An Introduction to the Standard Model of Electroweak Interactions

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1. Introduction

The aim of these lectures is that of describing the construction and the main phenomenological implications of the Glashow-Weinberg-Salam unified theory of weak and electromagnetic interactions (universally referred to as the *standard model*). Basic knowledge in quantum field theory[??] and elementary group theory[??] is assumed, as well as familiarity with the fundamental phenomenology of weak interactions[??].

No attempt will be made to give a full list of references. Such a list can be found in any standard text book of particle physics; see for example ref. [**??**] and ref. [**??**].

2. A gauge theory of weak interactions

Our starting point will be Fermi's theory of β and muon decay. This theory is based on an effective four-fermion lagrangian, which is usually written as follows[2]:

$$\mathcal{L} = -\frac{G_\beta}{\sqrt{2}}\overline{p}\gamma^\alpha(1-a\gamma_5)n\overline{e}\gamma_\alpha(1-\gamma_5)\nu_e - \frac{G_\mu}{\sqrt{2}}\overline{\nu}_\mu\gamma^\alpha(1-\gamma_5)\mu\overline{e}\gamma_\alpha(1-\gamma_5)\nu_e, \tag{2.1}$$

with

$$G_\mu \simeq 1.16639\times 10^{-5}\ GeV^{-2}; \quad G_\beta \simeq G_\mu; \quad a \simeq 1.239\pm 0.09. \tag{2.2}$$

As is well known, the lagrangian in eq. (**??**) is not renormalizable (it contains only operators with mass dimension 6, while a renormalizable theory must contain operators whose mass dimension is at most 4; see appendix **??**), and it gives rise to a non-unitary S matrix. However, it contains all the physical information needed to build a renormalizable and unitary theory of weak interactions.

The idea is that of building a theory which possesses local invariance under the action of some group, a *gauge* theory, in analogy with quantum electrodynamics (see appendix **??**). We will then require that the new theory reduce to eq. (**??**) in the low-energy limit, in the sense that the local four-fermion interaction of the Fermi lagrangian will be interpreted as the exchange of a massive vector boson with a momentum much smaller than its mass. In this way, both problems of renormalizability and unitarity will be solved, since gauge theories are known to be renormalizable, and

[2] Throughout these lectures, particle fields will be denoted by the letter usually adopted for the corresponding particle: e for the electron, ν_e for the electron neutrino, and so on.

the mass of the intermediate vector boson will act as a cut-off that stops the growth of cross sections with energy, in order to ensure unitarity of the scattering matrix.

In order to complete this program, we must choose the group of local invariance, and then assign particle fields to representations of this group. Both these steps are made with the help of the information contained in the Fermi lagrangian. Let us first consider the electron and the electron neutrino. They participate in the weak interaction via the current

$$J_\mu = \overline{\nu}_e \frac{1}{2}\gamma_\mu(1-\gamma_5)e. \tag{2.3}$$

We want to to rewrite J_μ in the form of a Noether current,

$$\overline{\psi}_i \gamma_\mu T^A_{ij} \psi_j, \tag{2.4}$$

where ψ_i are the components of some multiplet of the gauge group, and T^A_{ij} are the corresponding group generators. In the case of J_μ, this can be done in the following way. We observe that the current J_μ can be written as

$$J_\mu = \overline{L}\gamma_\mu \tau^+ L, \tag{2.5}$$

where

$$\tau^+ = \frac{1}{2}(\tau_1 + i\tau_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{2.6}$$

$$L = \frac{1}{2}(1-\gamma_5)\begin{pmatrix} \nu_e \\ e \end{pmatrix} \equiv \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}. \tag{2.7}$$

and τ_i are the usual Pauli matrices. The hermitian conjugate current $J^\dagger_\mu$ will also participate in the interaction,

$$J^\dagger_\mu = \overline{L}\gamma_\mu \tau^- L, \tag{2.8}$$

where $\tau^- = (\tau_1 - i\tau_2)/2$. In the context of gauge theories, currents are in one-to-one correspondence with the generators of the symmetry group. The group generators, in turn, form a closed set with respect to the commutation operation, that is, the commutator of two generators is also a generator. Therefore, the current

$$J^\mu_3 = \overline{L}\gamma^\mu[\tau^+, \tau^-]L = \overline{L}\gamma_\mu \tau_3 L \tag{2.9}$$

will also be present. No other current must be introduced, since $[\tau_3, \tau^\pm] = 2\tau^\pm$. We have then interpreted the current J_μ as being one of the three conserved currents of a theory with $SU(2)$ gauge invariance, the Pauli matrices being the $SU(2)$ generators in the fundamental representation, and we have assigned the left-handed neutrino and electron fields to an $SU(2)$ doublet. The right-handed neutrino and electron components, ν_{eR} and e_R, do not take part in the weak-interaction phenomena described by the Fermi lagrangian, so they must be assigned to the singlet (or scalar) representation. Of course, this is not the only possible choice, but it is the simplest possibility (and also the correct one, as we will see!) since it does not require the introduction of fermion fields other than the observed ones.

The current J_3^μ is a *neutral* current, it contains creation and destruction operators of particles with the same charge (actually, of the same particle). Neutral currents do not appear in the Fermi lagrangian, no neutral current phenomenon is observed in low-energy weak interactions. As we will see, the experimental observation of phenomena induced by weak neutral currents is a crucial test of the validity of the standard model. Notice also that the neutral current J_3^μ cannot be identified with the only other neutral current we know of, the electromagnetic one. This is for two reasons: first, the electromagnetic current involves both left-handed and right-handed fermion fields with the same weight; and second, the electromagnetic current does not contain a neutrino term, the neutrino being chargeless. We will come back later to the problem of neutral currents, that will end up with the inclusion of the electromagnetic current in the theory. For the moment, we go on with the construction of our $SU(2)$ gauge theory. We must introduce vector meson fields W_i^μ, one for each of the three $SU(2)$ generators, and build a covariant derivative

$$D^\mu = \partial^\mu - igW_i^\mu T_i, \tag{2.10}$$

where we have introduced, as is customary in gauge theories, a coupling constant g. The matrices T_i are generators of $SU(2)$ in the representation of the multiplet the covariant derivative is acting on. For example, when D^μ acts on the doublet L, we have $T_i \equiv \tau_i/2$, and when it acts on the gauge singlet e_R we have $T_i \equiv 0$. We are now ready to write the gauge-invariant lagrangian for the fermion fields:

$$\mathcal{L} = i\overline{L}\hat{D}L + i\overline{\nu}_{eR}\hat{D}\nu_{eR} + i\overline{e}_R\hat{D}e_R, \tag{2.11}$$

where $\hat{D} = \gamma_\mu D^\mu$. The lagrangian $\mathcal{L}$ contains the usual kinetic term $\mathcal{L}^{kin}$ for fermions,

$$\mathcal{L}^{kin} = i\overline{L}\hat{\partial}L + i\overline{\nu}_{eR}\hat{\partial}\nu_{eR} + i\overline{e}_R\hat{\partial}e_R, \tag{2.12}$$

and a term $\mathcal{L}^W$ that describes the interaction of fermions with the gauge bosons W_i^μ. The interaction term can be split into two parts, corresponding to neutral-current and charged-current interaction respectively:

$$\mathcal{L}^W = \mathcal{L}_c^W + \mathcal{L}_n^W , \tag{2.13}$$

where

$$\mathcal{L}_c^W = gW_1^\mu \overline{L}\gamma_\mu \frac{\tau_1}{2} L + gW_2^\mu \overline{L}\gamma_\mu \frac{\tau_2}{2} L \tag{2.14}$$

and

$$\mathcal{L}_n^W = gW_3^\mu \overline{L}\gamma_\mu \frac{\tau_3}{2} L = \frac{g}{2} W_3^\mu \left(\overline{\nu}_{eL}\gamma_\mu \nu_{eL} - \overline{e}_L \gamma_\mu e_L \right). \tag{2.15}$$

The charged-current interaction $\mathcal{L}_c^W$ is usually expressed in terms of the fields

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2) \tag{2.16}$$

as follows:

$$\mathcal{L}_c^W = \frac{g}{\sqrt{2}} \overline{L}\gamma^\mu \tau^+ L W_\mu^+ + \frac{g}{\sqrt{2}} \overline{L}\gamma^\mu \tau^- L W_\mu^- . \tag{2.17}$$

We have already observed that the neutral current $J_3^\mu = \overline{L}\gamma^\mu \tau_3 L$ cannot be identified with the electromagnetic current, and correspondingly that the gauge vector boson W_3^μ cannot be interpreted as the photon. The construction of the model can therefore proceed in two different directions: either we modify the multiplet structure of the theory, in order to make J_3^μ equal to the electromagnetic current, or we extend the gauge group in order to accomodate also the electromagnetic current in addition to the weak neutral current J_3^μ. We proceed to describe the second possibilty, which is the one that turned out to be correct, after the discovery of weak neutral currents. Nevertheless, it must be reminded that this was not at all obvious to physicists before the observation of weak-neutral-current effects.

