rangle\,\langle 4\,5\rangle}\frac{\mathrm{L}_2\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}^3}$$

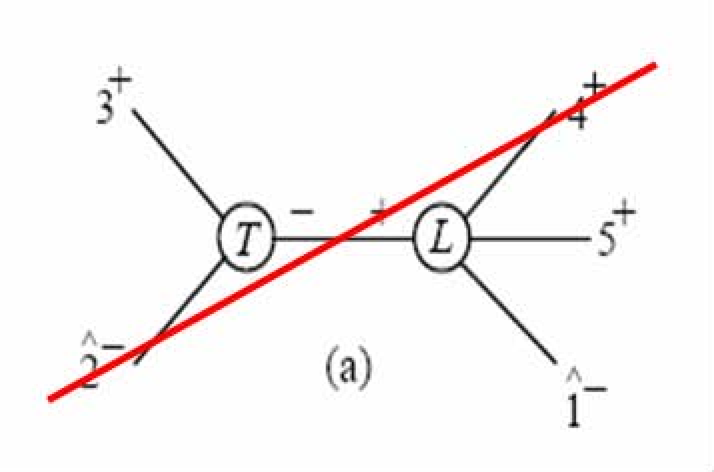
- have required large-z behavior

$$\mathrm{L}_0(r)=\frac{\ln r}{1-r},\qquad \mathrm{L}_2(r)=\frac{\ln r-(r-1/r)/2}{(1-r)^3}$$

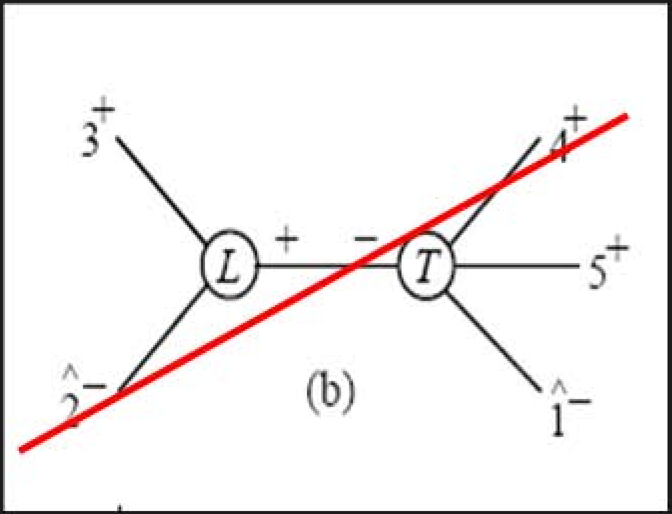
Five-Point Example

- Look at $F_5^s(1^-, 2^-, 3^+, 4^+, 5^+)$
- Recursive diagrams: use $[1, 2\rangle$ shift

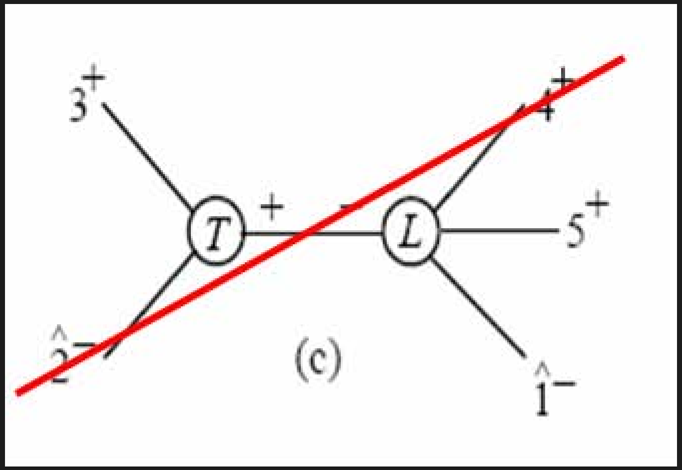




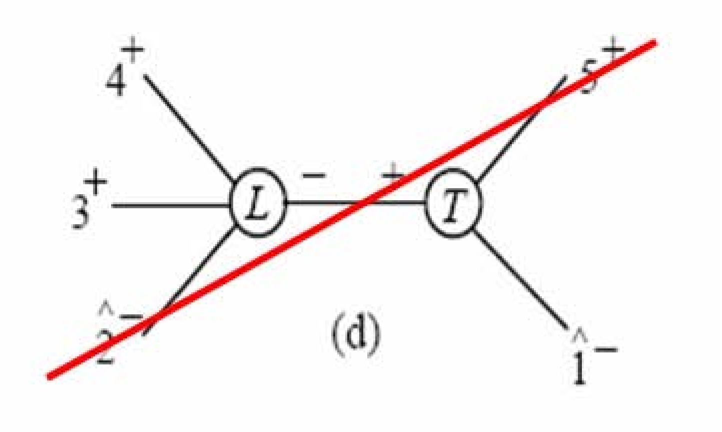
(a) Tree vertex $A(\hat{2}^-, 3^+, -\widehat{P}^-)$ vanishes