The simplest way of extending the gauge group $SU(2)$ to include another neutral generator is to include an abelian factor $U(1)$:

$$SU(2) \to SU(2) \otimes U(1). \tag{2.18}$$

We will require our lagrangian to be invariant also under the $U(1)$ gauge transformations

$$\psi \to \psi' = \exp\left[ig'\alpha \frac{Y(\psi)}{2} \right] \psi, \tag{2.19}$$

where ψ is a generic field of the theory, g' is the coupling constant associated with the $U(1)$ factor of the gauge group, and $Y(\psi)$ is a quantum number, usually called the *weak hypercharge*, to be specified for each field ψ. A new gauge vector field B^μ must be introduced, and the covariant derivative becomes

$$D^\mu = \partial^\mu - igW_i^\mu T_i - ig'\frac{Y}{2}B^\mu, \tag{2.20}$$

where Y is a diagonal matrix with the hypercharge values in its diagonal entries. Y being diagonal, only the term $\mathcal{L}_n^W$ is modified. We have now

$$\begin{aligned} \mathcal{L}_n^W &= \frac{g}{2}W_3^\mu\,(\overline{\nu}_{eL}\gamma_\mu\nu_{eL} - \overline{e}_L\gamma_\mu e_L) \\ &+ \frac{g'}{2}B^\mu\,[Y(L)\,(\overline{\nu}_{eL}\gamma_\mu\nu_{eL} + \overline{e}_L\gamma_\mu e_L) + Y(\nu_{eR})\overline{\nu}_{eR}\gamma_\mu\nu_{eR} + Y(e_R)\overline{e}_R\gamma_\mu e_R] \end{aligned} \tag{2.21}$$

We can now assign the quantum numbers Y in such a way that the electromagnetic interaction term appear in eq. (**??**). To do this, we first perform a rotation of an angle θ_W in the space of the two neutral gauge fields W_3^μ, B^μ:

$$A^\mu = \quad B^\mu\cos\theta_W + W_3^\mu\sin\theta_W \tag{2.22}$$

$$Z^\mu = -B^\mu\sin\theta_W + W_3^\mu\cos\theta_W. \tag{2.23}$$

To identify one of the two neutral vector fields, say A^μ, with the photon field, we must choose $Y(L)$, $Y(\nu_{eR})$ and $Y(e_R)$ so that A^μ couples to the electromagnetic current, $-e\overline{e}\gamma_\mu eA^\mu$. The remaining terms of the lagrangian will define the weak neutral current coupled to the other neutral vector boson Z_μ. After some algebra (a useful exercise!), we find

$$\begin{aligned} g\sin\theta_W &= e \\ g'\cos\theta_W &= e, \end{aligned} \tag{2.24}$$

where e is the positron charge, and

$$Y(L) = -1, \quad Y(\nu_{eR}) = 0, \quad Y(e_R) = -2. \tag{2.25}$$

In general, for a generic fermion field with charge Q (in units of e) and third component of weak isospin T_3 (1/2 for ν_{eL}, $-1/2$ for e_L, 0 for ν_{eR} and e_R), we have

$$Y = 2(Q - T_3). \tag{2.26}$$

Notice that the right-handed neutrino has zero charge and zero hypercharge, and it is an $SU(2)$ singlet: it does not take part in electroweak interactions. Notice also that the above hypercharge assignments can be rescaled by a common factor, provided the coupling constant g' is correspondingly redefined (only the product $g'Y$ appears in the lagrangian). The choice we made in eqs. (**??**) and (**??**) is universally adopted.

If we form a column vector Ψ with all the fermionic fields present in the theory (with left and right-handed components of the same particle counted separately), we can write the neutral-current electroweak-interaction lagrangian in the following, general form:

$$\mathcal{L}_n^W = e\overline{\Psi}\gamma_\mu Q\Psi A^\mu + \overline{\Psi}\gamma_\mu Q_Z\Psi Z^\mu, \tag{2.27}$$

where e is the positron charge, Q is the diagonal matrix of electromagnetic charges, and Q_Z is a diagonal matrix given by

$$Q_Z = \frac{e}{\cos\theta_W \sin\theta_W}\left(T_3 - Q\sin^2\theta_W\right). \tag{2.28}$$

The extension of the theory to other lepton doublets is straightforward.

We must now include hadrons in the theory. We will do this in terms of quark fields, taking as a starting point the hadronic current responsible for β decay and strange particle decays:

$$J^\mu_{had} = \cos\theta_c\overline{u}\gamma^\mu\frac{1}{2}(1-\gamma_5)d + \sin\theta_c\overline{u}\gamma^\mu\frac{1}{2}(1-\gamma_5)s, \tag{2.29}$$

where θ_c is the Cabibbo angle ($\theta_c \sim 13°$) and u, d, s are the up, down and strange quark fields respectively. We are tempted to proceed as in the case of leptons: define

$$Q = \frac{1}{2}(1-\gamma_5)\begin{bmatrix} u \\ d \\ s \end{bmatrix} \equiv \begin{bmatrix} u_L \\ d_L \\ s_L \end{bmatrix} \tag{2.30}$$

and

$$T^+ = \begin{bmatrix} 0 & \cos\theta_c & \sin\theta_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.31}$$

so that

$$J^\mu_{had} = \overline{Q}\gamma^\mu T^+ Q. \tag{2.32}$$

This leads to a system of currents which is in contrast with experimental observations. In fact, we find that

$$[T^+, T^-] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos^2\theta_c & -\cos\theta_c\sin\theta_c \\ 0 & -\cos\theta_c\sin\theta_c & -\sin^2\theta_c \end{bmatrix}, \tag{2.33}$$

and, because of the non-zero off-diagonal entries, the corresponding current contains flavour-changing neutral terms, like for example $\overline{d}_L\gamma^\mu s_L$, with a weight of the same order of magnitude of flavour-conserving ones. This would induce processes such as for example the decay $K^0 \to \pi^0 e^+ e^-$, which are not observed at the expected rate. We must then modify our theory in order to avoid the introduction of flavour-changing neutral currents. The solution to this puzzle was found by Glashow, Iliopoulos and Maiani. They suggested to introduce a fourth quark c (for *charm*) with charge 2/3 like the up quark, and to assume that its couplings to down and strange quarks are given by

$$\begin{aligned} J^\mu_{had} &= \cos\theta_c\overline{u}\gamma^\mu\frac{1}{2}(1-\gamma_5)d + \sin\theta_c\overline{u}\gamma^\mu\frac{1}{2}(1-\gamma_5)s \\ &- \sin\theta_c\overline{c}\gamma^\mu\frac{1}{2}(1-\gamma_5)d + \cos\theta_c\overline{c}\gamma^\mu\frac{1}{2}(1-\gamma_5)s. \end{aligned} \tag{2.34}$$

The c quark being not observed at the time, they had to assume that its mass was much larger than those of u, d and s quarks, and therefore outside the energy range of available experimental devices. The current J^μ_{had} can still be put in the form (**??**), where now

$$Q = \begin{bmatrix} u_L \\ c_L \\ d_L \\ s_L \end{bmatrix} \tag{2.35}$$

and

$$T^+ = \begin{bmatrix} 0 & 0 & \cos\theta_c & \sin\theta_c \\ 0 & 0 & -\sin\theta_c & \cos\theta_c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.36}$$

No flavour-changing neutral current is now present. In fact,

$$[T^+, T^-] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{2.37}$$

thanks to the fact that the upper right 2×2 block of T^+ has been cleverly chosen to be an orthogonal matrix. The existence of the quark c was later confirmed by the discovery of the J/ψ particle. The current J^μ_{had} is usually written in the following form, analogous to the corresponding leptonic current:

$$J^\mu_{had} = (\overline{u}_L \overline{d}'_L)\gamma^\mu \tau^+ \begin{pmatrix} u_L \\ d'_L \end{pmatrix} + (\overline{c}_L \overline{s}'_L)\gamma^\mu \tau^+ \begin{pmatrix} c_L \\ s'_L \end{pmatrix}, \tag{2.38}$$

where

$$\begin{pmatrix} d'_L \\ s'_L \end{pmatrix} = K \begin{pmatrix} d_L \\ s_L \end{pmatrix}, \qquad K = \begin{bmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{bmatrix}. \tag{2.39}$$

The pairs (u, d), (c, s) are called *quark families*. Actually, there is a correspondence between quark families and lepton families, a correspondence that will become clear in section **??**. The structure outlined above can be extended to an arbitrary number of quark families. With n families, K becomes an $n \times n$ matrix, and it must be unitary in order to ensure the absence of flavour-changing neutral currents.

The final form for the charged-current interaction term, including n families of leptons and quarks, is then

$$\mathcal{L}^W_c = \frac{g}{\sqrt{2}} \sum_{f=1}^{n} \left[\overline{L}_f \gamma^\mu \tau^+ L_f + \overline{Q}_f \gamma^\mu \tau^+ Q_f \right] W^+_\mu + h.c., \tag{2.40}$$

where

$$L_f = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \ldots \tag{2.41}$$

$$Q_f = \begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \begin{pmatrix} c_L \\ s'_L \end{pmatrix}, \ldots, \tag{2.42}$$

while the neutral-current lagrangian in eq. (??) is directly generalizable to include quark fields.

To conclude the construction of the gauge-invariant part of the standard model lagrangian, we must consider the pure Yang-Mills term

$$\mathcal{L}_{YM} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W_i^{\mu\nu}, \tag{2.43}$$

where

$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$$
$$W_i^{\mu\nu} = \partial^\mu W_i^\nu - \partial^\nu W_i^\mu + g\epsilon_{ijk} W_j^\mu W_k^\nu. \tag{2.44}$$

The corresponding expression in terms of the physical fields $W^\pm_\mu$, Z_μ and A_μ, can be easily worked out with the help of eqs. (??), (??) and (??). Try to derive it as an exercise!

3. Masses

In order to make contact with the Fermi theory, which is known to correctly describe low-energy weak interactions, we must give a mass to the gauge vector bosons of weak interactions. In fact, we are even able to estimate the order of magnitude of the W mass. Consider, for example, the amplitude for down-quark β decay. In the Fermi theory, it is simply given by

$$\frac{G_\beta}{\sqrt{2}} \overline{u} \gamma^\mu (1-\gamma_5) d \overline{e} \gamma_\mu (1-\gamma_5) \nu_e. \tag{3.1}$$

In the context of the standard model, the same process is induced by the exchange of a W boson, giving rise to the amplitude

$$\left(\frac{g}{\sqrt{2}} \overline{u}_L \gamma^\mu d_L \right) \frac{1}{q^2 - m_W^2} \left(\frac{g}{\sqrt{2}} \overline{e}_L \gamma_\mu \nu_{eL} \right), \tag{3.2}$$

where q is the momentum of the virtual W (we are neglecting Cabibbo mixing for simplicity). We see that, for eq. (??) to be equal to the Fermi amplitude in the $q \to 0$

limit, m_W must be non zero, and

$$\frac{G_\beta}{\sqrt{2}} = \left(\frac{g}{2\sqrt{2}}\right)^2 \frac{1}{m_W^2}. \tag{3.3}$$

Recalling that $g = e/\sin\theta_W$, eq. (**??**) gives us the lower bound

$$m_W \geq 37.3 \ \ GeV, \tag{3.4}$$

quite a large value! So, we know since the beginning that, if weak interactions are to be mediated by vector bosons, these must be quite heavy. On the other hand, we also know that gauge theories are incompatible with mass terms for the vector bosons. One possibility is to break gauge invariance explicitly and insert a mass term for the W boson by hand, but this leads to a non-renormalizable theory. Let us investigate this point in more detail. Consider for simplicity the lagrangian of a pure abelian gauge theory, with a mass term for the gauge vector field:

$$\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2} m_A^2 A^\mu A_\mu, \tag{3.5}$$

and work out the propagator $\Delta^{\mu\nu}$ for A^μ in momentum space. We get

$$\Delta^{\mu\nu} = \frac{i}{k^2 - m_A^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_A^2}\right). \tag{3.6}$$

The propagator $\Delta^{\mu\nu}$ has not the correct behaviour for large values of the momentum k: for $k \to \infty$ it becomes a constant, rather than vanishing as k^{-2}, thus violating power-counting and making the theory unrenormalizable.

To see how one can introduce a mass term for gauge vector bosons without spoiling renormalizability, we first consider a simple example where this happens, and then we generalize our considerations to the standard model. The simple theory we consider is scalar electrodynamics, that is, a gauge theory based on $U(1)$ invariance, coupled to one complex scalar field ϕ with charge $-e$. The lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\phi)^\dagger D_\mu\phi - V(\phi), \tag{3.7}$$

where $D^\mu = \partial^\mu + ieA^\mu$, and $V(\phi)$ is the so-called scalar potential, which is constrained by gauge invariance and renormalizability to be of the form

$$V(\phi) = m^2 \mid \phi \mid^2 + \lambda \mid \phi \mid^4 . \tag{3.8}$$

We look for field configurations that minimize the energy of the system. Because of the requirement of translational invariance, they must be constant configurations, so we can neglect the derivative terms and look for the minimum of the potential V. Now, if $m^2 \geq 0$, then V has a minimum for $\phi = 0$. If, on the other hand, $m^2 < 0$, then m^2 can no longer be interpreted as a mass squared for the field ϕ; furthermore, the potential has now an infinite number of degenerate minima, given by all those field configurations for which

$$| \phi |^2 = -\frac{m^2}{2\lambda} \equiv \frac{1}{2} v^2. \tag{3.9}$$

All these minimum configurations (in the language of quantum theory, all these ground states) are connected by gauge transformations, that change the phase of the complex field ϕ without affecting its modulus. When the system chooses one of the minimum configurations, the gauge symmetry is broken. This phenomenon is usually called *spontaneous breaking of the gauge symmetry*, but the symmetry is not actually broken. In fact, the Lagrangian is still gauge invariant, and all the properties connected with this invariance (such as, for example, current conservation) are still there. It is important to stress this point, because at the quantum level this is essentially what guarantees the renormalizability of the theory, which would instead be lost in the case of an explicit breaking of the gauge symmetry.

Let us now expand the field ϕ around the minimum configuration. We introduce real scalar fields $H(x)$ and $G(x)$ by

$$\phi(x) = \frac{1}{\sqrt{2}} \left[v + H(x) + iG(x) \right], \tag{3.10}$$

where v is defined in eq. (**??**). One of the two fields H and G could in principle be removed from the lagrangian by an appropriate gauge transformation. For example, one could eliminate G by choosing a gauge transformation that brings ϕ to be real. For the moment, we keep both H and G in the lagrangian; we will come back to this point later. Up to an irrelevant constant, the scalar potential takes the form

$$V(\phi) = (m^2 v + \lambda v^3) H + \frac{1}{2}(m^2 + 3\lambda v^2) H^2 + \frac{1}{2}(m^2 + \lambda v^2) G^2 + \lambda v H (H^2 + G^2) + \frac{\lambda}{4}(H^2 + G^2)^2. \tag{3.11}$$

Using eq. (**??**), $\lambda v = -m^2$, we see that the terms proportional to H and G^2 vanish, which means that the field G is massless. The coefficient of the H^2 term is now

$(-2m^2)/2$, and has therefore the correct sign to be interpreted as a mass term.

What about the covariant derivative term? We have

$$\begin{aligned}(D^\mu\phi)^\dagger D_\mu\phi &= \frac{1}{2}\partial^\mu H\partial_\mu H + \frac{1}{2}\partial^\mu G\partial_\mu G + \frac{1}{2}e^2(H^2+G^2+2vH)A^\mu A_\mu \\ &+ eA_\mu(H\partial^\mu G - G\partial^\mu H) + evA^\mu\partial_\mu G + \frac{1}{2}e^2v^2A^\mu A_\mu. \end{aligned} \quad (3.12)$$

We see that the gauge field A_μ has acquired a mass $m_A = ev$, precisely the result we were looking for. The term $evA^\mu\partial_\mu G$ is unpleasant, because it mixes the gauge vector field A^μ with the unphysical field G; we will see in a moment how to get rid of it.

We must now check that the appearance of a mass term for A^μ via the spontaneous symmetry breaking mechanism has not spoiled the renormalizability of our theory, contrary to what happened when we tried to break the symmetry explicitly. It is well known that, in order to quantize a gauge theory, a gauge-fixing term must be added to the lagrangian (obviously, this was not necessary in the case of explicit gauge symmetry breaking). A convenient choice for the gauge-fixing term is

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial^\mu A_\mu + b\xi G)^2, \quad (3.13)$$

where ξ is an arbitrary constant (the gauge parameter). Equation (**??**) gives the gauge-fixing condition $\partial^\mu A_\mu + b\xi G = 0$. The constant b can now be adjusted in order to cancel the unwanted $A^\mu\partial_\mu G$ term we encountered above; it is easy to see that this happens for $b = ev = m_A$. Observe also that the gauge-fixing lagrangian introduces a term

$$-\frac{1}{2}\xi m_A^2 G^2 \quad (3.14)$$

which gives a squared mass ξm_A^2 to the unphysical field G.

The part of the lagrangian quadratic in A_μ is now

$$-\frac{1}{2}(\partial^\mu A^\nu\partial_\mu A_\nu - \partial^\mu A^\nu\partial_\nu A_\mu) + \frac{1}{2}m_A^2 A^\mu A_\mu - \frac{1}{2\xi}(\partial^\mu A_\mu)^2, \quad (3.15)$$

from which we obtain the following expression for the propagator in momentum space:

$$\Delta_\xi^{\mu\nu} = \frac{i}{k^2 - m_A^2}\left[-g^{\mu\nu} + \frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi m_A^2}\right]. \quad (3.16)$$

The propagator has now the correct behaviour at large momenta. The price we have paid for this is that the photon propagator has now, in addition to the pole at $k^2 = m_A^2$, an unphysical singularity at $k^2 = \xi m_A^2$. This singularity is located at the mass squared of the unphysical scalar field G. One can check that the contribution of this term of the photon propagator to physical quantities is exactly cancelled by the contribution of G exchange, thus giving a unitary picture. When we let ξ tend to infinity, the photon propagator takes the form of eq. (**??**). The theory is still renormalizable, but now renormalizability must arise as a consequence of cancellations among different contributions to the same Green function, since the propagator does not obey the power-counting rule. This is called the *unitary gauge*, in which renormalizability is not manifest. The advantage of the unitary gauge is that the theory contains only physical degrees of freedom. In fact, when $\xi \to \infty$ the gauge-fixing condition reduces to $G(x) = 0$ (see eq. (**??**)); it corresponds to the gauge choice that eliminates G from the theory since the very beginning. Two common choices are the Feynman gauge, $\xi = 1$, which gives

$$\Delta_F^{\mu\nu} = -\frac{ig^{\mu\nu}}{k^2 - m_A^2} \tag{3.17}$$

and the Landau gauge, $\xi = 0$, for which

$$\Delta_L^{\mu\nu} = \frac{i}{k^2 - m_A^2}\left[-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right]. \tag{3.18}$$

One last observation about the field $G(x)$. It looks like we lost a degree of freedom, since we started with a complex scalar field and we end up with one real scalar. Actually, the number of degrees of freedom stays the same, since the photon is now massive, and has therefore three polarization states instead of two. The field $G(x)$ is called a *would-be Goldstone boson.* This terminology reflects the fact that, in the absence of gauge invariance and of the gauge-fixing term, G would have simply been a physical, zero-mass state, which is always present when spontaneous symmetry breaking occurs. This mechanism is known as the Higgs mechanism. It is possible to extend it to the standard model, with a few modifications that we now describe in detail.

We have learned that, in order to break spontaneously a gauge symmetry, we must introduce scalar fields in the game. How should we do this in the standard model?