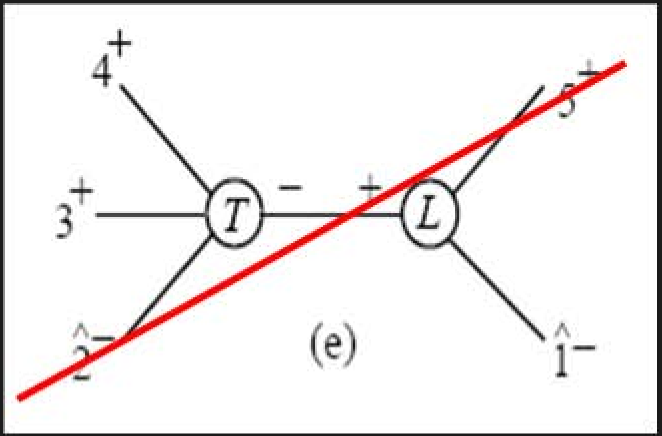
(b) Loop vertex $R(\hat{2}^-, 3^+, -\widehat{P}^-)$ vanishes



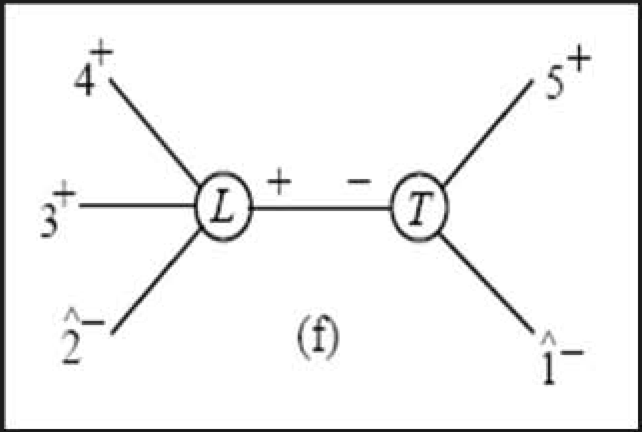
(c) Loop vertex $R_4(\hat{1}^-, \widehat{P}^-, 4^+, 5^+)$ vanishes



(d) Tree vertex $A(\hat{1}^-, 5^+, \widehat{P}^+)$ vanishes



(e) Loop vertex $R(\hat{1}^-, 5^+, \widehat{P}^+)$ vanishes



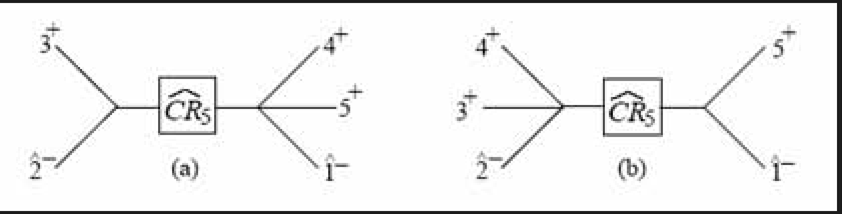
(f) Diagram doesn't vanish

$$D_5^{(\mathrm{f})} = A_3^{\mathrm{tree}}(5^+, \hat{1}^-, -\widehat{P}^-) \times \frac{i}{s_{51}} \times R_4(\hat{2}^-, 3^+, 4^+, \widehat{P}^+)$$