First, the scalar field must transform non-trivially under that part of the gauge group that we want to undergo spontaneous breaking. Secondly, we must be careful not to break the $U(1)$ invariance corresponding to electrodynamics, or, in other words, we want the photon to stay massless. This means that spontaneous symmetry breaking must take place in three of the four "directions" of the $SU(2) \times U(1)$ gauge group, the fourth one being that corresponding to electric charge. The simplest way to do this is to assign the scalar field ϕ to a doublet representation of $SU(2)$, and to impose that one the two components have zero electric charge:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}; \qquad Y(\phi) = 1. \tag{3.19}$$

The Higgs mechanism takes place in analogy with scalar electrodynamics. The most general scalar potential consistent with gauge invariance and renormalizability is

$$V(\phi) = m^2 \mid \phi \mid^2 + \lambda \mid \phi \mid^4, \tag{3.20}$$

which has a minimum at

$$\mid \phi \mid^2 = -\frac{m^2}{2\lambda} \equiv \frac{1}{2} v^2. \tag{3.21}$$

We can reparameterize ϕ in the following way:

$$\phi = \frac{1}{\sqrt{2}} e^{i\tau^i \theta^i(x)/v} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \tag{3.22}$$

with $\theta^i(x)$ and $H(x)$ real. This parametrization is not suited for renormalizable gauges, because it is non-linear and contains all powers of the fields θ_i. It is convenient, however, if we work in the unitary gauge; in fact, it is apparent that the fields θ_i can be rotated away by an $SU(2)$ gauge transformation. In this section, we will use the unitary gauge $\theta_i = 0$. The standard model lagrangian in a generic renormalizable gauge is given in appendix **??**.

The scalar potential takes the form

$$V = \frac{1}{2}(2\lambda v^2) H^2 + \lambda v H^3 + \frac{1}{4}\lambda H^4; \tag{3.23}$$

the Higgs scalar H has a squared mass $m_H^2 = 2\lambda v^2$. The term $(D^\mu \phi)^\dagger D_\mu \phi$ can be worked out using eq. (??) with $\theta^i = 0$. We get

$$D^\mu \phi = \left(\partial^\mu - i\frac{g}{2}\tau^i W^i_\mu - i\frac{g'}{2}B_\mu\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x)\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 \\ \partial^\mu H\end{pmatrix} - \frac{i}{2}(v+H)\begin{pmatrix} g(W_1^\mu - iW_2^\mu) \\ -gW_3^\mu + g'B^\mu\end{pmatrix}\right]$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \partial^\mu H\end{pmatrix} - \frac{i}{2}\left(1+\frac{H}{v}\right)\begin{pmatrix} gvW^{\mu+} \\ -\sqrt{(g^2+g'^2)/2}vZ^\mu\end{pmatrix} \tag{3.24}$$

where, to obtain the last line, we have used eqs. (??), (??) and (??). We have therefore

$$(D^\mu\phi)^\dagger D_\mu\phi = \frac{1}{2}\partial^\mu H\partial_\mu H + \left[\frac{1}{4}g^2v^2W^{\mu+}W^-_\mu + \frac{1}{8}(g^2+g'^2)v^2Z^\mu Z^-_\mu\right]\left(1+\frac{H}{v}\right)^2. \tag{3.25}$$

We see that the W and Z bosons have acquired masses given by

$$m_W^2 = \frac{1}{4}g^2v^2 \tag{3.26}$$

$$m_Z^2 = \frac{1}{4}(g^2+g'^2)v^2. \tag{3.27}$$

Notice that the photon stays massless. With the scalar field ϕ transforming as a doublet of $SU(2)$, there is always a linear combination of B^μ and W_3^μ that does not receive a mass term, but only if $Y(\phi) = 1$ (or -1) does this linear combination coincide with the one in eq. (??). The lagrangian in a generic renormalizable gauge is much more complicated, since it also involves kinetic and interaction terms for non-physical Higgs scalars, the would-be Goldstone bosons. It is described in appendix ??. The value of v, the *vacuum expectation value* of the neutral component of the Higgs doublet, can be deduced combining eqs. (??) and (??), and using the measured valued of the Fermi constant. We get

$$v = \sqrt{\frac{1}{G_\mu\sqrt{2}}} \simeq 246.22\ GeV. \tag{3.28}$$

Fermion masses are also forbidden by the gauge symmetry of the standard model.

In fact, the mass term for a fermion field ψ has the form

$$-m\overline{\psi}\psi = -m(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L), \tag{3.29}$$

and cannot be invariant under a chiral transformation, that is, a transformation that acts differenly on left-handed and right-handed fields. The gauge transformations of the standard model are precisely of this kind. Again, this difficulty can be circumvented by means of the Higgs doublet ϕ.

We first consider the hadronic sector. We have seen in section **??** that the interaction lagrangian is not diagonal in terms of quark mass eigenstates. Therefore, to be general, we introduce quark fields u'_f and d'_f, which are generic linear combination of the mass eigenstates u_f, d_f (the index f runs over the n fermion generations). We also define

$$Q'_f = \begin{pmatrix} u'_{fL} \\ d'_{fL} \end{pmatrix} \quad U'_f = u'_{fR} \quad D'_f = d'_{fR}. \tag{3.30}$$

A Yukawa interaction term can be added to the lagrangian:

$$\mathcal{L}_Y^{hadr} = -(\overline{Q'}\phi h'_D D' + \overline{D'}\phi^\dagger h'^\dagger_D Q') - (\overline{Q'}\phi_c h'_U U' + \overline{U'}\phi_c^\dagger h'^\dagger_U Q'), \tag{3.31}$$

where h'_U and h'_D are generic $n \times n$ constant matrices in the generation space, and

$$\phi_c = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}. \tag{3.32}$$

It easy to check that $\mathcal{L}_Y^{hadr}$ is Lorentz-invariant, gauge-invariant[3] and renormalizable, and therefore it can (actually, it must!) be included in the lagrangian. Now, we define new quark fields u and d by

$$u'_L = K_L^U u_L, \quad u'_R = K_R^U u_R \tag{3.33}$$

$$d'_L = K_L^D d_L, \quad d'_R = K_R^D d_R, \tag{3.34}$$

where $K_{L,R}^{U,D}$ are unitary matrices, chosen so that

$$h_U \equiv K_L^{U\dagger} h'_U K_R^U \tag{3.35}$$

[3]In fact, if ϕ transforms as an $SU(2)$ doublets, so does $\phi_c = \epsilon\phi^*$, where ϵ is the antisymmetric tensor; check it as an exercise.

and

$$h_D \equiv K_L^{D\dagger} h'_D K_R^D \tag{3.36}$$

are diagonal with real, non-negative entries (it is always possible). In the unitary gauge, eq. (??) becomes

$$\mathcal{L}_Y^{hadr} = -\frac{1}{\sqrt{2}}(v+H)\sum_{f=1}^{n}(h_D^f \overline{d}_f d_f + h_U^f \overline{u}_f u_f), \tag{3.37}$$

where $h^f_{U,D}$ are the diagonal entries of $h_{U,D}$. As we already know (see eq. (??), the charged hadronic weak current takes the form

$$J^\mu_{hadr} = \overline{Q'}\gamma^\mu \tau^+ Q' = \sum_{f,f'} \overline{u}_L^f \gamma^\mu K_{ff'} d_L^{f'}, \tag{3.38}$$

where

$$K = K_L^{U\dagger} K_L^D. \tag{3.39}$$

We can now identificate the quark masses by

$$m_U^f = \frac{v h_U^f}{\sqrt{2}}, \quad m_D^f = \frac{v h_D^f}{\sqrt{2}}. \tag{3.40}$$

The case of leptons is much simpler, since there are no right-handed neutrinos. The most general Yukawa interaction term is therefore

$$\mathcal{L}_Y^{lept} = -\sum_{f=1}^{n} h_L^f(\overline{L}_f \phi e_R^f + \overline{e}_R^f \phi^\dagger L_f). \tag{3.41}$$

The values of the Yukawa couplings h_L^f are determined by the values of the observed lepton masses. In fact, using eq. (??), we find

$$\mathcal{L}_Y^{lept} = -\sum_{f=1}^{n} \frac{h_L^f}{\sqrt{2}}(v+H)\overline{e}_f e_f, \tag{3.42}$$

thus allowing the identifications

$$m_L^f = \frac{v h_L^f}{\sqrt{2}}. \tag{3.43}$$

As in the case of vector bosons, in renormalizable gauges there are also interaction terms between quarks and non-physical scalars; the details are given in appendix **??**.

4. The problem of anomalies

We have seen in the previous sections that the renormalizability of the standard model in strictly connected with gauge invariance. In particular, we have seen that the massive vector boson propagators show unphysical singularities, that are cancelled by the presence of would-be Goldstone bosons. In turn, gauge invariance manifests itself in the form of identities between Green functions, called Slavnov-Taylor identities, that are consequences of current conservation, and that must hold at all perturbative orders for the theory to be renormalizable. In this section, we will show that this might not be the case for theories with axial currents, as the standard model itself. It might happen that current conservation is spoiled at the quantum level, unless the spectrum of the theory satisfies particular conditions. In the language of quantum field theory, terms that spoil the validity of Slavnov-Taylor identities are called *anomalies*. We will illustrate the problem of anomalies in the context of a simple example, and we will then state under which conditions the standard model is anomaly-free and renormalizable.

We consider quantum electrodynamics with one massive fermion, ψ. We consider the operators

$$V^\mu = \overline{\psi}\gamma^\mu\psi \tag{4.1}$$

$$A^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi \tag{4.2}$$

$$P = \overline{\psi}\gamma_5\psi. \tag{4.3}$$

It is easy to show, using the equations of motion, that

$$\partial_\mu V^\mu = 0 \tag{4.4}$$

$$\partial_\mu A^\mu = 2imP, \tag{4.5}$$

where m is the fermion mass. The interpretation of eqs. (**??**) and (**??**) is well known. Equation (**??**) is simply the conservation of the electromagnetic current, which reflects the gauge-invariance of the theory. The curren A^μ, on the other hand, is the current associated with axial transformations,

$$\psi \to e^{i\alpha\gamma_5}\psi. \tag{4.6}$$

The lagrangian of massive QED is not invariant under axial transformations because of the presence of the mass term, and as a consequence the associated current A^μ is not conserved. Equation (**??**) precisely states this fact.

Now consider the Green functions

$$T^{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1 d^4x_2 e^{ik_1x_1+ik_2x_2} < 0|T[V^\mu(x_1)V^\nu(x_2)A^\rho(0)]|0> \tag{4.7}$$

and

$$T^{\mu\nu}(k_1,k_2) = i\int d^4x_1 d^4x_2 e^{ik_1x_1+ik_2x_2} < 0|T[V^\mu(x_1)V^\nu(x_2)P(0)]|0> . \tag{4.8}$$

They formally satisfy the Slavnov-Taylor identities

$$k_1^\mu T_{\mu\nu\rho} = k_2^\nu T_{\mu\nu\rho} = 0 \tag{4.9}$$

$$q^\rho T_{\mu\nu\rho} = 2mT_{\mu\nu}, \tag{4.10}$$

where $q = k_1 + k_2$. These identities can be worked out by exploiting eqs. (**??**) and (**??**), and the canonical commutation relations. We will now check explicitly whether esq. (**??**,**??**) are satisfied in perturbation theory or not. At the one-loop order, the diagrams to be computed are those of fig. (1). We have

$$T^{\mu\nu\rho}(k_1,k_2) = T_1^{\mu\nu\rho}(k_1,k_2) + T_2^{\mu\nu\rho}(k_1,k_2) \tag{4.11}$$

$$T^{\mu\nu}(k_1,k_2) = T_1^{\mu\nu}(k_1,k_2) + T_2^{\mu\nu}(k_1,k_2), \tag{4.12}$$

where

$$T_1^{\mu\nu\rho} = -i\int \frac{d^4p}{(2\pi)^4} Tr\left[\frac{i}{\hat{p}-m}\gamma^\rho\gamma_5\frac{i}{\hat{p}-\hat{q}-m}\gamma^\nu\frac{i}{\hat{p}-\hat{k_1}-m}\gamma^\mu\right] \tag{4.13}$$

$$T_1^{\mu\nu} = -i\int \frac{d^4p}{(2\pi)^4} Tr\left[\frac{i}{\hat{p}-m}\gamma_5\frac{i}{\hat{p}-\hat{q}-m}\gamma^\nu\frac{i}{\hat{p}-\hat{k_1}-m}\gamma^\mu\right] \tag{4.14}$$

and

$$T_2^{\mu\nu\rho}(k_1,k_2) = T_1^{\nu\mu\rho}(k_2,k_1) \tag{4.15}$$

$$T_2^{\mu\nu}(k_1,k_2) = T_1^{\nu\mu}(k_2,k_1). \tag{4.16}$$

The overall minus sign is due to the presence of a fermion loop.

The loop integrals in eq. (??) and (??) are superficially divergent. We must therefore choose a regularization scheme before proceeding. Dimensional regularization is not suited here, because of the presence of γ_5, which has an intrinsically four-dimensional meaning and cannot be generalized to other space-time dimensions in a simple way. We will make a different choice, keeping in mind, however, that it is indeed possible, although quite complicated, to treat this problem within dimensional regularization. The regularization scheme we choose is the following. We subtract from each integrand in eqs. (??) and (??) the same expression, but with m replaced by an arbitrary regularization parameter M. In the limit $M \to \infty$ the original expression is recovered, while, for finite M, the integrals are now convergent. We will indicate with a subscript M the regularized quantities.

Equations (??), that state the conservation of the vector current, are satisfied by $T^{\mu\nu\rho}$ as given in eqs. (??) and (??). In fact, writing

$$\hat{k}_1 = -(\hat{p} - \hat{k}_1 - m) + (\hat{p} - m) \tag{4.17}$$

in $T_1^{\mu\nu\rho}$, and

$$\hat{k}_1 = -(\hat{p} - \hat{k}_1 - \hat{k}_2 - m) + (\hat{p} - \hat{k}_2 - m) \tag{4.18}$$

in $T_2^{\mu\nu\rho}$, (and similarly in the regularizing part of the integrands), we find $[k_1^\mu T_{\mu\nu\rho}]_M = [k_2^\nu T_{\mu\nu\rho}]_M = 0$. The limit $M \to \infty$ can then be taken safely.

We turn now to eq. (??). We first consider $T^{\mu\nu}(k_1, k_2)$. Using the Feynman parametrization

$$\begin{aligned}\frac{1}{d_1^{\alpha_1} \ldots d_n^{\alpha_n}} &= \frac{\Gamma(\alpha_1 + \ldots + \alpha_n)}{\Gamma(\alpha_1) \ldots \Gamma(\alpha_n)} \\ &\times \int_0^1 dx_1 \ldots \int_0^1 dx_n \frac{x_1^{\alpha_1 - 1} \ldots x_n^{\alpha_n - 1} \delta(1 - x_1 - \ldots - x_n)}{(x_1 d_1 + \ldots + x_n d_n)^{\alpha 1 + \ldots + \alpha_n}},\end{aligned} \tag{4.19}$$

we find

$$\begin{aligned}&[T_1^{\mu\nu}]_M \\ = \; & 2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^4 p}{(2\pi^4)} \left[\frac{4im\epsilon_{\rho\nu\sigma\mu} k_2^\rho k_1^\sigma}{[p^2 - 2p(qx + k_1(1 - x - y)) + q^2 x - m^2]^3} - (m \to M) \right],\end{aligned} \tag{4.20}$$

where we have set $k_1^2 = 0$. The simple expression in the numerator is obtained by dropping all products of γ_5 with two, three and five γ matrices, and exploiting the antisymmetry of $\epsilon_{\rho\nu\sigma\mu}$. The integration over the loop momentum p can be easily performed by shifting the integration variable

$$p \to p + qx + k_1(1-x-y), \tag{4.21}$$

with the result

$$[T_{\mu\nu}]_M = \frac{1}{2\pi^2}\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma \int_0^1 dx \int_0^{1-x} dy \left[\frac{m}{m^2 - q^2xy} - \frac{M}{M^2 - q^2xy}\right]. \tag{4.22}$$

We are now ready to check the identity [**??**]. Using the identity

$$\hat{q}\gamma_5 = 2m\gamma_5 + \gamma_5(\hat{p} - \hat{q} - m) + (\hat{p} - m)\gamma_5, \tag{4.23}$$

we get

$$[q_\rho T^{\mu\nu\rho}]_M = [2mT^{\mu\nu}]_M + [R^{\mu\nu}]_M\,, \tag{4.24}$$

where

$$R^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} Tr\left[\frac{i}{\hat{p}-m}\gamma_5\gamma^\nu \frac{i}{\hat{p}-\hat{k}_1-m}\gamma^\mu - \frac{i}{\hat{p}-\hat{k}_2-m}\gamma_5\gamma^\nu \frac{i}{\hat{p}-\hat{q}-m}\gamma^\mu \right.$$
$$\left. + \frac{i}{\hat{p}-m}\gamma_5\gamma^\mu \frac{i}{\hat{p}-\hat{k}_2-m}\gamma^\nu - \frac{i}{\hat{p}-\hat{k}_1-m}\gamma_5\gamma^\mu \frac{i}{\hat{p}-\hat{q}-m}\gamma^\nu\right]. \tag{4.25}$$

It is now easy to check that $[R^{\mu\nu}]_M$ vanishes. In fact, by shifting the loop momentum p to $p + k_2$ in the second term, and to $p + k_1$ in the fourth, they cancel against the first and the third respectively. The important point here is that these shifts in the integration variable can be performed only after regularizing the integrals. Therefore, using eq. [**??**],

$$[q_\rho T^{\mu\nu\rho}]_M = [2mT^{\mu\nu}]_M = 2mT^{\mu\nu} - \frac{1}{\pi^2}\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma \int_0^1 dx \int_0^{1-x} dy \frac{M^2}{M^2 - q^2xy}. \tag{4.26}$$

The limit $M \to \infty$ can now be taken safely, giving

$$q^\rho T_{\mu\nu\rho} = 2mT_{\mu\nu} - \frac{1}{2\pi^2}\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma. \tag{4.27}$$

The effect of the regularization is that the Slavnov-Taylor identity in eq. (**??**) is spoiled by an *anomalous* term, which is usually called the *axial anomaly*, or the Adler-Bardeen-Jackiw anomaly. This term arises because of the impossibility of regularizing the theory in a way that preserves both the vector and axial vector classical current divergence relations; one of the two must be given up.

The anomalous term can be taken into account by modifying eq. (**??**) at the one-loop level in the following way:

$$\partial_\mu A^\mu = 2imP + \frac{1}{(4\pi)^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{4.28}$$

where $F_{\mu\nu}$ is the field-strength tensor of QED. In other words, the axial current is not conserved, at the quantum level, even if $m = 0$. Notice in fact that the anomalous term is independent of the fermion mass. Furthermore, it can be proved that higher-order corrections do not modify the one-loop expression of the anomaly.