$$
\begin{aligned}
D_5^{(\mathrm{f})} &= -\frac{1}{3}\,\frac{\langle \hat{1}\,(-\widehat{P})\rangle^3}{\langle 5\,\hat{1}\rangle\,\langle(-\widehat{P})\,5\rangle}\,\frac{1}{s_{51}}\,\frac{\langle 3\,\widehat{P}\rangle\,[3\,\widehat{P}]^3}{[\hat{2}\,3]\,\langle 3\,4\rangle\,\langle 4\,\widehat{P}\rangle\,[\widehat{P}\,\hat{2}]} \\
&= \frac{1}{3}\,\frac{\left\langle 1^-\right|5\left|2^-\right\rangle^3}{\langle 5\,1\rangle\left\langle 5^-\right|1\left|2^-\right\rangle}\,\frac{1}{\langle 5\,1\rangle\,[1\,5]\left\langle 1^-\right|5\left|2^-\right\rangle^2} \\
&\quad \times\frac{\left\langle 3^-\right|4\left|2^-\right\rangle\left\langle 1^-\right|5\left|3^-\right\rangle^3}{[2\,3]\,\langle 3\,4\rangle\left\langle 4^-\right|3\left|2^-\right\rangle\left\langle 1^-\right|5\left|2^-\right\rangle} \\
&= -\frac{1}{3}\,\frac{[2\,4]\,[3\,5]^3}{\langle 3\,4\rangle\,[1\,2]\,[1\,5]\,[2\,3]^2}
\end{aligned}
$$

Five-Point Example (cont.)

- 'Overlap' contributions $F_5^s(1^-,2^-,3^+,4^+,5^+)$



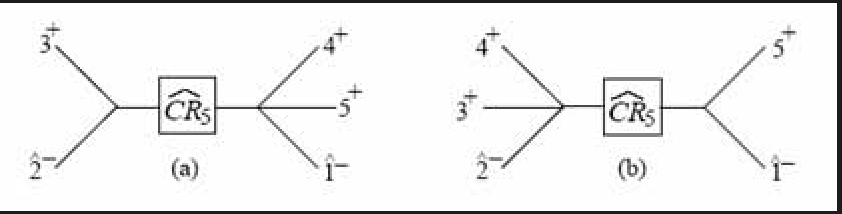
- Take rational terms in $\hat{C}$

$$\widehat{CR}_5 = -\frac{1}{6}\frac{s_{51}+s_{23}}{s_{23}s_{51}(s_{51}-s_{23})^2} \times \frac{[3\,4]\,\langle 4\,1\rangle\,\langle 2\,4\rangle\,[4\,5]\,\left(\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle + \langle 2\,4\rangle\,[4\,5]\,\langle 5\,1\rangle\right)}{\langle 3\,4\rangle\,\langle 5\,1\rangle}$$

- Apply shift, extract residues in each "channel"

$$
\begin{aligned}
\widehat{CR}_5(z) \quad = \quad & -\frac{1}{6}\frac{[3\,4]\,\langle 4\,1\rangle\,(\langle 2\,4\rangle + z\,\langle 1\,4\rangle)\,[4\,5]}{\langle 3\,4\rangle\,\langle 4\,5\rangle} \\
& \times \frac{\Big((\langle 2\,3\rangle + z\,\langle 1\,3\rangle)\,[3\,4]\,\langle 4\,1\rangle + (\langle 2\,4\rangle + z\,\langle 1\,4\rangle)\,[4\,5]\,\langle 5\,1\rangle\Big)}{\left(s_{51} - s_{23} - z\,\langle 1^-|\,(3+5)\,|2^-\rangle\right)^2} \\
& \times \frac{s_{51} + s_{23} - z\,\langle 1^-|\,5\,|2^-\rangle + z\,\langle 1^-|\,3\,|2^-\rangle}{(\langle 2\,3\rangle + z\,\langle 1\,3\rangle)\,[3\,2]\,\langle 1\,5\rangle\,([5\,1] - z\,[5\,2])}
\end{aligned}
$$

- 'Overlap' contributions $F_5^s(1^-, 2^-, 3^+, 4^+, 5^+)$



$$O_5^{(a)} = -\frac{1}{6}\frac{\langle 1\,2\rangle^2 \langle 1\,4\rangle [3\,4]}{\langle 1\,5\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle [2\,3]}$$

$$O_5^{(b)} = \frac{1}{6}\frac{\langle 1\,4\rangle [3\,4] [3\,5] \left(\langle 1\,4\rangle [3\,4] - \langle 1\,5\rangle [3\,5]\right)}{\langle 1\,5\rangle \langle 3\,4\rangle \langle 4\,5\rangle [1\,5] [2\,3]^2}$$

On-Shell Methods

- Physical states
- Use of properties of amplitudes as calculational tools
- Kinematics: Spinor Helicity Basis ⇔ Twistor space
- Tree Amplitudes: On-shell Recursion Relations ⇐ Factorization
- Loop Amplitudes: Unitarity (SUSY)
  Unitarity + On-shell Recursion QCD