The above considerations can be extended to the case of a theory with non-abelian gauge invariance. In this case, also fermion loops with four and five internal lines contribute to the anomaly. It can be shown that the anomalous term of the axial vector current in a non-abelian theory is proportional to

$$Tr(\{T^a, T^b\}T^c), \tag{4.29}$$

where T^a are the gauge group generators. In the standard model, fermions are either in the doublet or in the singlet representation of $SU(2)$; this means that the four quantities

$$Tr(\{\tau^a, \tau^b\}\tau^c) \tag{4.30}$$

$$Tr(\{\tau^a, \tau^b\}Y) \tag{4.31}$$

$$Tr(Y^2\tau^c) \tag{4.32}$$

$$Tr(Y^3) \tag{4.33}$$

must all vanish, for the axial anomaly to be cancelled. The first quantity is obviously zero:

$$Tr(\{\tau^a, \tau^b\}\tau^c) = 2\delta^{ab}Tr(\tau^c) = 0. \tag{4.34}$$

The second quantity requires more care. Since $\tau^a = 0$ for right-handed fermions, we have

$$Tr(\{\tau^a, \tau^b\}Y) = 2\delta^{ab} Tr(Y_L), \tag{4.35}$$

where Y_L is the hypercharge matrix restricted to left-handed fermions. Since $Y = 1/3$ for the doublets of left-handed quarks, and $Y = -1$ for the doublets of left-handed leptons, we find

$$Tr(Y_L) = n_q \times 3 \times 2 \times \frac{1}{3} + n_l \times 2 \times (-1) = 2(n_q - n_l), \tag{4.36}$$

where n_q (n_l) is the number of quark (lepton) families. The factor of 3 in front of the quark term is due to the colour degree of freedom, and the overall factor of 2 is due to the fact that left-handed fermions are $SU(2)$ doublets. We see that the cancellation of the axial anomaly requires that the numbers of quark and lepton families are equal! This is an important prediction of the standard model, which has been recently confirmed by the discovery of the *top* quark.

The third condition, $Tr(Y^2\tau^c) = 0$, is again trivially satisfied, since Y has the same value for both components of each doublet, and $Tr(\tau^c) = 0$ (for singlets, we have simply $\tau^c = 0$).

The last condition, $Tr(Y^3) = 0$, is also satisfied provided $n_q = n_l$. In fact,

$$Tr(Y^3) = Tr(Y_L^3) - Tr(Y_R^3). \tag{4.37}$$

There is a minus sign in front of Y_R^3 because fermions in the same diagram have the same chirality. Using $Y = 2(Q - T_3)$ we find

$$Tr(Y_L^3) = 6n_q \left(\frac{1}{3}\right)^3 + 2n_l(-1)^3 \tag{4.38}$$

$$Tr(Y_R^3) = 3n_q \left[\left(\frac{4}{3}\right)^3 + \left(-\frac{2}{3}\right)^3\right] + n_l(-2)^3, \tag{4.39}$$

and therefore

$$Tr(Y^3) = -6(n_q - n_l). \tag{4.40}$$

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APPENDIX A: Renormalizability and power counting

In this appendix, we describe the power-counting criterion for renormalizability of local field theories. Consider a Feynman diagram containing

- L loops;
- V vertices;
- F^I internal fermionic lines;
- F^E external fermionic lines;
- B^I internal bosonic lines;
- B^E external bosonic lines.

Let us assume that there are different types of vertices, each type being labelled by the index i, and that

$$V = \sum_i V^{(i)}, \tag{A.1}$$

where $V^{(i)}$ is the number of vertices of type i. Finally, let $n_f^{(i)}$, $n_b^{(i)}$, $d^{(i)}$ be the number of fermionic lines, bosonic lines and field derivatives in type-i vertices, respectively. The following relations hold:

$$2F^I + F^E = \sum_i n_f^{(i)} V^{(i)} \tag{A.2}$$

$$2B^I + B^E = \sum_i n_b^{(i)} V^{(i)}. \tag{A.3}$$

The number L of loops is equal to the number of independent internal momenta, which in turn is equal to the total number of internal lines, minus the number of independent momentum conservation equations. Therefore, we have

$$L = F^I + B^I - (V - 1). \tag{A.4}$$

We now define the degree of superficial divergence D of the diagram as the power of momenta in the numerator minus the power of momenta in denominator of the

Feynman diagram. Clearly,

$$D = 4L - F^I - 2B^I + \sum_i d^i V^i, \tag{A.5}$$

since fermion propagators behave as k^{-1}, boson propagator behave as k^{-2}, each field derivative corresponds to one power of momentum, and four powers of momentum are carried by each loop integration. Now, substituting eqs. (**??**) and (**??**) in eq. (**??**) and eliminating F^I and B^I via eqs. (**??**) and (**??**), we find

$$D = 4 - \frac{3}{2}F^E - B^E + \sum_i V^{(i)} \left[d^{(i)} + \frac{3}{2}n_f^{(i)} + n_b^{(i)} - 4 \right]. \tag{A.6}$$

If $D \geq 0$, the Feynman amplitude will be ultraviolet divergent. On the other hand, $D < 0$ is not a sufficient condition for convergence, since there can still be subdiagrams with $D \geq 0$. However, we notice that D decreases with increasing number of external lines. Therefore, if the last term in the r.h.s. of eq. (**??**) is zero or negative, then only a finite number of diagrams have $D \geq 0$, and the whole theory can be made finite by renormalizing only these *primitively divergent* amplitudes, at any order in perturbation theory. The condition for renormalizability then becomes

$$d^{(i)} + \frac{3}{2}n_f^{(i)} + n_b^{(i)} \leq 4, \tag{A.7}$$

and it must hold for each i separately (a diagram can contain only vertices of one type). Notice that the l.h.s. of eq. (**??**) is just the mass dimension of the operator that corresponds to type i vertices: in fact, fermion fields have dimension 3/2, boson fields have dimension 1 and derivatives have dimension 1. For this reason, the condition in eq. (**??**) can be rephrased in terms of coupling constant dimensionality: a renormalizable theory can contain only constants with mass dimension ≥ 0.

APPENDIX B: Gauge theories

The Dirac free lagrangian for a massive fermion,

$$\mathcal{L} = \overline{\psi}(i\hat{\partial} - m)\psi, \tag{B.1}$$

is invariant under the global (or first kind) $U(1)$ gauge transformation

$$\begin{aligned} \psi \to \psi' &= e^{-ie\alpha}\psi \\ \overline{\psi} \to \overline{\psi}' &= e^{ie\alpha}\overline{\psi}, \end{aligned} \tag{B.2}$$

where α is a real constant (the charge $-e$ of the field ψ has been inserted for later convenience). We want to promote this global symmetry to a local one, that is, we want to modify $\mathcal{L}$ in order to render it invariant under the field transformation (**??**), with $\alpha = \alpha(x)$. The derivative term is not invariant:

$$\overline{\psi}\partial^\mu\psi \to e^{ie\alpha}\overline{\psi}\partial^\mu(e^{-ie\alpha}\psi) = \overline{\psi}\partial^\mu\psi - ie\overline{\psi}(\partial^\mu\alpha)\psi. \tag{B.3}$$

The ordinary derivative must be replaced by a *covariant* derivative,

$$D^\mu = \partial^\mu + ieA^\mu, \tag{B.4}$$

where A^μ is a real vector field. The transformation property of A^μ must be fixed in such a way that

$$D^\mu\psi \to e^{-ie\alpha}D^\mu\psi. \tag{B.5}$$

We find

$$\begin{aligned} D^\mu\psi = (\partial^\mu + ieA^\mu)\psi &\to (\partial^\mu + ieA'^\mu)\psi' \\ &= (\partial^\mu + ieA'^\mu)e^{-ie\alpha}\psi \\ &= e^{-ie\alpha}\partial^\mu\psi - ie(\partial^\mu\alpha)e^{-ie\alpha}\psi + ieA'^\mu e^{-ie\alpha}\psi \\ &= e^{-ie\alpha}(\partial^\mu + ieA'^\mu - ie\partial^\mu\alpha)\psi \\ &= e^{-ie\alpha}(\partial^\mu + ieA^\mu)\psi, \end{aligned} \tag{B.6}$$

which implies

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu\alpha. \tag{B.7}$$

The lagrangian

$$\mathcal{L} = \overline{\psi}(i\hat{D} - m)\psi \tag{B.8}$$

is invariant under the local (or second kind) gauge transformation

$$\begin{aligned} &\psi \to \psi' = e^{-ie\alpha(x)}\psi \\ &\overline{\psi} \to \overline{\psi}' = e^{ie\alpha(x)}\overline{\psi}, \\ &A^\mu \to A'^\mu = A^\mu + \partial^\mu\alpha(x). \end{aligned} \tag{B.9}$$

Notice that the requirement of local gauge invariance has introduced the interaction term $e\overline{\psi}\gamma_\mu\psi A^\mu$.

A kinetic term for the vector field A^μ must now be introduced. It is uniquely fixed by the following requirements:

- Lorentz invariance
- Gauge invariance
- Presence of derivatives of the gauge fields
- Standard normalization of the propagator for A^μ

and it is given by

$$\mathcal{L}^{YM} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{B.10}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{B.11}$$

Notice that

$$(D^\mu D^\nu - D^\nu D^\mu)\psi = ieF^{\mu\nu}\psi, \tag{B.12}$$

and that $F^{\mu\nu}$ is invariant under a gauge transformation. Notice also that gauge invariance forbids the presence of a mass term for the gauge field A^μ. Finally, we observe that no self-interaction term for the vector field A^μ is present in the lagrangian. This is connected with the abelian nature of the invariance group.

Let us consider now the case when the invariance group of the theory is non-abelian. For definiteness, we consider the group $SU(N)$ of $N \times N$ unitary matrices with unit determinant. This group has N^2-1 hermitian generators T^A, that obey the commutation relations

$$[T^A, T^B] = if^{ABC}T^C, \qquad A, B, C = 1, ..., N^2-1, \tag{B.13}$$

where f^{ABC} is completely antisymmetric. A generic element U of $SU(N)$ can be expressed in terms of the generators T^A and of a set of real functions $\alpha^A(x)$ by

$$U \equiv U(\alpha) = \exp(ig\alpha^A T^A), \tag{B.14}$$

where we have inserted a coupling constant g in analogy with the abelian case. The covariant derivative is now given by

$$D^\mu = \partial^\mu I - igA^\mu, \tag{B.15}$$

where I is the unity matrix in the representation space, and the vector field A^μ is now a hermitian matrix

$$A^\mu = A^\mu_A T^A. \tag{B.16}$$

By the same reasoning of the abelian case, we obtain the transformation law for A^μ:

$$A^\mu \to A'^\mu = UA^\mu U^{-1} - \frac{i}{g}(\partial^\mu U)U^{-1}. \tag{B.17}$$

Consider now an infinitesimal gauge transformation

$$U(\alpha) = I + ig\alpha^A T^A + \mathcal{O}(\alpha^2). \tag{B.18}$$

At order α, eq. (**??**) becomes

$$\begin{aligned} A'^\mu &= A^\mu + i[\alpha^A T^A, A^\mu] - \frac{i}{g} ig\partial^\mu \alpha^A T^A \\ &= A^\mu_C T^C - \alpha^A A^\mu_B f^{ABC} T^C + \partial^\mu \alpha^C T^C, \end{aligned} \tag{B.19}$$

or

$$A'^\mu_C = A^\mu_C - \alpha^A A^\mu_B f^{ABC} + \partial^\mu \alpha^C. \tag{B.20}$$

To build a kinetic term for the gauge fields, we write the analogous of eq. (**??**):

$$(D^\mu D^\nu - D^\nu D^\mu)\psi = igF^{\mu\nu}\psi, \tag{B.21}$$

where ψ is a multiplet of some $SU(N)$ representation, and $F^{\mu\nu} = F_A^{\mu\nu} T^A$. We find

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu],$$
$$F_A^{\mu\nu} = \partial^\mu A_A^\nu - \partial^\nu A_A^\mu + gf^{ABC} A_B^\mu A_C^\nu. \tag{B.22}$$

The kinetic term is then given by

$$-\frac{1}{4} F_A^{\mu\nu} F_{\mu\nu}^A. \tag{B.23}$$

In the non-abelian case, self-interaction terms of the gauge fields are present. This is related to the fact that, contrary to the abelian case, the field strength $F^{\mu\nu}$ transforms non-trivially under a gauge transformation:

$$F^{\mu\nu} \to F'^{\mu\nu} = UF^{\mu\nu}U^{-1}. \tag{B.24}$$

For an infinitesimal gauge transformation, we find

$$F'^{\mu\nu}_A = F_A^{\mu\nu} - gf^{ABC} \alpha^B F_C^{\mu\nu}, \tag{B.25}$$

which means that the components $F_A^{\mu\nu}$ form a multiplet in the adjoint representation of the gauge group.

APPENDIX C: The standard model lagrangian

Let us consider the following part of the standard model lagrangian:

$$\mathcal{L}_D + \mathcal{L}_{GF} - V(\phi), \tag{C.1}$$

where

$$\mathcal{L}_D = (D^\mu\phi)^\dagger D_\mu\phi \tag{C.2}$$

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\left[\partial^\mu W^i_\mu - \xi f^i(\phi)\right]^2 - \frac{1}{2\xi}\left[\partial^\mu B_\mu - \xi f(\phi)\right]^2 \tag{C.3}$$

$$V(\phi) = m^2\left|\phi\right|^2 + \lambda\left|\phi\right|^4. \tag{C.4}$$

We define

$$\phi = \phi_1 + \phi_2, \tag{C.5}$$

where

$$\phi_1 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \qquad \phi_2 = \begin{pmatrix} G^+ \\ (H+iG)/\sqrt{2} \end{pmatrix} \tag{C.6}$$

and

$$v^2 = -\frac{m^2}{\lambda}. \tag{C.7}$$

We have

$$\begin{aligned} \mathcal{L}_D &= \left[\partial^\mu\phi^\dagger + \frac{i}{2}\phi^\dagger\left(\frac{g}{2}W^\mu_i\tau^i + \frac{g'}{2}B^\mu\right)\right]\left[\partial_\mu\phi - \frac{i}{2}\left(\frac{g}{2}W^j_\mu\tau^j + \frac{g'}{2}B_\mu\right)\phi\right] \\ &\equiv \mathcal{L}_{\phi\phi} + \mathcal{L}_{\phi\phi VV} + \mathcal{L}_{\phi\phi V}. \end{aligned} \tag{C.8}$$

The first term is simply the kinetic term for ϕ,

$$\mathcal{L}_{\phi\phi} = (\partial^\mu\phi)^\dagger\partial_\mu\phi = \partial^\mu G^+\partial_\mu G^- + \frac{1}{2}\partial^\mu H\partial_\mu H + \frac{1}{2}\partial^\mu G\partial_\mu G. \tag{C.9}$$

Next, we consider the $\phi\phi VV$ term:

$$\begin{aligned} \mathcal{L}_{\phi\phi VV} &= \frac{1}{4}(g^2 W^\mu_i W^i_\mu + g'^2 B^\mu B_\mu)\phi^\dagger\phi + \frac{1}{2}gg'B^\mu W^i_\mu\phi^\dagger\tau^i\phi \\ &= \frac{1}{2}g^2 W^{+\mu}W^-_\mu\phi^\dagger\phi + \frac{1}{\sqrt{2}}gg'B^\mu\phi^\dagger\begin{bmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{bmatrix}\phi \end{aligned}$$

$$+\frac{1}{4}(B^\mu \;\; W_3^\mu)\left[\begin{array}{cc} g'^2\phi^\dagger\phi & gg'\phi^\dagger\tau^3\phi \\ gg'\phi^\dagger\tau^3\phi & g^2\phi^\dagger\phi \end{array}\right]\left(\begin{array}{c} B_\mu \\ W_{3\mu} \end{array}\right), \tag{C.10}$$

where

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2). \tag{C.11}$$

To obtain the physical vector states, we must isolate the mass term in eq. (**??**) and put it in diagonal form. The mass term is given by replacing ϕ with ϕ_1 in eq. (**??**). We find

$$\mathcal{L}_{mass} = \frac{1}{4}g^2v^2W^{+\mu}W_\mu^- + \frac{1}{8}(B^\mu \;\; W_3^\mu)\left[\begin{array}{cc} g'^2 & -gg' \\ -gg' & g^2 \end{array}\right]\left(\begin{array}{c} B_\mu \\ W_{3\mu} \end{array}\right), \tag{C.12}$$

which is diagonalized by

$$\left(\begin{array}{c} B^\mu \\ W_3^\mu \end{array}\right) = \left[\begin{array}{cc} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{array}\right]\left(\begin{array}{c} A^\mu \\ Z^\mu \end{array}\right); \qquad \tan\theta_W = \frac{g'}{g}. \tag{C.13}$$

In terms of $W_\mu^\pm$, A_μ and Z_μ eq. (**??**) becomes

$$\begin{aligned}
\mathcal{L}_{\phi\phi VV} \;=\;& W^{+\mu}W_\mu^-\left(m_W + \frac{1}{2}gH\right)^2 + \frac{1}{2}Z^\mu Z_\mu\left(m_Z + \frac{1}{2}\frac{g}{\cos\theta_W}H\right)^2 \\
&+\frac{1}{2}g^2W^{+\mu}W_\mu^-(G^+G^- + \frac{1}{2}G^2) + \frac{1}{8}\frac{g^2}{\cos^2\theta_W}Z^\mu Z_\mu G^2 \\
&+\frac{1}{4}\frac{g^2}{\cos^2\theta_W}(A^\mu\sin 2\theta_W + Z^\mu\cos 2\theta_W)^2G^+G^- \\
&+g\sin\theta_W(m_WA^\mu - m_ZZ^\mu\sin\theta_W)\left(G^-W_\mu^+ + G^+W_\mu^-\right) \\
&+\frac{1}{2}g^2\sin\theta_W(A^\mu - Z^\mu\tan\theta_W)\left[G^-W_\mu^+(H+iG) + G^+W_\mu^-(H-iG)\right]
\end{aligned} \tag{C.14}$$

where

$$m_W = \frac{1}{4}g^2v^2; \qquad m_Z^2 = \frac{1}{4}(g^2+g'^2)v^2. \tag{C.15}$$

The third term in $\mathcal{L}_D$ must be considered in conjunction with the gauge-fixing term. We have

$$\mathcal{L}_{\phi\phi V} \;=\; -\frac{i}{2}gW_\mu^i\left[(\partial^\mu\phi_2)^\dagger\tau^i\phi_1 - \phi_1^\dagger\tau^i\partial^\mu\phi_2\right] - \frac{i}{2}g'B_\mu\left[(\partial^\mu\phi_2)^\dagger\phi_1 - \phi_1^\dagger\partial^\mu\phi_2\right]$$

$$-\frac{i}{2}gW^i_\mu\left[(\partial^\mu\phi_2)^\dagger\tau^i\phi_2-\phi_2^\dagger\tau^i\partial^\mu\phi_2\right]-\frac{i}{2}g'B_\mu\left[(\partial^\mu\phi_2)^\dagger\phi_2-\phi_2^\dagger\partial^\mu\phi_2\right], \tag{C.16}$$

Exploiting the fact that $\partial^\mu\phi_1=0$, we can integrate by parts the first row. Adding $\mathcal{L}_{GF}$, we find

$$\begin{aligned}\mathcal{L}_{\phi\phi V}+\mathcal{L}_{GF} &= -\frac{i}{2}gW^i_\mu\left[(\partial^\mu\phi_2)^\dagger\tau^i\phi_2-\phi_2^\dagger\tau^i\partial^\mu\phi_2\right]-\frac{i}{2}g'B_\mu\left[(\partial^\mu\phi_2)^\dagger\phi_2-\phi_2^\dagger\partial^\mu\phi_2\right]\\ &+\partial^\mu W^i_\mu\left[\frac{i}{2}g(\phi_2^\dagger\tau^i\phi_1-\phi_1^\dagger\tau^i\phi_2)+f^i(\phi)\right]\\ &+\partial^\mu B_\mu\left[\frac{i}{2}g'(\phi_2^\dagger\phi_1-\phi_1^\dagger\phi_2)+f(\phi)\right]\\ &-\frac{1}{2\xi}(\partial^\mu W^i_\mu)^2-\frac{1}{2\xi}(\partial^\mu B_\mu)^2-\frac{1}{2}\xi f^i(\phi)f^i(\phi)-\frac{1}{2}\xi f(\phi)f(\phi).\end{aligned} \tag{C.17}$$

With the choices

$$f^i(\phi) = -\frac{i}{2}g(\phi_2^\dagger\tau^i\phi_1-\phi_1^\dagger\tau^i\phi_2) \tag{C.18}$$

$$f(\phi) = -\frac{i}{2}g'(\phi_2^\dagger\phi_1-\phi_1^\dagger\phi_2) \tag{C.19}$$

the mixing between vector bosons and scalars disappears, and we remain with

$$\begin{aligned}\mathcal{L}_{\phi\phi V}+\mathcal{L}_{GF} &= -\frac{i}{2}gW^+_\mu\left[(H+iG)\partial^\mu G^- - G^-\partial^\mu(H+iG)\right]\\ &+\frac{i}{2}gW^-_\mu\left[(H-iG)\partial^\mu G^+ - G^+\partial^\mu(H-iG)\right]\\ &-\frac{i}{2}\left[2g\sin\theta_W A^\mu+(g\cos\theta_W-g'\sin\theta_W)Z^\mu\right](G^+\partial_\mu G^- - G^-\partial_\mu G^+)\\ &-\frac{1}{2}(g\cos\theta_W+g'\sin\theta_W)Z^\mu(G\partial_\mu H-H\partial_\mu G)\\ &-\frac{1}{2\xi}(\partial^\mu W^i_\mu)^2-\frac{1}{2\xi}(\partial^\mu B_\mu)^2-\xi m_W^2G^+G^- -\frac{1}{2}\xi m_Z^2G^2.\end{aligned} \tag{C.20}$$

We see that the would-be Goldstone bosons $G^\pm$ ang G have acquired squared masses equal to ξm_W^2 and ξm_Z^2, respectively, as is necessary in order to cancel the unphysical singularities in the vector boson propagators. These masses vanish in the Landau gauge, $\xi = 0$.

The last term to be considered is the scalar potential $V(\phi)$. After some algebra, we find

$$V(\phi) = \frac{1}{2} m_H^2 \left[H + \frac{H^2 + G^+ G^- + G^2}{2v^2} \right]^2 , \tag{C.21}$$

where

$$m_H^2 = 2\lambda v^2. \tag{C.22}$$

We consider now the interaction between fermions and scalars. From eqs. (**??**-**??**) and the definition in eq. (**??**), we get

$$\begin{aligned} \mathcal{L}_Y^{hadr} &= -G^+ \left(\overline{u}_L K h_D d_R - \overline{u}_R h_U K d_L \right) - G^- \left(\overline{d}_R h_D K^\dagger u_L - \overline{d}_L K^\dagger h_U u_R \right) \\ &\quad - \frac{1}{\sqrt{2}} (v + H) \left(\overline{d} h_D d + \overline{u} h_U u \right) - \frac{iG}{\sqrt{2}} \left(\overline{d} h_D \gamma_5 d - \overline{u} h_U \gamma_5 u \right) , \end{aligned} \tag{C.23}$$

and

$$\mathcal{L}_Y^{lept} = -\frac{1}{\sqrt{2}} (v + H) \overline{e} h_L e - G^+ \overline{\nu} h_L e_R - G^- \overline{e}_R h_L \nu, \tag{C.24}$$

where sums over generation indices are understood.

APPENDIX D: The $SU(2)$ custodial symmetry

We have seen in section 3 that in the standard model at tree level the masses of weak vector bosons, m_W and m_Z, satisfy the relationship

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = 1. \tag{D.1}$$

Equation (**??**) could in principle be modified at higher orders in perturbation theory. Actually, the measured value for ρ is very close to 1:

$$\rho_{exp} = 1.0048 \pm 0.0022, \tag{D.2}$$

thus suggesting that some symmetry property prevents the quantity ρ from receiving large radiative corrections. We will show that this is indeed the case in the standard model.

Preliminarly, we observe that, even after the inclusion of radiative corrections, the most general vector boson mass term is given by

$$\mathcal{L}_{mass} = \frac{1}{2}m_W^2(W^{1\mu}W_\mu^1 + W^{2\mu}W_\mu^2) + \frac{1}{2}(B^\mu \;\; W_3^\mu)\left[\begin{array}{cc} M^2 & M'^2 \\ M'^2 & M''^2 \end{array}\right]\left(\begin{array}{c} B_\mu \\ W_{3\mu} \end{array}\right). \tag{D.3}$$

Furthermore, the condition that the photon stays massless gives us $M'^2 = MM''$, and $M^2 + M''^2 = m_Z^2$. Therefore, the mass matrix in the neutral sector is completely fixed by the value of one parameter, M^2, and it is diagonalized by a rotation of an angle θ_W given by

$$\tan\theta_W = \frac{\sqrt{m_Z^2 - M^2}}{M}. \tag{D.4}$$

This in turn implies that

$$\rho = \frac{m_W^2}{m_Z^2\cos^2\theta_W} = \frac{m_W^2}{M^2}, \tag{D.5}$$

that is, $\rho = 1$ only if $M^2 = m_W^2$.

Next we notice that the scalar potential

$$V(\phi) = m^2 \mid \phi \mid^2 + \lambda \mid \phi \mid^4 \tag{D.6}$$

is invariant under a group of transformations which is larger than the standard model $SU(2)_L \times U(1)_Y$. In fact, defining

$$\phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{D.7}$$

we see that

$$\mid \phi \mid^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \tag{D.8}$$

can be interpreted as the squared length of a real four vector. Therefore, the scalar potential has an $O(4) \sim SU(2) \times SU(2)$ invariance. This symmetry property can be implemented in the following way. We define a 2×2 matrix

$$H = \begin{bmatrix} \phi^+ & \phi^{0*} \\ \phi^0 & -\phi^- \end{bmatrix}. \tag{D.9}$$

Recalling that the field $\phi_c = (\phi^{0*}, -\phi^-)^T$ transforms as an $SU(2)$ doublet, it follows that, under the action of a generic $SU(2)_L$ transformation U, we have

$$H \to UH. \tag{D.10}$$

On the other hand, it is easy to check that the scalar potential can be written in terms of H as

$$V(\phi) = \frac{1}{2} m^2 Tr\left(H^\dagger H\right) + \frac{1}{2}\lambda Tr\left(H^\dagger H\right)^2, \tag{D.11}$$

which is invariant under the $SU(2)_L \times SU(2)$ transformation

$$H \to UHV^\dagger, \tag{D.12}$$

where V is another $SU(2)$ constant matrix, independent of U. This is possible because the structure of H in eq. (**??**) is preserved also by right multiplication with an $SU(2)$ matrix. Equation (**??**) is a representation of the $O(4)$ symmetry we mentioned above. Is it possible to write also the kinetic term for the field ϕ in an $O(4)$-invariant way? The natural candidate is of course

$$\frac{1}{2} Tr \left(D_\mu H\right)^\dagger D^\mu H, \tag{D.13}$$

which is invariant under the transformations (**??**) since $D^\mu \to UD^\mu U^\dagger$. However, one readily realizes that (**??**) is not equal to $(D_\mu\phi)^\dagger D^\mu \phi$ (prove this statement as an exercise); this is because ϕ and ϕ_c have opposite values of the hypercharge quantum number. We conclude that the $O(4)$ symmetry is violated by the hypercharge interaction term contained in the covariant derivative.

Due to spontaneous breaking of $SU(2)_L$, the gound state is not invariant under $O(4)$; however, there is a residual $O(3) \sim SU(2)$ symmetry under transformations of the kind

$$H \to U H \tau_1 U^\dagger \tau_1, \tag{D.14}$$

that leave the vacuum expectation value $< H >= v\tau_1\sqrt{2}$ unchanged (U is now x-independent). We are almost at the end of the road: in fact, it is easy to check that the only mass term for the W^i_μ fields allowed by the symmetry in eq. (**??**) is of the form $W^i_\mu W^\mu_i$, that is, a scalar product in $O(3)$. In other words, $M^2 = m^2_W$ in the notation of eq. (**??**).

We have proven that $\rho = 1$ is a consequence of the so-called *custodial* $SU(2)$ symmetry defined in eq. (**??**), and therefore it is not spoiled by radiative corrections. The inclusion of the hypercharge interaction, that breaks $O(4)$ explicitly, does not change this conclusion, since radiative corrections to ρ due to the hypercharge coupling are very small.

Of course, fermion mass terms do not preserve the custodial symmetry; we expect corrections to eq. (**??**) of the order of $G_\mu m^2_f$. More precisely, one finds

$$\rho \simeq 1 + \frac{3G_\mu m_t^2}{8\pi^2\sqrt{2}}, \tag{D.15}$$

where we have included only the contribution from the top quark, for obvious reasons.